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ON APPLICATIONS OF INEQUALITIES FOR QUASIDEVIATION MEANS IN ACTUARIAL MATHEMATICS

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Abstract. Applying the results of Zs. Páles, concerning the inequalities for quasideviation means, we characterize some natural properties of implicitly defined functional stemming from actuarial mathematics.

1. Quasideviation means

A notion of quasideviation mean has been introduced in [11]. It is a generalization of a deviation mean, investigated in [3, 4, 5]. A related concept is that of implicit mean, studied in [10]. A series of properties of quasideviation means have been proved in [12]. In order to recall this notion, assume that $I \subseteq \mathbb{R}$ is an open interval. A function $D: I^2 \to \mathbb{R}$ is said to be *a quasideviation*, provided it satisfies the following three conditions:

(D1) for every $x, y \in I$, $D(x,y)$ is of the same sign as $x-y$;
(D2) for every $x \in I$, the function $I \ni t \to D(x,t)$ is continuous;
(D3) for every $x, y \in I$ with $x<y$, the function

$$(x,y) \ni t \to D(y,t)/D(x,t)$$

is strictly increasing.

For every $n \in \mathbb{N}$, let

$$\Delta_n := \{\overline{\lambda} = (\lambda_1, \dots, \lambda_n) : \lambda_1, \dots, \lambda_n \in [0,\infty) : \sum_{i=1}^{n} \lambda_i > 0\}.$$

According to [12, Theorem 1], if $D: I^2 \to \mathbb{R}$ is a quasideviation, then for every $n \in \mathbb{N}$, $\overline{x} = (x_1, \dots, x_n) \in I^n$ and $\overline{\lambda} = (\lambda_1, \dots, \lambda_n) \in \Delta_n$, the equation

$$\sum_{i=1}^{n} \lambda_i D(x_i, t) = 0 \tag{1}$$

has a unique solution t_0. Furthermore, we have

$$\min\{x_i : i \in \{1,..,n\}\} \leqslant t_0 \leqslant \max\{x_i : i \in \{1,..,n\}\}.$$

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Hence, equation (1) defines a mean. Following [11], we call it *a D-quasideviation mean of $\overline{x}$ weighted by $\overline{\lambda}$* and we denote it by $\tilde{\mathfrak{M}}_D(\overline{x};\overline{\lambda})$. If $\overline{\lambda}=(1,\ldots,1)$ then the mean is called *a D-quasideviation mean of $\overline{x}$* and it is denoted by $\mathfrak{M}_D(\overline{x})$.

In [12] several properties of quasideviation means have been proved. The following two results, concerning the comparison and subadditivity of the means, respectively, will play an important role in our considerations.

THEOREM 1. ([12, Theorem 7]) *Assume that $I\subset\mathbb{R}$ is an open interval and $D_1,D_2:I^2\to\mathbb{R}$ are quasideviations. Then the following statements are equivalent:*

(i) $\mathfrak{M}_{D_1}(\overline{x})\leqslant\mathfrak{M}_{D_2}(\overline{x})$ *for* $\overline{x}\in I^n$, $n\in\mathbb{N}$;

(ii) $\tilde{\mathfrak{M}}_{D_1}(\overline{x};\overline{\lambda})\leqslant\tilde{\mathfrak{M}}_{D_2}(\overline{x};\overline{\lambda})$ *for* $\overline{x}\in I^n$, $\overline{\lambda}\in\Delta_n$, $n\in\mathbb{N}$;

(iii) $\tilde{\mathfrak{M}}_{D_1}((x_1,x_2);(\lambda,1-\lambda))\leqslant\tilde{\mathfrak{M}}_{D_2}((x_1,x_2);(\lambda,1-\lambda))$ *for* $x_1,x_2\in I$, $\lambda\in[0,1]$;

(iv) *there exists a function* $A:I\to(0,\infty)$ *such that*

$$D_1(x,t)\leqslant A(t)D_2(x,t)\quad \textit{for}\quad x,t\in I;$$

(v) $D_1(x,t)D_2(y,t)\leqslant D_1(y,t)D_2(x,t)$ *for* $x,y,t\in I$, $x\leqslant t\leqslant y$.

