Mathematical
Inequalities
& Applications
Volume 21, Number 3 (2018), 611–627 doi:10.7153/mia-2018-21-45

ON THE REVERSE CONVOLUTION INEQUALITIES FOR THE KONTOROVICH–LEBEDEV, FOURIER COSINE TRANSFORMS AND APPLICATIONS

PHAM VAN HOANG

(*Communicated by M. Praljak*)

Abstract. In this paper, we investigate some reverse weighted L_p-norm ($p > 1$) inequalities for convolutions related to Kontorovich-Lebedev, Fourier cosine transforms. A class of intergro-diffirential equations involing in Bessel operator are considered. The estimate of scratted acoustic field is established.

1. Introduction

For the Fourier transform, beside the fundamental Young's inequality, the weighted L_p-norm inequalities for Fourier convolution were considered by S. Saitoh *et all* (see [8, 11] and references there in). Inequalities of these types were considered by N. D. V. Nhan, D. T. Duc, V. K. Tuan (see [6] and references there in). Inequalities for Fourier cosine convolution was studied by N. T. Hong (see [3]). The reverse weighted L_p-norm convolution inequalities for Fourier transform and its applications also investigated in [9, 10].

PROPOSITION 1. ([9]) *Let F_1 and F_2 be positive functions satisfying*

$$0 < m_1^{\frac{1}{p}} \leqslant F_1(x) \leqslant M_1^{\frac{1}{p}} < \infty, \;\; 0 < m_2^{\frac{1}{p}} \leqslant F_2(x) \leqslant M_1^{\frac{1}{p}} < \infty, \;\; p > 1, \;\; x \in \mathbb{R}. \quad (1)$$

Then for any positive continuous functions ρ_1 and ρ_2, we have the reverse L_p-weighted convolution inequality

$$\begin{aligned} &||((F_1\rho_1) \underset{F}{*} (F_2\rho_2))(\rho_1 \underset{F}{*} \rho_2)^{\frac{1}{p}-1}||_{L_p(\mathbb{R})} \\ &\geqslant A_{p,q}\left(\frac{m_1 m_2}{M_1 M_2}\right)^{-1} ||F_1||_{L_p(\mathbb{R};\rho_1)}||F_2||_{L_p(\mathbb{R};\rho_2)}, \end{aligned} \quad (2)$$

where

$$\left(f \underset{F}{*} g\right)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy, \quad x \in \mathbb{R}. \quad (3)$$

Mathematics subject classification (2010): 35A22, 44A35, 26D15, 26D10, 45E10, 45J05, 33E30, 65R10.

Keywords and phrases: Integral transforms, weighted L_p inequality, reverse Hölder inequality, Kontorovich-Lebedev transform, integro-differential equations.

© ELEMENT, Zagreb
Paper MIA-21-45

The main key to prove these results is the reverse Hölder's inequality.

PROPOSITION 2. ([13]) *For two positive functions f and g satisfying $0 < m \leqslant \frac{f}{g} \leqslant M < \infty$ on the set X, and for $p,q > 1$, $p^{-1}+q^{-1}=1$,*

$$\left(\int_X f d\mu\right)^{\frac{1}{p}}\left(\int_X g d\mu\right)^{\frac{1}{q}} \leqslant A_{p,q}\left(\frac{m}{M}\right)\int_X f^{\frac{1}{p}} g^{\frac{1}{q}} d\mu, \tag{4}$$

if the right hand side integral converges, where $A_{p,q}(t) = p^{-\frac{1}{p}} q^{-\frac{1}{q}} \dfrac{t^{-\frac{1}{pq}}(1-t)}{(1-t^{\frac{1}{p}})^{\frac{1}{p}}(1-t^{\frac{1}{q}})^{\frac{1}{q}}}$.

The convolution for Kontorovich-Lebedev transform was first introduced by V. A. Kakichev in [5]

$$(f \underset{\mathscr{K}\mathscr{L}}{*} g)(x) = \int_0^\infty\int_0^\infty \frac{1}{2x} e^{-\frac{1}{2}\left(\frac{uv}{x}+\frac{ux}{v}+\frac{vx}{u}\right)} f(u)g(v)dudv, \quad x>0, \tag{5}$$

where $\mathscr{K}\mathscr{L}$ denotes the Kontorovich-Lebedev transform (see [14, 17])

$$\mathscr{K}\mathscr{L}[f](y) = \int_0^\infty K_{iy}(x) f(x)dx, \quad y>0. \tag{6}$$

Its kernel consists of the Macdonald function $K_\nu(x)$ of the pure imaginary index $\nu = iy$. This function satifies the differential equation

$$z^2\frac{d^2u}{dz^2} + z\frac{du}{dz} - (z^2+\nu^2)u = 0. \tag{7}$$

The inequalities of Young's type as well as the boundedness in weighted L_p for the Kontorovich-Lebedev transform and its convolution were investigated in [4, 15, 16]. Combining the Kontorovich-Lebedev transform together with the Fourier cosine transform, the following convolution was introduced (see [17])

$$(f \underset{1}{*} g)(x) = \int_0^\infty\int_0^\infty \frac{1}{2\pi x}[e^{-x\cosh(u+v)} + e^{-x\cosh(u-v)}] f(u)g(v)dudv, \quad x>0. \tag{8}$$

In the present paper, we will establish the reverse weighted L_p-norm for convolutions (5), (8). In the mentioned applications, we are looking solution of integro-differential equations involing in Bessel operator in the convolution form and estimate it basing on the reverse norm inequalities of convolution. The estimate for the diffraction of an acoustic is given.

2. Reverse convolution inequalities for Kontorovich-Lebedev, Fourier cosine transforms

The aim of this section is drawing a parallel results of reverse weighted L_p-norm inequalities for convolutions related to Kontorovich-Lebedev, Fourier cosine transforms as the Fourier convolution (see [9, 10]).

THEOREM 3. *Let F_1 and F_2 be positive functions satisfying*

$$0 < m_1^{\frac{1}{p}} \leqslant F_1(x) \leqslant M_1^{\frac{1}{p}} < \infty,\ \ 0 < m_2^{\frac{1}{p}} \leqslant F_2(x) \leqslant M_2^{\frac{1}{p}} < \infty,\ \ p > 1,\ \ x > 0. \tag{9}$$

