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ON THE MIXED (ℓ_1,ℓ_2)–LITTLEWOOD INEQUALITIES AND INTERPOLATION

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Abstract. It is well-known that the optimal constant of the bilinear Bohnenblust–Hille inequality (i.e., Littlewood's $4/3$ inequality) is obtained by interpolating the bilinear mixed (ℓ_1,ℓ_2)-Littlewood inequalities. We remark that this cannot be extended to the 3-linear case and, in the opposite direction, we show that the asymptotic growth of the constants of the m-linear Bohnenblust–Hille inequality is the same of the constants of the mixed $\left(\ell_{\frac{2m+2}{m+2}},\ell_2\right)$-Littlewood inequality. This means that, contrary to what the previous works seem to suggest, interpolation does not play a crucial role in the search of the exact asymptotic growth of the constants of the Bohnenblust–Hille inequality. In the final section we use mixed Littlewood type inequalities to obtain the optimal cotype constants of certain sequence spaces.

1. Introduction

The mixed (ℓ_1,ℓ_2)-Littlewood inequality for $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$ asserts that

$$\sum_{j_1=1}^{\infty}\left(\sum_{j_2,\ldots,j_m=1}^{\infty}\left|U(e_{j_1},\ldots,e_{j_m})\right|^2\right)^{\frac{1}{2}}\leqslant\left(\sqrt{2}\right)^{m-1}\|U\|, \tag{1}$$

for all continuous m-linear forms $U:c_0\times\cdots\times c_0\to\mathbb{K}$, where $(e_i)_{i=1}^{\infty}$ denotes the sequence of canonical vectors of c_0. It is well-known that arguments of symmetry combined with an inequality due to Minkowski yields that for each $k\in\{2,\ldots,m\}$ we have

$$\left(\sum_{j_1,\ldots,j_{k-1}=1}^{\infty}\left(\sum_{j_k=1}^{\infty}\left(\sum_{j_{k+1},\ldots,j_m=1}^{\infty}\left|U(e_{j_1},\ldots,e_{j_m})\right|^2\right)^{\frac{1}{2}\times 1}\right)^{\frac{1}{1}\times 2}\right)^{\frac{1}{2}}\leqslant\left(\sqrt{2}\right)^{m-1}\|U\|, \tag{2}$$

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which is also called mixed (ℓ_1,ℓ_2)-Littlewood inequality. For the sake of simplicity we can say that we have m inequalities with "multiple" exponents $(1,2,2,\ldots,2),\ldots,(2,\ldots,2,1)$. These inequalities are in the heart of the proof of the famous Bohnenblust–Hille inequality for multilinear forms [6] which states that there exists a sequence of positive scalars $\left(B_m^{\mathbb{K}}\right)_{m=1}^{\infty}$ in $[1,\infty)$ such that

$$\left(\sum_{i_1,\ldots,i_m=1}^{\infty}\left|U\left(e_{i_1},\ldots,e_{i_m}\right)\right|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}}\leqslant B_m^{\mathbb{K}}\|U\| \tag{3}$$

for all continuous m-linear forms $U:c_0\times\cdots\times c_0\to\mathbb{K}$. This inequality is essentially a result of the successful theory of nonlinear absolutely summing operators (for more details on summing operators see, for instance, [5, 12, 13] and references therein). To prove the Bohnenblust–Hille inequality using the mixed (ℓ_1,ℓ_2)-Littlewood inequalities it suffices to observe that the exponent $\frac{2m}{m+1}$ can be seen as a multiple exponent $\left(\frac{2m}{m+1},\ldots,\frac{2m}{m+1}\right)$ and this exponent is precisely the interpolation of the exponents $(1,2,2,\ldots,2),\ldots,(2,\ldots,2,1)$ with weights $\theta_1=\cdots=\theta_m=1/m$. Mixed Littlewood inequalities are also crucial to prove Hardy–Littlewood inequalities for multilinear forms (see [3, 10] and the references therein).