THEOREM 2. ([12, Theorem 11]) *Assume that $D_1,D_2,D_3:(0,\infty)\to\mathbb{R}$ are quasideviations. Then the following statements are equivalent:*

(i) $\mathfrak{M}_{D_1}(\overline{x}+\overline{y})\leqslant\mathfrak{M}_{D_2}(\overline{x})+\mathfrak{M}_{D_3}(\overline{y})$ *for* $\overline{x},\overline{y}\in(0,\infty)^n$, $n\in\mathbb{N}$;

(ii) *there exist functions* $A,B:(0,\infty)^2\to\mathbb{R}$ *such that*

$$D_1(x+y,s+t)\leqslant A(s,t)D_2(x,s)+B(s,t)D_3(y,t)\quad \textit{for}\quad x,y,s,t\in(0,\infty).$$

We conclude this section with one more result, which is a particular case of [12, Theorem 4].

THEOREM 3. *Assume that $D:(0,\infty)\to\mathbb{R}$ is a quasideviation. Then the following statements are equivalent:*

(i) $\mathfrak{M}_D\left(\frac{\overline{x}+\overline{y}}{2}\right)\leqslant\frac{\mathfrak{M}_D(\overline{x})+\mathfrak{M}_D(\overline{y})}{2}$ *for* $\overline{x},\overline{y}\in(0,\infty)^n$, $n\in\mathbb{N}$;

(ii) *there exist functions* $A,B:(0,\infty)^2\to\mathbb{R}$ *such that*

$$D\left(\frac{x+y}{2},\frac{s+t}{2}\right)\leqslant A(s,t)D(x,s)+B(s,t)D(y,t)\ \textit{for}\ x,y,s,t\in(0,\infty).$$

2. Zero utility principle

Assume that $\mathscr{X}_+$ is a family of all non-negative bounded random variables on a given probability space. Consider an insurance company, covering the risks represented by the elements of $\mathscr{X}_+$. If $w \in \mathbb{R}$ is an initial wealth of the insurance company and $v: \mathbb{R} \to \mathbb{R}$ is its continuous and strictly increasing utility function, then for every $X \in \mathscr{X}_+$ there exists a unique real number $H_{v,w}(X)$ such that

$$E[v(w + H_{v,w}(X) - X)] = v(w). \tag{2}$$

In this way, equation (2) determines a functional $H_{v,w}$ on $\mathscr{X}_+$, called *the principle of equivalent utility* (cf. [1, 2, 6]). It belongs to the so-called economic methods of insurance contracts pricing. The principle, proposed by Bühlmann [2], involves the notion of a utility function and is defined in such a way that the insurance company is indifferent between rejecting the contract and entering into it. Let $u: \mathbb{R} \to \mathbb{R}$ be given by $u(x) = v(x+w) - v(w)$ for $x \in \mathbb{R}$. Then u is strictly increasing, continuous and $u(0) = 0$. Moreover, setting $H_u := H_{v,w}$, from (2) we derive that

$$E[u(H_u(X) - X)] = 0 \quad \text{for} \quad X \in \mathscr{X}_+. \tag{3}$$

A functional H_u on $\mathscr{X}_+$, defined by equation (3), is called *the zero utility principle* (cf. [2, 14]). In general, (3) has no explicit solution. The most prominent exceptions are the cases where $u(x) = cx$ for $x \in \mathbb{R}$ with a $c > 0$ and $u(x) = a(1 - e^{-cx})$ for $x \in \mathbb{R}$ with some $a, c > 0$. In the first case, we get the net premium principle $H_u(X) = E[X]$ for $X \in \mathscr{X}_+$ and in the second one we obtain the exponential premium principle $H_u(X) = \frac{1}{c} \ln E[e^{cX}]$ for $X \in \mathscr{X}_+$. Several results concerning the properties of the principle of equivalent utility and the zero utility principle can be found e.g. in [1, 2, 6, 14]. Recently, the properties of the principle of equivalent utility under various theories of choice have been investigated in [7] and [8].