Then for any positive functions ρ_1 and ρ_2 we have the following reverse L_p-weighted convolution inequalities

$$\begin{aligned} &||((F_1\rho_1) \underset{\mathscr{K}\mathscr{L}}{*} (F_2\rho_2))(\rho_1 \underset{\mathscr{K}\mathscr{L}}{*} \rho_2)^{\frac{1}{p}-1}||_{L_p(\mathbb{R}_+)} \\ &\geqslant \left(\frac{1}{3}K_0(\frac{\sqrt{2}}{3})\right)^{\frac{1}{p}} \left\{A_{p,q}(\frac{m_1m_2}{M_1M_2})\right\}^{-1} ||F_1||_{L_p(\mathbb{R}_+;\rho_1(u)\varphi(u))}||F_2||_{L_p(\mathbb{R}_+;\rho_2(v)\varphi(v))}, \end{aligned} \tag{10}$$

$$\begin{aligned} &||((F_1\rho_1) \underset{1}{*} (F_2\rho_2))(\rho_1 \underset{1}{*} \rho_2)^{\frac{1}{p}-1}||_{L_p(\mathbb{R}_+;\pi x)} \\ &\qquad\geqslant \left\{A_{p,q}(\frac{m_1m_2}{M_1M_2})\right\}^{-1} ||F_1||_{L_p(\mathbb{R}_+;\frac{\rho_1(u)}{\cosh u})}||F_2||_{L_p(\mathbb{R}_+;\frac{\rho_2(v)}{\cosh v})}. \end{aligned} \tag{11}$$

Proof. First, using the AM-GM inequality for three positive real u, v, $\frac{1}{x}$, we obtain $\frac{uv}{x} = uv\frac{1}{x} \leqslant u^3 + v^3 + \frac{1}{x^3}$. Similarly, we have $\frac{xu}{v} \leqslant x^3 + u^3 + \frac{1}{v^3}$; $\frac{xv}{u} \leqslant x^3 + v^3 + \frac{1}{u^3}$. Therefore,

$$\frac{1}{2}\left(\frac{uv}{x} + \frac{xu}{v} + \frac{xv}{u}\right) \leqslant \left(\frac{x^3}{3} + \frac{1}{6x^3}\right) + \left(\frac{u^3}{3} + \frac{1}{6u^3}\right) + \left(\frac{v^3}{3} + \frac{1}{6v^3}\right).$$

So, we obtain the remarkable inequality

$$K(x,u,v) := \frac{1}{2x}e^{-\frac{1}{2}\left(\frac{uv}{x}+\frac{ux}{v}+\frac{vx}{u}\right)} \geqslant \frac{\varphi(x)\varphi(u)\varphi(v)}{2x}, \tag{12}$$

where $\varphi(t) = e^{-(\frac{t^3}{3}+\frac{1}{6t^3})}$. Put $f(u,v) = F_1^p(u)F_2^p(v)K(x,u,v)\rho_1(u)\rho_2(v)$, and put $g(u,v) = K(x,u,v)\rho_1(u)\rho_2(v)$. The condition (9) implies

$$m_1m_2 \leqslant \frac{f(u,v)}{g(u,v)} \leqslant M_1M_2,\ \forall u > 0, v > 0. \tag{13}$$

Applying the reverse Hölder's inequality (4) for f and g on $X = \mathbb{R}^2_+$, we get

$$
\begin{aligned}
&A_{p,q}\left(\frac{m_1m_2}{M_1M_2}\right)((F_1\rho_1) \underset{\mathscr{K}\mathscr{L}}{*} (F_2\rho_2))(x) \\
&= A_{p,q}\left(\frac{m_1m_2}{M_1M_2}\right)\int_0^\infty\int_0^\infty K(x,u,v)F_1(u)\rho_1(u)F_2(v)\rho_2(v)dudv \\
&= A_{p,q}\left(\frac{m_1m_2}{M_1M_2}\right)\int_0^\infty\int_0^\infty f^{\frac{1}{p}}(u,v)g^{\frac{1}{q}}(u,v)dudv \\
&\geqslant \left(\int_0^\infty\int_0^\infty f(u,v)dudv\right)^{\frac{1}{p}}\left(\int_0^\infty\int_0^\infty g(u,v)dudv\right)^{\frac{1}{q}} \\
&= \left(\int_0^\infty\int_0^\infty F_1^p(u)F_2^p(v)K(x,u,v)\rho_1(u)\rho_2(v)dudv\right)^{\frac{1}{p}}\left(\int_0^\infty\int_0^\infty K(x,u,v)\rho_1(u)\rho_2(v)dudv\right)^{\frac{1}{q}}.
\end{aligned} \tag{14}
$$

Therefore,

$$
\begin{aligned}
&\left(((F_1\rho_1) \underset{\mathscr{K}\mathscr{L}}{*} (F_2\rho_2))(x)\right)^p((\rho_1 \underset{\mathscr{K}\mathscr{L}}{*} \rho_2)(x))^{1-p} \\
\geqslant & A_{p,q}\left(\frac{m_1m_2}{M_1M_2}\right)^{-p}\left(\int_0^\infty\int_0^\infty F_1^p(u)F_2^p(v)K(x,u,v)\rho_1(u)\rho_2(v)dudv\right) \\
&\times\left(\int_0^\infty\int_0^\infty K(x,u,v)\rho_1(u)\rho_2(v)dudv\right)^{\frac{p}{q}}\left(\int_0^\infty\int_0^\infty K(x,u,v)\rho_1(u)\rho_2(v)dudv\right)^{1-p} \\
\geqslant & A_{p,q}\left(\frac{m_1m_2}{M_1M_2}\right)^{-p}\int_0^\infty\int_0^\infty \frac{\varphi(x)\varphi(u)\varphi(v)}{2x}F_1^p(u)F_2^p(v)\rho_1(u)\rho_2(v)dudv.
\end{aligned} \tag{15}
$$

From inequality (12) and using the Fubini theorem to interchange the order of intergration, we have

$$
\begin{aligned}
&||((F_1\rho_1) \underset{\mathscr{K}\mathscr{L}}{*} (F_2\rho_2))(\rho_1 \underset{\mathscr{K}\mathscr{L}}{*} \rho_2)^{\frac{1}{p}-1}||^p_{L_p(\mathbb{R}_+)} \\
&= \int_0^\infty\left(((F_1\rho_1) \underset{\mathscr{K}\mathscr{L}}{*} (F_2\rho_2))(x)\right)^p\left((\rho_1 \underset{\mathscr{K}\mathscr{L}}{*} \rho_2)(x)\right)^{1-p}dx \\
&\geqslant \left\{A_{p,q}\left(\frac{m_1m_2}{M_1M_2}\right)\right\}^{-p}\int_0^\infty\left(\int_0^\infty\int_0^\infty F_1^p(u)F_2^p(v)K(x,u,v)\rho_1(u)\rho_2(v)dudv\right)dx
\end{aligned}
$$

$$\geqslant \left\{A_{p,q}\left(\frac{m_1m_2}{M_1M_2}\right)\right\}^{-p}\int_0^\infty\int_0^\infty\int_0^\infty \frac{\varphi(x)\varphi(u)\varphi(v)}{2x}F_1^p(u)F_2^p(v)\rho_1(u)\rho_2(v)dxdudv$$
$$\geqslant \left\{A_{p,q}\left(\frac{m_1m_2}{M_1M_2}\right)\right\}^{-p}\int_0^\infty F_1^p(u)\rho_1(u)\varphi(u)du\int_0^\infty F_2^p(v)\rho_2(v)\varphi(v)dv\int_0^\infty \frac{\varphi(x)}{2x}dx$$
$$=\frac{1}{3}K_0\left(\frac{\sqrt{2}}{3}\right)\left\{A_{p,q}\left(\frac{m_1m_2}{M_1M_2}\right)\right\}^{-p}\int_0^\infty F_1^p(u)\rho_1(u)\varphi(u)du\int_0^\infty F_2^p(v)\rho_2(v)\varphi(v)$$
$$=\frac{1}{3}K_0\left(\frac{\sqrt{2}}{3}\right)\left\{A_{p,q}\left(\frac{m_1m_2}{M_1M_2}\right)\right\}^{-p}||F_1||^p_{L_p(\mathbb{R}_+;\rho_1(u)\varphi(u))}||F_2||^p_{L_p(\mathbb{R}_+;\rho_2(v)\varphi(v))}. \quad (16)$$