2. Mixed Littlewood inequalities and interpolation

The optimal constant of the 3-linear mixed (ℓ_1,ℓ_2)-Littlewood inequality for real scalars with multiple exponents $(1,2,2)$ and $(2,1,2)$ were obtained in [7, 11] (these constants are precisely 2). Curiously, the arguments could not be extended to obtain the optimal constant associated to the multiple exponent $(2,2,1)$. However, using the 3-linear form

$$U(x,y,z)=(z_1+z_2)(x_1y_1+x_1y_2+x_2y_1-x_2y_2)+(z_1-z_2)(x_3y_3+x_3y_4+x_4y_3-x_4y_4)$$

it is easy to show that the optimal constant associated to the multiple exponent $(2,2,1)$ is not smaller than $\sqrt{2}$. So, interpolating the three inequalities we obtain the estimate $2^{1/3}\times 2^{1/3}\times\sqrt{2}^{1/3}$ for the 3-linear Bohnenblust–Hille inequality, i.e., $2^{5/6}$, but it is well-known that the optimal constant of the 3-linear Bohnenblust–Hille inequality is not bigger than $2^{3/4}$. So we conclude that the optimal constant of the 3-linear Bohnenblust–Hille inequality cannot be obtained by interpolating the optimal constants of the multiple exponents $(1,2,2)$, $(2,1,2)$ and $(2,2,1)$.

In the paper [2], Albuquerque *et al.* have shown that the Bohnenblust–Hille inequality is a very particular case of the following theorem:

THEOREM 1. *Let* $1\leqslant k\leqslant m$ *and* $n_1,\ldots,n_k\geqslant 1$ *be positive integers such that* $n_1+\cdots+n_k=m$, *let* $q_1,\ldots,q_k\in[1,2]$. *The following assertions are equivalent:*

(A) There is a constant $C^{\mathbb{K}}_{q_1\dots q_k} \geqslant 1$ *such that*

$$\left(\sum_{i_1=1}^{\infty}\left(\sum_{i_2=1}^{\infty}\left(\cdots\left(\sum_{i_{k-1}=1}^{\infty}\left(\sum_{i_k=1}^{\infty}\left|A\left(e_{i_1}^{n_1},\dots,e_{i_k}^{n_k}\right)\right|^{q_k}\right)^{\frac{q_{k-1}}{q_k}}\right)^{\frac{q_{k-2}}{q_{k-1}}}\cdots\right)^{\frac{q_2}{q_3}}\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}} \leqslant C^{\mathbb{K}}_{q_1\dots q_k}\|A\| \tag{4}$$

for all continuous m-linear forms $A: c_0\times\cdots\times c_0 \to \mathbb{K}$.

(B) $\frac{1}{q_1}+\cdots+\frac{1}{q_k} \leqslant \frac{k+1}{2}$.

The inequalities (4) when $k=m$, $q_j=2$ and $q_l=\frac{2m-2}{m}$ for all $l\in\{1,\dots,j-1,j+1,\dots,m\}$ can be called mixed $\left(\ell_{\frac{2m-2}{m}},\ell_2\right)$-Littlewood inequality for short (see [11]). The best constants $C^{\mathbb{K}}_{\frac{2m}{m+1}\cdots\frac{2m}{m+1}}$ ($C^{\mathbb{K}}_m$ for short) are unknown (even its asymptotic growth is unknown). We stress that it is even unknown if the sequence $\left(C^{\mathbb{K}}_m\right)_{m=1}^{\infty}$ is increasing. By the Khinchin inequality it can be proved (see [4]) that

$$C^{\mathbb{K}}_{2,\frac{2m-2}{m},\dots,\frac{2m-2}{m}} \leqslant A^{-1}_{\frac{2m-2}{m}} C^{\mathbb{K}}_{m-1}. \tag{5}$$

where A_p are the optimal constants of the Khinchin inequality. Using an interpolative procedure, or the Hölder inequality for mixed sums, this means that

$$C^{\mathbb{K}}_m \leqslant A^{-1}_{\frac{2m-2}{m}} C^{\mathbb{K}}_{m-1}.$$

We shall prove the following asymptotic equivalences:

$$C^{\mathbb{K}}_{m-1} \sim C^{\mathbb{K}}_{2,\frac{2m-2}{m},\dots,\frac{2m-2}{m}} \sim \cdots \sim C^{\mathbb{K}}_{\frac{2m-2}{m},\dots,\frac{2m-2}{m},2} \tag{6}$$

that seem to have been overlooked until now. This means that the search of the precise asymptotic growth of the best constants of the Bohnenblust–Hille inequality is equivalent to the search of the precise asymptotic growth of, for instance, the sequence $\left(C^{\mathbb{K}}_{2,\frac{2m-2}{m},\dots,\frac{2m-2}{m}}\right)_{m=1}^{\infty}$ and no interpolative procedure is needed. As a corollary conclude that the inequality (5) is asymptotically sharp.