The aim of this paper is to show that the zero utility principle is a particular case of the quasideviation mean. In this way we establish a tool for dealing with the properties of the principle. Furthermore, we show how this tool can be effectively applied to the characterization of such properties of the principle as: comparison, equality, positive homogeneity, risk loading, subadditivity and convexity.

3. Results

In what follows we assume that (Ω, Σ, P) is a probability space with a non-atomic P. This implies that for any probability distribution μ on $\mathbb{R}$ there is a random variable $X: \Omega \to \mathbb{R}$ with the distribution μ (cf. e.g. [13, Lemma 2.7.1]). Let $\mathscr{X}_+ \subset L^\infty(\Omega, \Sigma, P)$ be a family of all non-negative bounded random variables and, for every $n \in \mathbb{N}$, let

$$\Gamma_n := \{\overline{x} = (x_1, \ldots, x_n) \in [0, \infty)^n : x_1 < x_2 < \ldots < x_n\}$$

and

$$\mathscr{P}_n := \{\overline{p} = (p_1, \ldots, p_n) : p_1, \ldots, p_n \in (0, 1] : \sum_{i=1}^{n} p_i = 1\}.$$

Furthermore, for every $n \in \mathbb{N}$, $\overline{x} = (x_1, \ldots, x_n) \in \Gamma_n$ and $\overline{p} = (p_1, \ldots, p_n) \in \mathscr{P}_n$, let $\langle \overline{x}; \overline{p} \rangle$ denote the random variable taking the values $x_1, \ldots, x_n$ with probabilities $p_1, \ldots, p_n$, respectively. Put

$$\mathscr{X}_+^{(n)} := \{\langle \overline{x}; \overline{p} \rangle : \overline{x} \in \Gamma_n, \overline{p} \in \mathscr{P}_n\} \quad \text{for} \quad n \in \mathbb{N}$$

and

$$\mathscr{X}_+^{fin} := \bigcup_{n \in \mathbb{N}} \mathscr{X}_+^{(n)}.$$

We begin with the result showing that the zero utility principle is a particular case of a quasideviation mean.

THEOREM 4. *Assume that $u : \mathbb{R} \to \mathbb{R}$ is a continuous and strictly increasing utility function with $u(0) = 0$. Then a function $D_u : (0, \infty)^2 \to \mathbb{R}$, defined by*

$$D_u(x, y) = -u(y - x) \quad \textit{for} \quad (x, y) \in (0, \infty)^2, \tag{4}$$

is a quasideviation. Furthermore,

$$H_u(\langle \overline{x}; \overline{p} \rangle) = \tilde{\mathfrak{M}}_{D_u}(\overline{x}; \overline{p}) \quad \textit{for} \quad \overline{x} \in \Gamma_n, \ \overline{p} \in \mathscr{P}_n, \ n \in \mathbb{N}. \tag{5}$$

Proof. The properties (D1)–(D3) of D_u follow directly from the fact that u is strictly increasing, continuous and $u(0) = 0$. Moreover, in view of (3) and (4), for every $n \in \mathbb{N}$, $\overline{x} \in \Gamma_n$ and $\overline{p} \in \mathscr{P}_n$, we obtain

$$\sum_{i=1}^{n} p_i D_u(x_i, H_u(\langle \overline{x}; \overline{p} \rangle)) = -\sum_{i=1}^{n} p_i u(H_u(\langle \overline{x}; \overline{p} \rangle) - x_i) = -E[u(H_u(\langle \overline{x}; \overline{p} \rangle) - \langle \overline{x}; \overline{p} \rangle] = 0,$$

which implies (5). □

Now, we are going to present a result concerning a comparison of the zero utility principles.