To prove the inequality (11), we note that

$$T(x,u,v)=\frac{1}{2\pi x}\left(e^{-x\cosh(u+v)}+e^{-x\cosh(u-v)}\right)\geqslant\frac{1}{2\pi x}\left(2\sqrt{e^{-x\cosh(u+v)}e^{-x\cosh(u-v)}}\right)$$
$$=\frac{1}{2\pi x}2e^{-\frac{x}{2}(\cosh(u+v)+\cosh(u-v))}=\frac{1}{\pi x}e^{-x\cosh u\cosh v}. \quad (17)$$

Let $f(u,v)=F_1^p(u)F_2^p(v)T(x,u,v)\rho_1(u)\rho_2(v)$ and $g(u,v)=T(x,u,v)\rho_1(u)\rho_2(v)$. We have

$$m_1m_2\leqslant\frac{f(u,v)}{g(u,v)}\leqslant M_1M_2. \quad (18)$$

Applying the reverse Hölder's inequality (4) for f and g on $X=\mathbb{R}_+^2$, we get

$$A_{p,q}\left(\frac{m_1m_2}{M_1M_2}\right)((F_1\rho_1)\underset{1}{*}(F_2\rho_2))(x)$$
$$=A_{p,q}\left(\frac{m_1m_2}{M_1M_2}\right)\int_0^\infty\int_0^\infty T(x,u,v)F_1(u)\rho_1(u)F_2(v)\rho_2(v)dudv$$
$$=A_{p,q}\left(\frac{m_1m_2}{M_1M_2}\right)\int_0^\infty\int_0^\infty f^{\frac{1}{p}}(u,v)g^{\frac{1}{q}}(u,v)dudv$$
$$\geqslant\left(\int_0^\infty\int_0^\infty f(u,v)dudv\right)^{\frac{1}{p}}\left(\int_0^\infty\int_0^\infty g(u,v)dudv\right)^{\frac{1}{q}}$$
$$=\left(\int_0^\infty\int_0^\infty F_1^p(u)F_2^p(v)T(x,u,v)\rho_1(u)\rho_2(v)dudv\right)^{\frac{1}{p}}\left(\int_0^\infty\int_0^\infty T(x,u,v)\rho_1(u)\rho_2(v)dudv\right)^{\frac{1}{q}}. \quad (19)$$

Therefore,

$$\Big(((F_1\rho_1)\underset{1}{*}(F_2\rho_2))(x)\Big)^p((\rho_1\underset{1}{*}\rho_2)(x))^{1-p}$$
$$\geqslant A_{p,q}(\frac{m_1m_2}{M_1M_2})^{-p}\int_0^\infty\int_0^\infty\frac{1}{\pi x}e^{-x\cosh u\cosh v}F_1^p(u)F_2^p(v)\rho_1(u)\rho_2(v)dudv. \tag{20}$$

Similarly, we obtain

$$||((F_1\rho_1)\underset{1}{*}(F_2\rho_2))(\rho_1\underset{1}{*}\rho_2)^{\frac{1}{p}-1}||^p_{L_p(\mathbb{R}_+;\pi x)}$$
$$=\int_0^\infty\Big(((F_1\rho_1)\underset{1}{*}(F_2\rho_2))(x)\Big)^p\Big((\rho_1\underset{1}{*}\rho_2)(x)\Big)^{1-p}\pi xdx$$
$$\geqslant\left\{A_{p,q}\left(\frac{m_1m_2}{M_1M_2}\right)\right\}^{-p}\int_0^\infty\left(\int_0^\infty\int_0^\infty F_1^p(u)F_2^p(v)T(x,u,v)\rho_1(u)\rho_2(v)dudv\right)\pi xdx$$
$$\geqslant\left\{A_{p,q}\left(\frac{m_1m_2}{M_1M_2}\right)\right\}^{-p}\int_0^\infty\int_0^\infty\int_0^\infty e^{-x\cosh u\cosh v}F_1^p(u)F_2^p(v)\rho_1(u)\rho_2(v)dxdudv$$
$$\geqslant A_{p,q}\left(\frac{m_1m_2}{M_1M_2}\right)^{-p}\int_0^\infty F_1^p(u)\rho_1(u)du\int_0^\infty F_2^p(v)\rho_2(v)dv\int_0^\infty e^{-x\cosh u\cosh v}dx$$
$$=\left\{A_{p,q}\left(\frac{m_1m_2}{M_1M_2}\right)\right\}^{-p}\int_0^\infty F_1^p(u)\frac{\rho_1(u)}{\cosh u}du\int_0^\infty F_2^p(v)\frac{\rho_2(v)}{\cosh v}dv$$
$$=\left\{A_{p,q}\left(\frac{m_1m_2}{M_1M_2}\right)\right\}^{-p}||F_1||^p_{L_p(\mathbb{R}_+;\frac{\rho_1(u)}{\cosh u})}||F_2||^p_{L_p(\mathbb{R}_+;\frac{\rho_2(v)}{\cosh v})}. \tag{21}$$

The proof is complete. □

In various problems, the solutions can be presented in the convolution form. In the cases of interest, the following reverse inequalities could be used to prove the lower boundedness of these solutions.

Denote $\mathbb{R}_\alpha=(0;\alpha)$ for $\alpha>0$.

