A HILBERT–TYPE FRACTIONAL INTEGRAL INEQUALITY WITH THE KERNEL OF MITTAG–LEFFLER FUNCTION AND ITS APPLICATIONS

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Abstract. By using the theory of the local fractional calculus and the methods of weight function, a Hilbert-type fractional integral inequality with the kernel of Mittag-Leffler function and its equivalent form are given. Their constant factors are proved being the best possible, and its applications are discussed briefly.

1. Introduction

Assuming that $f, g \geqslant 0$, $0 < \int_0^\infty f^2(x)dx < \infty$, $0 < \int_0^\infty g^2(y)dy < \infty$, we have the famous Hilbert's integral inequality and its equivalent form (cf. [1, 2]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \pi \left\{ \int_0^\infty f^2(x)dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2(y)dy \right\}^{\frac{1}{2}}, \quad (1)$$

$$\int_0^\infty \left[\int_0^\infty \frac{f(x)}{x+y}dx \right]^2 dy < \pi^2 \int_0^\infty f^2(x)dx, \quad (2)$$

where the constant π and π^2 are the best possible. There have been a number of improvements and extensions on inequalities (1) and (2) (cf. [3–10]), which are important in the mathematical analysis and its applications (cf. [1, 2, 8]). In 2010, Yang [11] presented a new Hilbert-type integral inequality and its equivalent form, as follows:

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f, g > 0$, $0 < \int_0^\infty \frac{1}{x}f^p(x)dx < \infty$, $0 < \int_0^\infty \frac{1}{y}g^q(y)dy < \infty$, then

$$\int_0^\infty \int_0^\infty e^{-xy}f(x)g(y)dxdy < \left\{ \int_0^\infty \frac{1}{x}f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \frac{1}{y}g^q(y)dy \right\}^{\frac{1}{q}}, \quad (3)$$

$$\int_0^\infty y \left[\int_0^\infty e^{-xy}f(x)dx \right]^p dy < \int_0^\infty \frac{1}{x}f^p(x)dx. \quad (4)$$

In recent years, the theory of fractional calculus has been developed rapidly, and it has been widely used in the fields of science and engineering. Some researchers have used the fractal theory to discuss and generalize some classical inequalities (cf. [12–14]), but the research on the Hilbert-type fractional integral inequalities are still not involved. In this paper, by using the theory of local fractional calculus and the method of weight function to make a meaningful attempt, a Hilbert-type fractional integral inequality and its equivalent form are established.

2. Preliminaries

DEFINITION 1. (cf. [15, 16]) The α-type set of set Ω are defined as the set Ω^α, where $0 < \alpha \leqslant 1$. This set Ω^α is called a fractional set. The set Ω is called base set of fractional set.

Let $\mathbb{R}^\alpha$ $(0 < \alpha \leqslant 1)$ be the α-type set of real line numbers, if a^α, b^α, $c^\alpha \in \mathbb{R}^\alpha$, then (cf. [15, 16])

1° $a^\alpha + b^\alpha \in \mathbb{R}^\alpha$, $a^\alpha b^\alpha \in \mathbb{R}^\alpha$;

2° $a^\alpha + b^\alpha = b^\alpha + a^\alpha = (a+b)^\alpha = (b+a)^\alpha$;

3° $a^\alpha + (b+c)^\alpha = (a+b)^\alpha + c^\alpha$;

4° $a^\alpha b^\alpha = b^\alpha a^\alpha = (ab)^\alpha = (ba)^\alpha$;

5° $a^\alpha(b^\alpha c^\alpha) = (a^\alpha b^\alpha)c^\alpha$;

6° $a^\alpha(b^\alpha + c^\alpha) = a^\alpha b^\alpha + a^\alpha c^\alpha$;

7° $a^\alpha + 0^\alpha = a^\alpha$, $a^\alpha 1^\alpha = a^\alpha$, $(-a)^\alpha = -a^\alpha$;

8° If $a < b$, then $a^\alpha < b^\alpha$. If $a^\alpha < b^\alpha$, then $a < b$.

DEFINITION 2. (cf. [15, 16]) A non-differentiable function $f : \mathbb{R} \to \mathbb{R}^\alpha$ $(0 < \alpha \leqslant 1)$, $x \to f(x)$ is called to be local fractional continuous at x_0, if for any $\varepsilon > 0$, there exists $\delta > 0$, such that $|f(x) - f(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$. If $f(x)$ is local fractional continuous on the interval (a,b), we denote $f(x) \in C_\alpha(a,b)$.