THEOREM 5. *Let $u, v : \mathbb{R} \to \mathbb{R}$ be continuous and strictly increasing utility functions with $u(0) = v(0) = 0$. Then the following statements are equivalent:*

(*i*) $H_v(X) \leqslant H_u(X)$ *for* $X \in \mathscr{X}_+^{(2)}$;

(*ii*) $H_v(X) \leqslant H_u(X)$ *for* $X \in \mathscr{X}_+$;

(*iii*) *there exists* $M \in (0, \infty)$ *such that*

$$u(x) \leqslant Mv(x) \quad \textit{for} \quad x \in \mathbb{R}. \tag{6}$$

Proof. Assume that (i) holds. We claim that

$$\tilde{\mathfrak{M}}_{D_v}((x_1, x_2); (p, 1-p)) \leqslant \tilde{\mathfrak{M}}_{D_u}((x_1, x_2); (p, 1-p)) \ \text{for} \ x_1, x_2 \in (0, \infty), \ \ p \in (0, 1). \tag{7}$$

To this end, fix $x_1, x_2 \in (0,\infty)$ and $p \in (0,1)$. If $x_1 = x_2$ then both sides of (7) are equal. Suppose that $x_1 \neq x_2$, say $x_1 < x_2$. Then $(x_1,x_2) \in \Gamma_2$ and so, making use of (5), in view of (i), we get

$$\tilde{\mathfrak{M}}_{D_v}((x_1,x_2);(p,1-p)) = H_v(\langle (x_1,x_2);(p,1-p)\rangle)$$
$$\leqslant H_u(\langle (x_1,x_2);(p,1-p)\rangle) = \tilde{\mathfrak{M}}_{D_u}((x_1,x_2);(p,1-p)).$$

Thus, (7) is proved. Therefore, applying Theorem 1, we conclude that there exists a function $A:(0,\infty)\to(0,\infty)$ such that $D_v(x,y) \leqslant A(y)D_u(x,y)$ for $x,y \in (0,\infty)$. Since u and v are strictly increasing and $u(0)=v(0)=0$, replacing in the last inequality x by $y-x$, in view of (4), we obtain

$$\frac{u(x)}{v(x)} \leqslant \frac{1}{A(y)} \quad \text{for} \quad x \in (0,\infty), \quad y > x \tag{8}$$

and

$$\frac{u(x)}{v(x)} \geqslant \frac{1}{A(y)} \quad \text{for} \quad x \in (-\infty,0], \quad y \in (0,\infty). \tag{9}$$

From (9) it follows that $M := \sup\{1/A(y) : y \in (0,\infty)\} < \infty$. Furthermore, taking into account (8) and (9), we obtain (6). This proves that $(i) \Rightarrow (iii)$.

Now, assume that (iii) holds. Then, in view of (3) and (6), we get

$$E[u(H_v(X)-X)] \leqslant ME[v(H_v(X)-X)] = 0 \quad \text{for} \quad X \in \mathscr{X}_+. \tag{10}$$

Since, for every $X \in \mathscr{X}_+$, the function $\mathbb{R} \ni t \to E[u(t-X)]$ is nondecreasing and $H_u(X)$ is its unique zero, from (10) we derive that $H_v(X) \leqslant H_u(X)$ for $X \in \mathscr{X}_+$. In this way we have proved that $(iii) \Rightarrow (ii)$.

The implication $(ii) \Rightarrow (i)$ is obvious. □

The following result is a direct consequence of Theorem 5.

THEOREM 6. *Let $u,v:\mathbb{R}\to\mathbb{R}$ be continuous and strictly increasing utility functions with $u(0)=v(0)=0$. The following statements are equivalent:*

(i) $H_v(X) = H_u(X)$ *for* $X \in \mathscr{X}_+^{(2)}$;

(ii) $H_v(X) = H_u(X)$ *for* $X \in \mathscr{X}_+$;

(iii) *there exists* $M \in (0,\infty)$ *such that* $u(x) = Mv(x)$ *for* $x \in \mathbb{R}$.