THEOREM 4. *Let f and g be positive functions satisfying*

$$0<f(x)\leqslant M<\infty,\ \ 0<g(x)\leqslant N<\infty,\ \ x>0. \tag{22}$$

i) Assume that $f\in L_p(\mathbb{R}_+;\varphi)$, $g\in L_p(\mathbb{R}_+;\varphi)$, where $\varphi(t)=e^{-(\frac{t^3}{3}+\frac{1}{6t^3})}$. We have the inequality

$$(f\underset{\mathscr{K}\mathscr{L}}{*}g)(x)\geqslant\frac{\varphi(x)}{(M.N)^{p-1}2x}||f||^p_{L_p(\mathbb{R}_+;\varphi)}||g||^p_{L_p(\mathbb{R}_+;\varphi)}. \tag{23}$$

Moreover, the following inequality

$$||(f \underset{\mathcal{K}\mathcal{L}}{*} g)||_{L_r(\mathbb{R}_+)} \geqslant \left(\frac{2^{\frac{5+r}{6}} K_{\frac{1-r}{3}}(\frac{\sqrt{2}r}{3})}{3(MN)^{r(p-1)}} \right)^{\frac{1}{r}} ||f||_{L_p(\mathbb{R}_+;\varphi)} ||g||_{L_p(\mathbb{R}_+;\varphi)} \tag{24}$$

holds true for $r > p > 1$.

ii) Assume that $f \in L_p(\mathbb{R}_\alpha), g \in L_p(\mathbb{R}_\beta)$, *where* α, β *are positive real number. We have the inequality*

$$(f \underset{1}{*} g)(x) \geqslant \frac{e^{-x\cosh\alpha\cosh\beta}}{(MN)^{p-1}\pi x} ||f||^p_{L_p(\mathbb{R}_\alpha)} ||g||^p_{L_p(\mathbb{R}_\beta)}. \tag{25}$$

Moreover, the following inequality

$$||(f \underset{1}{*} g)||_{L_q(\mathbb{R}_r;x^\gamma)} \geqslant \left(\frac{r^{\gamma-q+1}}{\gamma-q+1} \right)^{\frac{1}{q}} \frac{e^{-r\cosh\alpha\cosh\beta}}{\pi(MN)^{p-1}} ||f||^p_{L_p(\mathbb{R}_\alpha)} ||g||^p_{L_p(\mathbb{R}_\beta)} \tag{26}$$

holds true for $\gamma \geqslant q > 1, r > 1$.

Proof. i) We have

$$\begin{aligned}(f^p \underset{\mathcal{K}\mathcal{L}}{*} g^p)(x) &= \int_0^\infty \int_0^\infty K(x,u,v) f^p(u) g^p(v) du dv \\ &\leqslant \int_0^\infty \int_0^\infty K(x,u,v) M^{p-1} N^{p-1} f(u) g(v) du dv = (MN)^{p-1} (f \underset{\mathcal{K}\mathcal{L}}{*} g)(x).\end{aligned} \tag{27}$$

On the other hand, we have

$$\begin{aligned}(f^p \underset{\mathcal{K}\mathcal{L}}{*} g^p)(x) &= \int_0^\infty \int_0^\infty K(x,u,v) f^p(u) g^p(v) du dv \\ &\geqslant \int_0^\infty \int_0^\infty \frac{\varphi(x)\varphi(u)\varphi(v)}{2x} f^p(u) g^p(v) du dv \\ &\geqslant \frac{\varphi(x)}{2x} \int_0^\infty \int_0^\infty f^p(u)\varphi(u) g^p(v)\varphi(v) du dv \\ &= \frac{\varphi(x)}{2x} ||f||^p_{L_p(\mathbb{R}_+;\varphi)} ||g||^p_{L_p(\mathbb{R}_+;\varphi)}.\end{aligned} \tag{28}$$

From (27) and (28), we obtain the estimate

$$(f \underset{\mathcal{K}\mathcal{L}}{*} g)(x) \geqslant \frac{\varphi(x)}{2x(M.N)^{p-1}} ||f||^p_{L_p(\mathbb{R}_+;\varphi)} ||g||^p_{L_p(\mathbb{R}_+;\varphi)}. \tag{29}$$

Therefore,

$$\begin{aligned} ||(f \underset{\mathscr{K}\mathscr{L}}{*} g)||^r_{L_r(\mathbb{R}_+)} &= \int_0^\infty |f \underset{\mathscr{K}\mathscr{L}}{*} g|^r(x)dx \\ &\geqslant ||f||^{pr}_{L_p(\mathbb{R}_+;\varphi)} ||g||^{pr}_{L_p(\mathbb{R}_+;\varphi)} \int_0^\infty \left(\frac{\varphi(x)}{2x(M.N)^{p-1}}\right)^r dx \\ &= \frac{2^{\frac{5+r}{6}} K_{\frac{1-r}{3}}(\frac{\sqrt{2}r}{3})}{3(MN)^{r(p-1)}} ||f||^{pr}_{L_p(\mathbb{R}_+;\varphi)} ||g||^{pr}_{L_p(\mathbb{R}_+;\varphi)}. \end{aligned} \tag{30}$$

Hence, one can obtain (24).

ii) Similarly, we have

$$\begin{aligned} (f^p \underset{1}{*} g^p)(x) &= \int_0^\infty \int_0^\infty T(x,u,v) f^p(u) g^p(v) du dv \\ &\leqslant \int_0^\infty \int_0^\infty T(x,u,v)(MN)^{p-1} f(u) g(v) du dv \\ &= (MN)^{p-1}(f \underset{1}{*} g)(x). \end{aligned} \tag{31}$$

On the other hand,

$$\begin{aligned} (f^p \underset{1}{*} g^p)(x) &= \int_0^\infty \int_0^\infty T(x,u,v) f^p(u) g^p(v) du dv \\ &\geqslant \int_0^\infty \int_0^\infty \frac{1}{\pi x} e^{-x\cosh u \cosh v} f^p(u) g^p(v) du dv \\ &\geqslant \int_0^\beta \int_0^\alpha \frac{1}{\pi x} e^{-x\cosh\alpha\cosh\beta} f^p(u) g^p(v) du dv \\ &= \frac{e^{-x.\cosh\alpha\cosh\beta}}{\pi x} ||f||^p_{L_p((\mathbb{R}_\alpha))} ||g||^p_{L_p((\mathbb{R}_\beta))}. \end{aligned} \tag{32}$$

From (31) and (32), one can obtain the estimate respective to x on $(0;r)$.

$$(f \underset{1}{*} g)(x) \geqslant \frac{e^{-x\cosh\alpha\cosh\beta}}{(MN)^{p-1}\pi x} ||f||^p_{L_p(\mathbb{R}_\alpha)} ||g||^p_{L_p(\mathbb{R}_\beta)}. \tag{33}$$