The proof of (6) is simple. If T_{m-1} is a $(m-1)$-linear form, we define

$$T_m(x^{(1)},\dots,x^{(m)}) = T_{m-1}(x^{(2)},\dots,x^{(m)})x_1^{(1)}.$$

Then

$$\begin{aligned}
&\left(\sum_{j_2,\dots,j_m=1}^{\infty}\left|T_{m-1}\left(e_{j_2,\dots,e_{j_m}}\right)\right|^{\frac{2m-2}{m}}\right)^{\frac{m}{2m-2}}\\
&\qquad=\left(\sum_{j_1=1}^{\infty}\left(\sum_{j_2,\dots,j_m=1}^{\infty}\left|T_m\left(e_{j_1,\dots,e_{j_m}}\right)\right|^{\frac{2m-2}{m}}\right)^{\frac{m}{2m-2}2}\right)^{\frac{1}{2}}\\
&\qquad\leqslant C^{\mathbb{K}}_{2,\frac{2m-2}{m},\dots,\frac{2m-2}{m}}\|T_m\|\\
&\qquad= C^{\mathbb{K}}_{2,\frac{2m-2}{m},\dots,\frac{2m-2}{m}}\|T_{m-1}\|.
\end{aligned}$$

We thus conclude that

$$C^{\mathbb{K}}_{m-1}\leqslant C^{\mathbb{K}}_{2,\frac{2m-2}{m},\dots,\frac{2m-2}{m}}.$$

Therefore

$$C^{\mathbb{K}}_{m-1}\leqslant C^{\mathbb{K}}_{2,\frac{2m-2}{m},\dots,\frac{2m-2}{m}}\leqslant A^{-1}_{\frac{2m-2}{m}}C^{\mathbb{K}}_{m-1}.$$

Since (for both real and complex scalars)

$$\lim_{m\to\infty}A^{-1}_{\frac{2m-2}{m}}=1,$$

we conclude that

$$C^{\mathbb{K}}_{m-1}\sim C^{\mathbb{K}}_{2,\frac{2m-2}{m},\dots,\frac{2m-2}{m}}.$$

The other equivalences are similar.

3. Cotype 2 constants of ℓ_p spaces

Let $2\leqslant q<\infty$ and $0<s<\infty$. A Banach space X has cotype q (see [1, page 138]) if there is a constant $C_{q,s}>0$ such that, no matter how we select finitely many vectors $x_1,\dots,x_n\in X$,

$$\left(\sum_{k=1}^{n}\|x_k\|^q\right)^{\frac{1}{q}}\leqslant C_{q,s}\left(\int_{[0,1]}\left\|\sum_{k=1}^{n}r_k(t)x_k\right\|^s dt\right)^{1/s}, \tag{7}$$

where r_k denotes the k-th Rademacher function. The smallest of all of these constants will be denoted by $C_{q,s}(X)$.

By the Kahane inequality we know that if (7) holds for a certain $s>0$ than it holds for all $s>0$. It is well-known that for all $p\geqslant 1$, the sequence space ℓ_p has cotype $\max\{p,2\}$. The optimal values of $C_{2,s}(\ell_p)$ for $1\leqslant p<2$ are perhaps known or at least folklore, but we were not able to find in the literature. Classical books like [1, 8, 9] do not provide this information.

In this section we shall show how the optimal cotype constant of ℓ_p spaces can be obtained using mixed inequalities similar to those treated in the previous section. From now on, p_0 is the solution of the following equality

$$\Gamma\left(\frac{p_0+1}{2}\right)=\frac{\sqrt{\pi}}{2}.$$

THEOREM 2. *Let* $1\leqslant r\leqslant p_0\approx 1.84742$. *Then*

$$C_{2,r}(\ell_r)=2^{\frac{1}{r}-\frac{1}{2}}.$$

Proof. It is not difficult to prove that $C_{2,r}(\ell_r)\leqslant 2^{\frac{1}{r}-\frac{1}{2}}$ (see [1, pages 141-142]). Now we prove that $2^{\frac{1}{r}-\frac{1}{2}}$ is the best constant possible.