From Theorem 5 we also derive two results, establishing characterizations of the risk loading property and positive homogeneity of the zero utility principle. Let us recall that a premium principle H has *the risk loading* property on a family of risks $\mathscr{X} \subseteq \mathscr{X}_+$, provided $H(X) \geqslant E[X]$ for $X \in \mathscr{X}$. Furthermore, if $\mathscr{X} \subseteq \mathscr{X}_+$ is such that $a\mathscr{X} \subset \mathscr{X}$ for $a \in (0,\infty)$, then a premium principle H is said to be *positively homogeneous* on $\mathscr{X}$, provided $H(aX) = aH(X)$ for $X \in \mathscr{X}$, $a \in (0,\infty)$.

THEOREM 7. *Let $u:\mathbb{R}\to\mathbb{R}$ be a continuous and strictly increasing utility function with $u(0)=0$. The following statements are equivalent:*

(i) $H_u(X)\geqslant E[X]$ *for* $X\in\mathscr{X}_+^{(2)}$*;*

(ii) $H_u(X)\geqslant E[X]$ *for* $X\in\mathscr{X}_+$*;*

(iii) *there exists* $M\in(0,\infty)$ *such that* $u(x)\leqslant Mx$ *for* $x\in\mathbb{R}$*.*

Proof. Let v be the identity on $\mathbb{R}$. Then, in view of (3), we have $H_v(X)=E[X]$ for $X\in\mathscr{X}_+$. Hence, applying Theorem 5, we get the assertion. □

REMARK 1. If $u:\mathbb{R}\to\mathbb{R}$ is a strictly increasing continuous concave function with $u(0)=0$ then the right-sided derivation of u at 0 exists and it is positive. Moreover, we have $u(x)\leqslant u'_+(0)x$ for $x\in\mathbb{R}$. Therefore, according to Theorem 7, the concavity of u implies the risk loading property of H_u on $\mathscr{X}_+$ (see e.g. [14, Theorem 3.2.7]).

THEOREM 8. *Let $u:\mathbb{R}\to\mathbb{R}$ be a continuous and strictly increasing utility function with $u(0)=0$. The following statements are equivalent:*

(i) H_u *is positively homogeneous on* $\mathscr{X}_+^{(2)}$*;*

(ii) H_u *is positively homogeneous on* $\mathscr{X}_+$*;*

(iii) *there exist* $\alpha,\beta,r\in(0,\infty)$ *such that*

$$u(x)=\begin{cases}-\alpha(-x)^r & \text{for } x\in(-\infty,0],\\ \beta x^r & \text{for } x\in(0,\infty).\end{cases}\tag{11}$$

Proof. Assume that (i) holds. For every $t\in(0,\infty)$, define a function $u_t:\mathbb{R}\to\mathbb{R}$ by $u_t(x)=u(tx)$ for $x\in\mathbb{R}$. Then, in view of (3), for every $X\in\mathscr{X}_+^{(2)}$ and $t\in(0,\infty)$, we have

$$E[u_t(H_u(X)-X)]=E[u(t(H_u(X)-X))]=E[u(H_u(tX)-tX)]=0,$$

that is $H_{u_t}(X)=H_u(X)$. Therefore, according to Theorem 6, for every $t\in(0,\infty)$ there exists $\alpha(t)\in(0,\infty)$ such that $u_t(x)=\alpha(t)u(x)$ for $x\in\mathbb{R}$. Thus

$$u(tx)=\alpha(t)u(x)\quad\text{for}\quad x\in\mathbb{R},\ t\in(0,\infty).\tag{12}$$

In particular, we have $u(tx)=\alpha(t)u(x)$ for $t,x\in(0,\infty)$. Moreover, as u is strictly increasing and continuous, so is α. Hence, applying [9, Theorem 13.3.8], we conclude that there exist $\beta,r\in(0,\infty)$ such that $u(x)=\beta x^r$ for $x\in(0,\infty)$ and $\alpha(t)=t^r$ for $t\in(0,\infty)$. Furthermore, setting in (12) $x=-1$, we get $u(-t)=u(-1)t^r$ for $t\in(0,\infty)$, whence $u(x)=u(-1)(-x)^r$ for $x\in(-\infty,0)$. Consequently, as $u(0)=0$, we obtain (11) with $\alpha:=-u(-1)>0$. Thus, $(i)\Rightarrow(iii)$.