Therefore,

$$
\begin{aligned}
|(f \underset{1}{*} g)||_{L_q(\mathbb{R}_r;x^\gamma)} &= \left(\int_0^r (f \underset{1}{*} g)^q(x) x^\gamma dx \right)^{\frac{1}{q}} \\
&\geqslant \frac{e^{-r.\cosh\alpha\cosh\beta}}{(MN)^{p-1}\pi} ||f||^p_{L_p(\mathbb{R}_\alpha)} ||g||^p_{L_p(\mathbb{R}_\beta)} \left(\int_0^r x^{\gamma-q} dx \right)^{\frac{1}{q}} \\
&= \left(\frac{r^{\gamma-q+1}}{\gamma-q+1} \right)^{\frac{1}{q}} \frac{e^{-r.\cosh\alpha\cosh\beta}}{\pi(MN)^{p-1}} ||f||^p_{L_p(\mathbb{R}_\alpha)} ||g||^p_{L_p(\mathbb{R}_\beta)}, \qquad (34)
\end{aligned}
$$

where $\gamma \geqslant q > 1$, $r > 1$. □

3. Applications

3.1. A class of integro-differential equations involing the Bessel operator

Inspite of having many useful applications, not many integro-differential equations can be solved in closed form. In this section, we consider a class of the integro-differential equation

$$
f(x) + \frac{1}{2x} D \int_0^\infty \int_0^\infty e^{-\frac{1}{2}\left(\frac{uv}{x} + \frac{ux}{v} + \frac{vx}{u}\right)} f(u) h(v) du dv = g(x), \quad x > 0, \qquad (35)
$$

which are arisen naturally from the integral equation (see [4, 14, 15])

$$
f(x) + \frac{1}{2x} \int_0^\infty \int_0^\infty e^{-\frac{1}{2}\left(\frac{uv}{x} + \frac{ux}{v} + \frac{vx}{u}\right)} f(u) h(v) du dv = g(x), \quad x > 0, \qquad (36)
$$

where D is a differential operator.

In [14], S. B. Yakubovich introduced the following function space

$$
L^\alpha(\mathbb{R}_+) \equiv L(\mathbb{R}_+, K_\alpha(x)), \quad \alpha \geqslant 0, \qquad (37)
$$

with the norm

$$
||f||_{L^\alpha(\mathbb{R}_+)} = \int_0^\infty K_\alpha(x) |f(x)| dx. \qquad (38)
$$

In this function space, the following analog of the Wiener theorem was established.

THEOREM 5. ([14]) *Let* $f \in L^\alpha(\mathbb{R}_+)$. *If* $\mathscr{F}(s) = \lambda + \mathscr{K}\mathscr{L}[f](-is) \neq 0$ *for all* s *in the closed strip* $|\Re e(s)| \leqslant \alpha$, *including infinity then there is a unique* q *from* L^α *such that*

$$
\frac{1}{\lambda + \mathscr{K}\mathscr{L}[f](-is)} = \lambda + \mathscr{K}\mathscr{L}[q](-is). \qquad (39)
$$

LEMMA 1. [16] *Let* $f \in L_2(\mathbb{R}_+;x)$ *and* $h \in L^0(\mathbb{R}_+)$. *Then,* $(f \underset{\mathscr{K}\mathscr{L}}{*} h) \in L_2(\mathbb{R}_+;x)$ *and satisfies the factorization equation*

$$\mathscr{K}\mathscr{L}[f \underset{\mathscr{K}\mathscr{L}}{*} h](y) = \mathscr{K}\mathscr{L}[f](y)\mathscr{K}\mathscr{L}[h](y). \tag{40}$$

Now, we consider the case operator D is the Bessel operator $\mathscr{B}$ of the form

$$\mathscr{B}[\omega](x) := \left(x^2\frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial x} - x^2\right)\omega(x). \tag{41}$$

The equation (35) can be written in the form

$$f(x) + \frac{1}{2x}\mathscr{B}\left[\int_0^\infty\int_0^\infty e^{-\frac{1}{2}\left(\frac{xu}{v}+\frac{xv}{u}+\frac{uv}{x}\right)} f(u)h(v)dudv\right] = g(x). \tag{42}$$

We will find the solution of equation (42) in the $L_2(R_+;x) \cap L^0(\mathbb{R}_+)$.

THEOREM 6. *Let* $g \in L_2(\mathbb{R}_+;x)$ *and* $h \in L^0(\mathbb{R}_+)$ *are given functions, satifying*

$$1 - y^2\mathscr{K}\mathscr{L}[h](y) \neq 0,\ \forall y > 0; \tag{43}$$

$$y^2\mathscr{K}\mathscr{L}[h](y) \in L_2(\mathbb{R}_+; y\sinh \pi y); \mathscr{K}\mathscr{L}^{-1}(y^2\mathscr{K}\mathscr{L}[h](y)) = h_1(y) \in L^0(\mathbb{R}+). \tag{44}$$

The equation (42) *has a unique solution in* $L^0(\mathbb{R}_+) \cap L_2(\mathbb{R}_+;x)$ *which can be presented in the convolution form*

$$f(x) = g(x) + (g \underset{\mathscr{K}\mathscr{L}}{*} l)(x), \tag{45}$$

where l *exists uniquely via the following*

$$\frac{y^2\mathscr{K}\mathscr{L}[f](y)}{1 - y^2\mathscr{K}\mathscr{L}[h](y)} = \mathscr{K}\mathscr{L}[l](y). \tag{46}$$

Assuming that g, l *belong to* $L_p(\mathbb{R}_+;\varphi(x))$ *and* $0 < g(x) \leqslant M < \infty,\ 0 < l(x) \leqslant N < \infty$, $\forall x > 0$, *we obtain*

$$f(x) \geqslant g(x) + \frac{\varphi(x)}{(M.N)^{p-1}2x}\|g\|_{L_p(\mathbb{R}_+;\varphi(x))}\|l\|_{L_p(\mathbb{R}_+;\varphi(x))},\ \ x > 0. \tag{47}$$

Furthermore,

i) the asymptotics of the solution

$$f(x) = g(x)\left(1 + \bigcirc\left(\frac{1}{x}\right)\right),\ \ x \to \infty \tag{48}$$

is valid if we add the condition $g, l \in L_1(\mathbb{R}_+)$.

ii) a norm estimate

$$\|f\|_{L_{2p}(\mathbb{R}_+;x^p)} \geqslant \left[2(MN)^{1-p}\|g\|_{L_p(\mathbb{R}_+;\varphi(x))}\|g\|_{L_p(\mathbb{R}_+;(\varphi(x))^p)}\|l\|_{L_p(\mathbb{R}_+;\varphi(x))}\right]^{\frac{1}{2}} \tag{49}$$

holds on if we add condition $g \in L_p(\mathbb{R}+;(\varphi(x))^p)$ *and* $f \in L_{2p}(\mathbb{R}_+;x^p)$.