Let $A:c_0\times c_0\to\mathbb{R}$ a bilinear form and define, for all positive integers n,

$$A_{n,e}:c_0\to\ell_r$$

by

$$A_{n,e}(x)=\left(A\left(x,e_k\right)\right)_{k=1}^{n}.$$

It is simple to verify that

$$\|A_{n,e}\|\leqslant\|A\|.$$

In fact,

$$\begin{aligned}\|A_{n,e}\|&=\sup_{\|x\|\leqslant 1}\|A_{n,e}(x)\|=\sup_{\|x\|\leqslant 1}\left(\sum_{j=1}^{n}\left|A\left(x,e_j\right)\right|^{r}\right)^{1/r}\\&\leqslant\sup_{\|x\|\leqslant 1}\pi_{(r,r)}\left(A\left(x,\cdot\right)\right)\sup_{\varphi\in B_{(c_0)^*}}\left(\sum_{j=1}^{n}\left|\varphi\left(e_j\right)\right|^{r}\right)^{1/r}\\&\leqslant\sup_{\|x\|\leqslant 1}\|A\left(x,\cdot\right)\|\sup_{\varphi\in B_{(c_0)^*}}\sum_{j=1}^{n}\left|\varphi\left(e_j\right)\right|\\&=\|A\|.\end{aligned}$$

It is also well-known that $A_{n,e}$ is absolutely $(2,1)$-summing and

$$\pi_{(2,1)}\left(A_{n,e}\right)\leqslant C_{2,r}(\ell_r)\|A_{n,e}\|.$$

In fact, for any continuous linear operator $u:c_0\to\ell_r$ we have

$$\begin{aligned}\left(\sum_{j=1}^{n}\left\|u\left(x_j\right)\right\|^{2}\right)^{\frac{1}{2}}&\leqslant C_{2,r}(\ell_r)\left(\int_{[0,1]}\left\|\sum_{j=1}^{n}r_j(t)u\left(x_j\right)\right\|^{r}dt\right)^{\frac{1}{r}}\\&\leqslant C_{2,r}(\ell_r)\sup_{t\in[0,1]}\left\|\sum_{j=1}^{n}r_j(t)u\left(x_j\right)\right\|\\&\leqslant C_{2,r}(\ell_r)\|u\|\sup_{\varphi\in B_{(c_0)^*}}\sum_{j=1}^{n}\left|\varphi\left(x_j\right)\right|.\end{aligned}$$

We have

$$\left(\sum_{j_1=1}^{n}\left(\sum_{j_2=1}^{n}\left|A(e_{j_1},e_{j_2})\right|^r\right)^{\frac{1}{r}\times 2}\right)^{\frac{1}{2}} = \left(\sum_{j_1=1}^{n}\left\|A_{n,e}\left(e_{j_1}\right)\right\|^2\right)^{\frac{1}{2}} \tag{8}$$
$$\leqslant C_{2,r}(\ell_r)\left\|A_{n,e}\right\| \sup_{\varphi\in B_{(c_0)^*}} \sum_{j=1}^{n}\left|\varphi\left(e_j\right)\right|$$
$$\leqslant C_{2,r}(\ell_r)\left\|A\right\|.$$

But, plugging

$$A(x,y) = x_1y_1 + x_1y_2 + x_2y_1 - x_2y_2$$

into (8) we conclude that

$$\left(2\cdot 2^{\frac{2}{r}}\right)^{\frac{1}{2}} \leqslant 2C_{2,r}(\ell_r)$$

and thus

$$C_{2,r}(\ell_r) \geqslant \frac{2^{\frac{1}{2}+\frac{1}{r}}}{2} = 2^{\frac{1}{r}-\frac{1}{2}}. \quad \square$$

A simple adaptation of the above proof gives us:

PROPOSITION 1. *Let* $1 \leqslant r \leqslant 2$. *Then*

$$C_{2,s}(\ell_r) \geqslant 2^{\frac{1}{r}-\frac{1}{2}}$$

for all $s > 0$.

The same argument of the previous result provides:

COROLLARY 1. *Let* $p_0 \approx 1.84742 < r \leqslant 2$. *Then*

$$2^{\frac{1}{r}-\frac{1}{2}} \leqslant C_{2,r}(\ell_r) \leqslant \frac{1}{\sqrt{2}}\left(\frac{\Gamma(\frac{r+1}{2})}{\sqrt{\pi}}\right)^{-1/r}.$$

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