If (iii) holds then $u(tx) = t^r u(x)$ for $x \in \mathbb{R}$, $t \in (0,\infty)$. Hence, for every $X \in \mathscr{X}_+$ and $t \in (0,\infty)$, we obtain

$$E[u(tH_u(X) - tX)] = t^r E[u(H_u(X) - X)] = 0 = E[u(H_u(tX) - tX)],$$

which gives $H_u(tX) = tH_u(X)$. Therefore $(iii) \Rightarrow (ii)$.

Obviously, $(ii) \Rightarrow (i)$. $\square$

In the next two theorems we characterize the subadditivity and convexity of the zero utility principle, respectively.

THEOREM 9. *Let $u : \mathbb{R} \to \mathbb{R}$ be a continuous and strictly increasing utility function with $u(0) = 0$. The following statements are equivalent:*

(i) *$H_u(X+Y) \leqslant H_u(X) + H_u(Y)$ for $X, Y \in \mathscr{X}_+^{fin}$;*

(ii) *$H_u(X+Y) \leqslant H_u(X) + H_u(Y)$ for $X, Y \in \mathscr{X}_+$;*

(iii) *u is superadditive, that is $u(x+y) \geqslant u(x) + u(y)$ for $x, y \in \mathbb{R}$.*

Proof. Assume that (i) holds. We show that

$$\mathfrak{M}_{D_u}(\overline{x}+\overline{y}) \leqslant \mathfrak{M}_{D_u}(\overline{x}) + \mathfrak{M}_{D_u}(\overline{y}) \ \text{ for } \overline{x},\, \overline{y} \in (0,\infty)^n, \ \ n \in \mathbb{N}, \tag{13}$$

where D_u is given by (4). To this end, fix $n \in \mathbb{N}$, and $\overline{x} = (x_1, \dots, x_n)$, $\overline{y} = (y_1, \dots, y_n) \in (0,\infty)^n$. Since P is non-atomic, there exist $A_1, \dots, A_n \in \Sigma$ such that $\bigcup_{i=1}^n A_i = \Omega$, $A_i \cap A_j = \emptyset$ for $i, j \in \{1, \dots, n\}$, $i \neq j$ and $P(A_i) = 1/n$ for $i \in \{1, \dots, n\}$. Let $X = \sum_{i=1}^n \mathbb{1}_{A_i} x_i$. Then $X \in \mathscr{X}_+^{fin}$ and, in view of (3), we get

$$\begin{aligned}
\sum_{i=1}^n D_u(x_i, H_u(X)) &= -n \sum_{i=1}^n \frac{1}{n} u(H_u(X) - x_i) = -n \sum_{i=1}^n E[\mathbb{1}_{A_i} u(H_u(X) - x_i)] \\
&= -nE\left[\sum_{i=1}^n \mathbb{1}_{A_i} u(H_u(X) - x_i)\right] = -nE[u(H_u(X) - X)] = 0.
\end{aligned}$$

Thus $H_u(X) = \mathfrak{M}_{D_u}(\overline{x})$. Furthermore, taking $Y = \sum_{i=1}^n \mathbb{1}_{A_i} y_i$, in the same way we obtain $H_u(Y) = \mathfrak{M}_{D_u}(\overline{y})$ and $H_u(X+Y) = \mathfrak{M}_{D_u}(\overline{x}+\overline{y})$. Therefore, making use of (i), we get $\mathfrak{M}_{D_u}(\overline{x}+\overline{y}) \leqslant \mathfrak{M}_{D_u}(\overline{x}) + \mathfrak{M}_{D_u}(\overline{y})$. In this way we have proved (13). Hence, according to Theorem 2, there exist functions $A, B : (0,\infty)^2 \to \mathbb{R}$ such that

$$D_u(x+y, s+t) \leqslant A(s,t) D_u(x,s) + B(s,t) D_u(y,t) \quad \text{for} \ \ x, y, s, t \in (0,\infty).$$

Thus, taking into account (4), we obtain

$$u(s+t-x-y) \geqslant A(s,t) u(s-x) + B(s,t) u(t-y) \quad \text{for} \ \ x, y, s, t \in (0,\infty). \tag{14}$$

Setting in (14) $x = s/2$ and $y = t$, we get $A(s,t) \leqslant 1$ for $s, t \in (0,\infty)$. On the other hand, applying (14) with $x = 2s$ and $y = t$, we obtain $A(s,t) \geqslant 1$ for $s, t \in (0,\infty)$.