Proof. Since $h \in L^0(\mathbb{R}_+)$, $g \in L_2(\mathbb{R}_+)$, then viture the (1), we have $\omega(x) := (f \underset{\mathscr{K}\mathscr{L}}{*} h)(x) \in L_2(\mathbb{R}_+;x)$. For $\omega(x) \in L_2(\mathbb{R}_+;x)$, by using the inverse formula for the Kontorovich-Lebedev transform between two spaces $L_2(\mathbb{R}_+;x)$ and $L_2(\mathbb{R}_+;\frac{\pi^2}{2}y\sinh\pi y)$, we have

$$\omega(x) = \mathscr{K}\mathscr{L}^{-1}[\Psi(y)] := \frac{2}{\pi^2 x}\int_0^\infty K_{iy}(x) y \sinh\pi y\, \Psi(y) dy, \tag{50}$$

therefore we obtain

$$2x\omega(x) = \frac{4}{\pi^2}\int_0^\infty K_{iy}(x) y \sinh\pi y\, \Psi(y) dy, \tag{51}$$

where $\Psi(y) = \mathscr{K}\mathscr{L}[\omega](y)$. We have

$$\mathscr{B}[2x\omega(x)] = \frac{4}{\pi^2}\int_0^\infty \mathscr{B}[K_{iy}(x)] y \sinh\pi y\, \Psi(y) dy, \tag{52}$$

if integral (51) and

$$\int_0^\infty \left(\frac{\partial}{\partial x} K_{iy}(x)\right) y \sinh\pi y\, \Psi(y) dy, \tag{53}$$

$$\int_0^\infty \left(\frac{\partial^2}{\partial x^2} K_{iy}(x)\right) y \sinh\pi y\, \Psi(y) dy, \tag{54}$$

converge uniformly on any compact subset of $\mathbb{R}_+$. Since the Macdonald function is a solution of the Bessel equation, we obtain (see [4, 12, 16])

$$\mathscr{B}[K_{iy}(x)] = -y^2 K_{iy}(x). \tag{55}$$

Using (41), (50), (52), we have

$$\begin{aligned}\mathscr{K}\mathscr{L}\left[\frac{1}{2x}\mathscr{B}[2x\omega(x)]\right](y) &= \int_0^\infty \frac{1}{2x}K_{iy}(x)\frac{4}{\pi^2}\int_0^\infty \mathscr{B}[K_{i\tau}(x)]\tau\sinh\pi\tau\Psi(\tau)\, d\tau\, dx \\ &= \mathscr{K}\mathscr{L}\left[\mathscr{K}\mathscr{L}^{-1}[-\tau^2\Psi(\tau)]\right](y,t) = -y^2\Psi(y).\end{aligned} \tag{56}$$

Formula (56) holds true if

$$y^2\Psi(y) \in L_2(\mathbb{R}_+; y\sinh\pi y). \tag{57}$$

By taking the Kontorovich-Lebedev transform to both side of (42), we obtain

$$\mathscr{K}\mathscr{L}[f](y) - y^2\mathscr{K}\mathscr{L}[f](y)\mathscr{K}\mathscr{L}[h](y) = \mathscr{K}\mathscr{L}[g](y), \quad y > 0. \tag{58}$$

Combining with (43), we have

$$\mathscr{K}\mathscr{L}[f](y) = \frac{\mathscr{K}\mathscr{L}[g](y)}{1 - y^2\mathscr{K}\mathscr{L}[h](y)}. \tag{59}$$

The condition (44) implies that exsits a unique h_1 belongs to $L^0(\mathbb{R}_+)$ such that $\mathscr{K}\mathscr{L}[h_1](y) = y^2\mathscr{K}\mathscr{L}[h](y)$. In virtue of Wiener-Levy theorem [14], there exists a unique function $l \in L^0(\mathbb{R}_+)$ satifying $\dfrac{\mathscr{K}\mathscr{L}[h_1](y)}{1-\mathscr{K}\mathscr{L}[h_1](y)} = \mathscr{K}\mathscr{L}[l](y)$. The equation (59) can be rewriten $\mathscr{K}\mathscr{L}[f](y) = \mathscr{K}\mathscr{L}[g](y) + \mathscr{K}\mathscr{L}[g](y)\mathscr{K}\mathscr{L}[l](y)$. Therefore, the solution of equation (42) can represented in the convolution form (45).

On the other hand, using formula (233), page 73 in [2]

$$z\frac{\partial K_\nu(z)}{\partial z} = \nu K_\nu(z) - zK_{\nu+1}(z), \tag{60}$$

we have

$$\frac{\partial K_{iy}(x)}{\partial x} = \frac{1}{x}\left(iyK_{iy}(x) - xK_{iy+1}(x)\right), \tag{61}$$

and

$$\frac{\partial^2 K_{iy}(x)}{\partial^2 x} = -\frac{iy+y^2}{x^2}K_{iy}(x) - \frac{2iy+1}{x}K_{iy+1}(x) + K_{iy+2}(x). \tag{62}$$

Applying the integral representation 9.6.25, page 376 in [1]

$$K_\nu(x) = \frac{2^\nu\Gamma(\nu+1/2)}{\sqrt{\pi}x^\nu}\int_0^\infty \cos(xt)(t^2+1)^{-\nu-1/2}dt, \quad \Re\nu > -1/2,$$

with $\nu = 1+iy$ and $\nu = 2+iy$, and recalling that

$$\Gamma(\nu+1) = \nu\Gamma(\nu), \quad |\Gamma(1/2+iy)| = \sqrt{\frac{\pi}{\cosh(\pi y)}},$$

we obtain

$$|K_{iy+1}(x)| \leqslant \frac{C(1+y)e^{-\pi y/2}}{x}, \quad |K_{iy+2}(x)| \leqslant \frac{C(1+y^2)e^{-\pi y/2}}{x^2}, \text{ for any } y > 0,$$

uniformly in x on any compact subset of $\mathbb{R}_+$. For $K_{iy}(x)$ the following estimate holds [14]

$$|K_{iy}(x)| \leqslant e^{-\delta y}K_0(x\cos\delta), \quad 0 < \delta < \frac{\pi}{2}. \tag{63}$$

On the other hand,

$$|\mathscr{K}\mathscr{L}[f](y)| = |\int_0^\infty K_{iy}(v)f(v)dv| \leqslant \int_0^\infty K_0(v)|f(v)|dv = \|f\|_{L_1(\mathbb{R}_+;K_0(v))}. \tag{64}$$

We have

$$\begin{aligned}
&\left|\int_0^\infty K_{iy+\nu}(x)y^m \sinh \pi y\, \Psi(y)dy\right| \\
&\leqslant C\int_0^\infty \left|K_{iy+\nu}(x)\right| y^m \sinh \pi y\, |\mathscr{K}\mathscr{L}[f](y)||\mathscr{K}\mathscr{L}[h](y)|dy \\
&\leqslant C\int_0^\infty \left|K_{iy+\nu}(x)\right| y^m dy < C_{m,\nu}, \quad \nu = 0,1,2, \quad m = 0,1,2. \qquad (65)
\end{aligned}$$