DEFINITION 3. (cf. [15, 16]) The local fractional derivative of $f(x)$ of order α $(0 < \alpha \leqslant 1)$ at x_0 is defined by

$$f^{(\alpha)}(x_0) = \frac{d^\alpha f(x)}{dx^\alpha}\Big|_{x=x_0} = \lim_{x \to x_0} \frac{\Gamma(1+\alpha)(f(x) - f(x_0))}{(x - x_0)^\alpha}.$$

If for all $x \in I \subset \mathbb{R}$, there exists $f^{(k+1)\alpha}(x) = \overbrace{D_x^\alpha \cdots D_x^\alpha}^{k+1} f(x)$, then we denote $f \in D_{(k+1)\alpha}(I)$, where $k = 0, 1, 2, \cdots$.

DEFINITION 4. (cf. [15, 16]) Let $f(x) \in C_\alpha(a,b)$. Then the local fractional integral is defined by

$$_aI_b^\alpha f(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)(dt)^\alpha = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{i=1}^N f(t_i)(\Delta t_i)^\alpha,$$

with $\Delta t_i = t_i - t_{i-1}$ $(i = 1, \cdots N)$, $\Delta t = \max_{1 \leqslant i \leqslant N}\{\Delta t_i\}$, and $a = t_0 < t_1 < \cdots < t_{N-1} < t_N = b$ is partition of interval $[a,b]$. Here, it follows that $_aI_b^\alpha f(x) = 0$ if $a = b$, $_aI_b^\alpha f(x) = -_bI_a^\alpha f(x)$ if $a < b$.

DEFINITION 5. (cf. [15, 16]) Mittag-Leffler function defined on fractal set of fractal dimension α is by the expression:

$$E_\alpha(x^\alpha) := \sum_{k=0}^\infty \frac{x^{\alpha k}}{\Gamma(1+\alpha k)}, \quad x \in \mathbb{R}, \text{ and } 0 < \alpha \leqslant 1.$$

DEFINITION 6. If $s \in \mathbb{C}$, let $f(x)$ denote a function which vanishes for negative values of x. Local fractional Laplace's transform $L_\alpha\{f(x)\}$ of order α is defined by the following expression, if it is finite:

$$L_\alpha\{f(x)\}(s) := \frac{1}{\Gamma(1+\alpha)} \int_0^\infty E_\alpha(-s^\alpha x^\alpha) f(x)(dx)^\alpha,$$

DEFINITION 7. Gamma function on fractal set of fractal dimension α is defined by the following expression:

$$\Gamma_\alpha(x) := \frac{1}{\Gamma(1+\alpha)} \int_0^\infty E_\alpha(-t^\alpha) t^{(x-1)\alpha}(dt)^\alpha, \ \ 0 < \alpha \leqslant 1$$

LEMMA 1. *If* $m > -1$, $Re(s) > 0$, *then we have*

$$L_\alpha\{x^{m\alpha}\}(s) = \frac{\Gamma_\alpha(m+1)}{s^{(m+1)\alpha}}.$$

Proof. Setting $t = sx$, by Definition 6 and Definition 7, we obtain

$$\begin{aligned}
L_\alpha\{x^{m\alpha}\}(s) &= \frac{1}{\Gamma(1+\alpha)} \int_0^\infty E_\alpha(-s^\alpha x^\alpha) x^{m\alpha}(dx)^\alpha \\
&= \frac{1}{s^{(m+1)\alpha}\Gamma(1+\alpha)} \int_0^\infty E_\alpha(-t^\alpha) t^{m\alpha}(dt)^\alpha \\
&= \frac{\Gamma_\alpha(m+1)}{s^{(m+1)\alpha}}. \quad \square
\end{aligned}$$

LEMMA 2. (cf. [18, 19, 20]) *If* $f, g \ (\geqslant 0) \in C_\alpha(a,b)$, $F, G, h \ (\geqslant 0) \in C_\alpha(S^{(\beta)})$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $S^{(\beta)}$ *is a fractal surface, then we have*
$1°$ *Hölder inequality on the fractal set*

$$_aI_b^\alpha f(x)g(x) \leqslant \left\{ _aI_b^\alpha f^p(x) \right\}^{\frac{1}{p}} \left\{ _aI_b^\alpha g^q(x) \right\}^{\frac{1}{q}},$$