Hence $A(s,t) = 1$ for $s,t \in (0,\infty)$. Similarly, we have $B(s,t) = 1$ for $s,t \in (0,\infty)$. Therefore, (14) becomes

$$u(s+t-x-y) \geqslant u(s-x) + u(t-y) \quad \text{for} \quad x,y,s,t \in (0,\infty). \tag{15}$$

Let $x,y \in \mathbb{R}$ and $s,t \in (0,\infty)$ be such that $s > x$ and $t > y$. Then, replacing in (15) x and y by $s-x$ and $t-y$, respectively, we conclude that $u(x+y) \geqslant u(x) + u(y)$. Hence, u is superadditive and so $(i) \Rightarrow (iii)$.

If u is superadditive then, in view of (3), for every $X,Y \in \mathscr{X}_+$, we get

$$\begin{aligned} E[u(H_u(X)+H_u(Y)-(X+Y))] &\geqslant E[u(H_u(X)-X)] + E[u(H_u(Y)-Y)] \\ &= 0 = E[u(H_u(X+Y)-(X+Y))], \end{aligned}$$

which implies that $H_u(X+Y) \leqslant H_u(X) + H_u(Y)$. Therefore, $(iii) \Rightarrow (ii)$. The implication $(ii) \Rightarrow (i)$ is obvious. □

THEOREM 10. *The following statements are equivalent:*

(i)

$$H_u\left(\frac{X+Y}{2}\right) \leqslant \frac{H_u(X)+H_u(Y)}{2} \quad \textit{for} \quad X,Y \in \mathscr{X}_+^{fin};$$

(ii)

$$H_u((1-\alpha)X + \alpha Y) \leqslant (1-\alpha)H_u(X) + \alpha H_u(Y) \quad \textit{for} \quad X,Y \in \mathscr{X}_+, \alpha \in (0,1);$$

(iii) *u is concave.*

Proof. Assume that (i) holds. Then, arguing as in the proof of Theorem 9, we conclude that

$$\mathfrak{M}_{D_u}\left(\frac{\overline{x}+\overline{y}}{2}\right) \leqslant \frac{\mathfrak{M}_{D_u}(\overline{x}) + \mathfrak{M}_{D_u}(\overline{y})}{2} \quad \text{for } \overline{x},\overline{y} \in (0,\infty)^n, \quad n \in \mathbb{N},$$

where D_u is given by (4). Hence, according to Theorem 3, there exist functions $A,B : (0,\infty)^2 \to \mathbb{R}$ such that

$$D_u\left(\frac{x+y}{2}, \frac{s+t}{2}\right) \leqslant A(s,t)D_u(x,s) + B(s,t)D_u(y,t) \quad \text{for } x,y,s,t \in (0,\infty).$$

Thus, in view of (4), we get

$$u\left(\frac{s-x+t-y}{2}\right) \geqslant A(s,t)u(s-x) + B(s,t)u(t-y) \quad \text{for} \quad x,y,s,t \in (0,\infty). \tag{16}$$

Fix $t \in (0,\infty)$. Replacing in (16) s, x and y by t, $t-x$ and $t-y$, respectively, we obtain

$$u\left(\frac{x+y}{2}\right) \geqslant A(t,t)u(x) + B(t,t)u(y) \quad \text{for} \quad x,y \in (-\infty,t). \tag{17}$$