Thus, integrals (51), (53), and (54) converge uniformly on any compact subset of $\mathbb{R}_+$, and (52) holds. We have

$$\begin{aligned}
\int_0^\infty |y^2\Psi(y)|^2 y\sinh \pi y dy &= \int_0^\infty |\mathscr{K}\mathscr{L}[f](y)|^2 |y^2\mathscr{K}\mathscr{L}[h](y)|^2 y\sinh \pi y dy \\
&\leqslant \|f\|^2_{L_1(\mathbb{R}_+;K_0(\nu))}\int_0^\infty |\mathscr{K}\mathscr{L}[h_1](y)|^2 y\sinh \pi y dy \\
&= \|f\|^2_{L_1(\mathbb{R}_+;K_0(\nu))}\|\mathscr{K}\mathscr{L}[h_1]\|^2_{L_2(\mathbb{R}_+;y\sinh \pi y)} < \infty.
\end{aligned}$$

Consequently, $y^2\Psi(y) \in L_2(\mathbb{R}_+;y\sinh \pi y)$.

i) We obtain the inequality (47) by applying directly the inequality (29). Using the elementary inequality

$$a^2 + b^2 \geqslant 2ab, \qquad (66)$$

we obtain

$$|K(x,u,v)| := \frac{1}{2x}e^{-\frac{1}{2}(\frac{uv}{x}+\frac{u^2+v^2}{uv}x)} \leqslant \frac{e^{-x}}{2x}. \qquad (67)$$

Therefore,

$$f(x) = g(x) + (g \underset{\mathscr{K}\mathscr{L}}{*} l)(x) \leqslant g(x) + \int_0^\infty\int_0^\infty \frac{e^{-x}}{2x} g(u)l(v)dudv \qquad (68)$$

$$= g(x) + \frac{e^{-x}}{2x}\|g\|_{L_1(\mathbb{R}+)}\|l\|_{L_1(\mathbb{R}+)}. \qquad (69)$$

From (47) and (69), one can obtain (48).

ii) Recalling the inequality (66), (47) we have

$$\begin{aligned}\left(\int_0^\infty |f(x)|^{2p}x^p dx\right)^{\frac{1}{2p}} &= \left(\int_0^\infty |g(x)+(g \underset{\mathscr{K}\mathscr{L}}{*} l)(x)|^{2p}x^p dx\right)^{\frac{1}{2p}}\\ &\geqslant \left(\int_0^\infty 2^{2p}|g(x)|^p|(g \underset{\mathscr{K}\mathscr{L}}{*} l)(x)|^p dx\right)^{\frac{1}{2p}}\\ &\geqslant \left(\int_0^\infty 2^p|g(x)|^p \frac{(\varphi(x))^p}{(M.N)^{p(p-1)}}\|g\|^{p^2}_{L_p(\mathbb{R}_+;\varphi(x))}\|l\|^{p^2}_{L_p(\mathbb{R}_+;\varphi(x))}dy\right)^{\frac{1}{2p}}\\ &= \left[2(MN)^{1-p}\|g\|_{L_p(\mathbb{R}_+;\varphi(x))}\|g\|^p_{L_p(\mathbb{R}_+;(\varphi(x))^p)}\|l\|^p_{L_p(\mathbb{R}_+;\varphi(x))}\right]^{\frac{1}{2}}.\end{aligned}$$

The proof is complete. □

Next, we consider the operator D is the differential operator of the infinite order, related to Bessel operator

$$\begin{aligned}D[\omega](x) &= \lim_{N\to\infty}\prod_{k=1}^{N}\left(-1+\frac{\mathscr{B}}{(2k-1)^2}\right)[\omega](x)\\ &= -\lim_{N\to\infty}\prod_{k=1}^{N}\left(1+\frac{x\left(x-\frac{d}{dx}-x\frac{d^2}{dx^2}\right)}{(2k-1)^2}\right)[\omega](x). \qquad (70)\end{aligned}$$

The equation (35) can be rewritten in the form

$$f(x)-\frac{1}{2x}\lim_{N\to\infty}\prod_{k=1}^{N}\left(1+\frac{x\left(x-\frac{d}{dx}-x\frac{d^2}{dx^2}\right)}{(2k-1)^2}\right)\int_0^\infty\int_0^\infty e^{-\frac{1}{2}\left(\frac{xu}{v}+\frac{xv}{u}+\frac{uv}{x}\right)}f(u)h(v)dudv = g(x). \qquad (71)$$

Assuming that $h\in L^0(\mathbb{R}_+)$ and $g\in L_2(\mathbb{R}_+;x)$ are given functions. Recalling the results in [16], we have

$$\mathscr{K}\mathscr{L}[G](y) = \mathscr{K}\mathscr{L}[f](y)\mathscr{K}\mathscr{L}[h](y)\cosh\frac{\pi y}{2}, \qquad (72)$$

where

$$G(x) = \frac{1}{2x}\lim_{N\to\infty}\prod_{k=1}^{N}\left(1+\frac{x\left(x-\frac{d}{dx}-x\frac{d^2}{dx^2}\right)}{(2k-1)^2}\right)\int_0^\infty\int_0^\infty e^{-\frac{1}{2}\left(\frac{xu}{v}+\frac{xv}{u}+\frac{uv}{x}\right)}f(u)h(v)dudv. \qquad (73)$$

By taking the Kontorovich-Lebedev transform to the both side of (71), we obtain

$$\mathscr{K}\mathscr{L}[f](y)-\mathscr{K}\mathscr{L}[f](y)\mathscr{K}\mathscr{L}[h](y)\cosh\frac{\pi y}{2}=\mathscr{K}\mathscr{L}[g](y),\quad x>0. \tag{74}$$

Therefore,

$$\mathscr{K}\mathscr{L}[f](y)=\frac{\mathscr{K}\mathscr{L}[g](y)}{1-\cosh\frac{\pi y}{2}\mathscr{K}\mathscr{L}[h](y)} \tag{75}$$

if

$$1-\cosh\frac{\pi y}{2}\mathscr{K}\mathscr{L}[h](y)\neq 0,\forall y>0. \tag{76}$$

Assuming that exsits $q_1\in L^0(\mathbb{R}_+)$ such that $\mathscr{K}\mathscr{L}[q_1](y)=\cosh\frac{\pi y}{2}\mathscr{K}\mathscr{L}[h](y)$. In virtue of Wiener-Levy theorem [14], there exists a unique function $q\in L^0(\mathbb{R}_+)$ satisfying $\frac{\mathscr{K}\mathscr{L}[q_1](y)}{1-\mathscr{K}\mathscr{L}[q_1](y)}=\mathscr{K}\mathscr{L}[q](y)$, then

$$\frac{\cosh\frac{\pi y}{2}\mathscr{K}\mathscr{L}[h](y)}{1-\cosh\frac{\pi y}{2}\mathscr{K}\mathscr{L}[h](y)}=\mathscr{K}\mathscr{L}[q](y). \tag{77}$$

The equation (75) can be rewriten

$$\mathscr{K}\mathscr{L}[f](y)=\mathscr{K}\mathscr{L}[g](y)+\mathscr{K}\mathscr{L}[g](y)\mathscr{K}\mathscr{L}[q](y).$$