Interchanging in (17) the role of x and y and adding obtained in this way inequality side by side to (17), we get

$$2u\left(\frac{x+y}{2}\right) \geqslant (A(t,t)+B(t,t))(u(x)+u(y)) \quad \text{for} \quad x,y\in(-\infty,t). \tag{18}$$

Applying (18) with $x=y=t/2$ and then with $x=y=-t/2$, we conclude that $A(t,t)+B(t,t)=1$. Hence, (18) becomes

$$u\left(\frac{x+y}{2}\right) \geqslant \frac{u(x)+u(y)}{2} \quad \text{for} \quad x,y\in(-\infty,t).$$

Since $t\in(0,\infty)$ is arbitrarily fixed, this means that

$$u\left(\frac{x+y}{2}\right) \geqslant \frac{u(x)+u(y)}{2} \quad \text{for} \quad x,y\in\mathbb{R},$$

that is u is Jensen-concave. Therefore, as u is continuous, it is concave. Hence $(i)\Rightarrow(iii)$.

If (iii) holds then, in view of (3), for every $X,Y\in\mathscr{X}_+$ and $\alpha\in(0,1)$, we get

$$\begin{aligned}
&E[u((1-\alpha)H_u(X)+\alpha H_u(Y)-((1-\alpha)X+\alpha Y))]\\
&\geqslant (1-\alpha)E[u(H_u(X)-X)]+\alpha E[u(H_u(Y)-Y)]\\
&=0=E[u(H_u((1-\alpha)X+\alpha Y)-((1-\alpha)X+\alpha Y))],
\end{aligned}$$

which implies that $H_u((1-\alpha)X+\alpha Y)\leqslant(1-\alpha)H_u(X)+\alpha H_u(Y)$. Therefore, $(iii)\Rightarrow(ii)$. The implication $(ii)\Rightarrow(i)$ is trivial. □

REFERENCES

[1] N. L. BOWERS, H. U. GERBER, J. C. HICKMAN, D. A. JONES AND C. J. NESBITT, *Actuarial Mathematics*, The Society of Actuaries, Itasca Illinois, 1986.

[2] H. BÜHLMANN, *Mathematical Models in Risk Theory*, Springer, Berlin, 1970.

[3] Z. DARÓCZY, *A general inequality for means*, Aequationes Math. **7** (1971), 16–21.

[4] Z. DARÓCZY, *Über eine Klasse von Mittelwerten*, Publ. Math. Debrecen **19** (1972), 211–217.

[5] Z. DARÓCZY AND ZS. PÁLES, *On comparison of mean values*, Publ. Math. Debrecen **29** (1982), 107–115.

[6] H. U. GERBER, *An Introduction to Mathematical Risk Theory*, S. S. Huebner Foundation, Homewood Illinois: R. D. Irwin Inc., 1979.

[7] S. HEILPERN, *A rank-dependent generalization of zero utility principle*, Insurance: Mathematics and Economics **33** (2003), 67–73.

[8] M KAŁUSZKA AND M. KRZESZOWIEC, *Pricing insurance contracts under Cumulative Prospect Theory*, Insurance: Mathematics and Economics **50** (2012), 159–166.

[9] M. KUCZMA, *An Introduction to the Theory of Functional Equations and Inequalities*, Birkhäuser, Berlin, 2009.

[10] L. LOSONCZI, *General inequalities for nonsymmetric means*, Aequationes Math. **9** (1973), 221–235.

[11] ZS. PÁLES, *Characterization of quasideviation means*, Acta. Math. Sci. Hungar. **40** (1982), 243–260.

[12] ZS. PÁLES, *General inequalities for quasideviation means*, Aequationes Math. **36** (1988), 32–56.

[13] S. T. RACHEV, L. B. KLEBANOV, S. V. STOYANOV AND F. J. FABOZZI, *The Methods of Distances in the Theory of Probability and Statistics*, Springer, New York, 2013.

[14] T. ROLSKI, H. SCHMIDLI, V. SCHMIDT AND J. TEUGELS, *Stochastic Processes for Insurance and Finance*, John Wiley & Sons, New York, 1999.

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