Therefore, the solution of equation (71) can represented in the convolution form

$$f(x)=g(x)+(g\underset{\mathscr{K}\mathscr{L}}{*}q)(x). \tag{78}$$

3.2. Estimate the diffraction of an acoustic via inequalities of generalized convolution

We will use the similar statement about a particular of the scattered acoustic field considered in [12] with the spectral potential function $g(u)$ is defined

$$(F_c g)(t)=\sinh(2\pi t)u(t). \tag{79}$$

Hence, we have

$$\begin{aligned}U(x)&=\frac{-\pi\sqrt{x}}{2}\frac{2}{\pi^2 x}\int_0^\infty tK_{it}(x)(F_c g)(t)(F_c h_1)(t)dt\\&=\frac{-\pi}{2}\sqrt{x}(g\underset{1}{*}h_1)(x).\end{aligned} \tag{80}$$

Formula (80) gives us another representation of $U(x)$ in a form related to the generalized convolution (8). Assuming that g is positive function and $g \in L_p(\mathbb{R}_\alpha), \alpha > 0$. By using the inequality (25), we have

$$|U(x)| = \frac{\pi}{2}\sqrt{x}(g \underset{1}{*} h_1)(x) \geqslant \frac{e^{-x\cosh\alpha\cosh\beta}}{2(MN)^{p-1}\sqrt{x}}||g||^p_{L_p(\mathbb{R}_\alpha)}||h_1||^p_{L_p(\mathbb{R}_\beta)}, \beta > 0. \tag{81}$$

In particular, $p = 2$, we obtain $\int\limits_0^\beta \frac{1}{\cosh^2\frac{v}{2}}dv = 2\,\mathrm{Tanh}\,\beta$, $\beta > 0$. Thus,

$$|U(x)| \geqslant 2\,\mathrm{Tanh}\,\beta\frac{e^{-x\cosh\alpha\cosh\beta}}{(MN)\sqrt{x}}||g||^2_{L_2(\mathbb{R}_\alpha)}.$$

If $U \in L_2(\mathbb{R}_r; x^2)$, $r > 1$ and $g \in L_2(\mathbb{R}_\alpha)$, $\alpha > 0$, $\beta > 0$ then we have estimate

$$\begin{aligned} \|U\|^2_{L_2(\mathbb{R}_r;x)} &= \int\limits_0^r |U(x)|^2 x dx = \int\limits_0^r \left|\frac{2U(x)}{\pi\sqrt{x}}\right|^2 \frac{\pi^2 x^2}{4}dx = \int\limits_0^r [(g \underset{1}{*} h_1)(x)]^2\frac{\pi^2x^2}{4}dx \\ &= \frac{\pi^2}{4}\|[(g \underset{1}{*} h_1)\|^2_{L_2(\mathbb{R}_r;x^2)} \geqslant \frac{r}{\pi M}\mathrm{Tanh}^2\,\beta e^{-2r\cosh\alpha\cosh\beta}\|g\|^4_{L_2(\mathbb{R}_\alpha)}, \end{aligned} \tag{82}$$

by using the inequality (26) with $\gamma = q = 2$, $p = 2$, $M = \sup\limits_{x\in\mathbb{R}_+} g(x) < \infty$ and $0 < h_1(x) = \frac{1}{\cosh\frac{x}{2}} < N = 1$, $x > 0$.

Therefore,

$$\|U\|_{L_2(\mathbb{R}_r;x^2)} \geqslant \sqrt{\frac{r}{\pi M}}\mathrm{Tanh}\,\beta e^{-r\cosh\alpha\cosh\beta}\|g\|^2_{L_2(\mathbb{R}_\alpha)}.$$

REFERENCES

[1] M. ABRAMOWITZ AND I. A. STEGUN, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series **55**, Washington, 1979.

[2] D. S JONES, *The Theory of Electromagnetism*, Pergamon Press, London, 1964.

[3] N. T. HONG, *Inequalities for Fourier cosine convolution and applications*, Integral Transforms Spec. Funct. **62**, 10 (2010), 755–763.

[4] N. T. HONG, P. V. HOANG, AND V. K. TUAN, *The convolution for the Kontorovich-Lebedev transform revisited*, J. Math. Anal. Appl. **440**, 1 (2016), 369–378.

[5] V. A. KAKICHEV, *On the convolution for integral transforms*, Izv. Vyssh. Uchebn. Zaved. Mat. **2**, 1 (1967), 53–62 (In Russian).

[6] N. D. V. NHAN, D. T. DUC, AND V. K. TUAN, *Reverse weighted l_p-norm inequalities for convolution type integrals*, Armenian. Journ. of. Amth **2**, 3 (2009), 77–93.

[7] S. SAITOH, *A fudamental inequality in convolution of L_2 functions on the half line*, Proc. Amer. Math. **91**, 2 (1984), 285–286.

[8] S. SAITOH, *Inequalities in the most simple Sobolev space and convolution of L_2 functions with weights*, Proc. Amer. Math. **118**, 2 (1993), 515–520.

[9] S. SAITOH, V. K. TUAN, AND M. YAMAMOTO, *Reverse weighted L_p-norm inequalities in convolutions*, J. of Ineq. In Pure and App. Math. **1**, 1 (2000), 1–11.

[10] S. SAITOH, V. K. TUAN, AND M. YAMAMOTO, *Reverse convolution inequalities and applications to inverse heat source problems*, J. of Ineq. In Pure and App. Math. **3**, 5 (2002), 1–11.
[11] S. SAITOH, V. K. TUAN, AND M. YAMAMOTO, *Convolution inequalities and Applications*, J. of Ineq. In Pure and App. Math. **4**, 3 (2003), 1–8.
[12] T. TUAN, N. T. HONG, AND P. V. HOANG, *Generalized convolution for the Kontorovich-Lebedev, Fourier transforms and applications to acoustic fields*, Acta Math. Vietnam. **42**, 2 (2016), 355–367.
[13] L. XIAO-HUA, *On the inverse of Hölder inequality*, Math. Practice and Theory **1**, 1 (1990), 84–88.
[14] S. B. YAKUBOVICH, *Index Transforms*, World Scisentific Publishing Company, Singapore, New Jersey, London and Hong Kong, 1996.
[15] S. B. YAKUBOVICH, *Boundedness and inversion properties of certain convolution transforms*, J. Korean Math. Soc. **40**, 6 (2003), 999–1014.
[16] S. B. YAKUBOVICH, *Integral transforms of the Kontorovich-Lebedev convolution type*, Collect. Math. **54**, 2 (2003), 99–110.
[17] S. B. YAKUBOVICH, L. E. BRITVINA, *Convolution related to the Fourier and Kontorovich-Lebedev transforms revisited*, Integral Transforms Spec. Funct. **21**, 4 (2010), 259–276.

(Received November 1, 2017)

Pham Van Hoang
Kim Lien High School
Hanoi Department of Education and Training
Hanoi, Vietnam
e-mail: phamhoang0103@gmail.com