$2°$ *Hölder weighted inequality on the fractal set*

$$\frac{1}{\Gamma^2(1+\alpha)} \iint\limits_{S^{(\beta)}} h(x,y)F(x,y)G(x,y)(dx)^\alpha(dy)^\alpha$$

$$\leqslant \left\{ \frac{1}{\Gamma^2(1+\alpha)} \iint\limits_{S^{(\beta)}} h(x,y)F^p(x,y)(dx)^\alpha(dy)^\alpha \right\}^{\frac{1}{p}}$$

$$\times \left\{ \frac{1}{\Gamma^2(1+\alpha)} \iint\limits_{S^{(\beta)}} h(x,y)G^q(x,y)(dx)^\alpha(dy)^\alpha \right\}^{\frac{1}{q}}.$$

The inequality keeps the form of equality, then there exist constants A and B, such that they are not all zero and $AF^p(x,y) = BG^q(x,y)$.a.e. on $S^{(\beta)}$.

LEMMA 3. *If $0 < \alpha \leqslant 1$, $\beta, \lambda > 0$, we define the weight function $\omega(\alpha, \beta, \lambda, x)$ as follows:*

$$\omega(\alpha, \beta, \lambda, x) := \frac{1}{\Gamma(1+\alpha)} \int_0^\infty E_\alpha(-\lambda^\alpha (xy)^\alpha) x^{\beta\alpha} y^{(\beta-1)\alpha} (dy)^\alpha, \quad x \in (0, +\infty),$$

then we have

$$\omega(\alpha, \beta, \lambda, x) = \frac{\Gamma_\alpha(\beta)}{\lambda^{\beta\alpha}}, \tag{5}$$

Proof. Setting $xy = u$, by Lemma 1, we obtain

$$\begin{aligned}
\omega(\alpha, \beta, \lambda, x) &= \frac{1}{\Gamma(1+\alpha)} \int_0^\infty E_\alpha(-\lambda^\alpha (xy)^\alpha) x^{\beta\alpha} y^{(\beta-1)\alpha} (dy)^\alpha \\
&= \frac{1}{\Gamma(1+\alpha)} \int_0^\infty E_\alpha(-\lambda^\alpha u^\alpha) u^{(\beta-1)\alpha} (du)^\alpha \\
&= L_\alpha\{u^{(\beta-1)\alpha}\}(\lambda) \\
&= \frac{\Gamma_\alpha(\beta)}{\lambda^{\beta\alpha}} \quad \square
\end{aligned}$$

LEMMA 4. *If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \alpha \leqslant 1$, $\beta, \lambda > 0$, and ε small enough, we define two functions $\tilde{f}(x)$, $\tilde{g}(y)$ as follows:*

$$\tilde{f}(x) := \begin{cases} 0, & x \in (0,1) \\ x^{(\beta-1)\alpha - \frac{\varepsilon\alpha}{p}}, & x \in [1, \infty) \end{cases},$$

$$\tilde{g}(y) := \begin{cases} 0, & y \in (1, \infty) \\ y^{(\beta-1)\alpha + \frac{\varepsilon\alpha}{q}}, & y \in (0, 1] \end{cases},$$

then we have

$$\tilde{J}\varepsilon^\alpha = \left\{ {}_0I_\infty^\alpha(x^{[p(1-\beta)-1]\alpha} \tilde{f}^p(x)) \right\}^{\frac{1}{p}} \left\{ {}_0I_\infty^\alpha(y^{[q(1-\beta)-1]\alpha} \tilde{g}^q(y)) \right\}^{\frac{1}{q}} \varepsilon^\alpha = \frac{1}{\Gamma(1+\alpha)}, \tag{6}$$

$$\tilde{I}\varepsilon^\alpha = \varepsilon^\alpha \cdot {}_0I_\infty^\alpha \left[{}_0I_\infty^\alpha(E_\alpha(-\lambda^\alpha (xy)^\alpha) \tilde{f}(x) \tilde{g}(y)) \right] > \frac{\Gamma_\alpha(\beta)}{\lambda^{\beta\alpha}\Gamma(1+\alpha)} (1 - o(1)) \ (\varepsilon \to 0^+). \tag{7}$$

Proof. We easily obtain:

$$\begin{aligned}
\tilde{J}\varepsilon^\alpha &= \left\{ {}_0I_\infty^\alpha(x^{[p(1-\beta)-1]\alpha} \tilde{f}^p(x)) \right\}^{\frac{1}{p}} \left\{ {}_0I_\infty^\alpha(y^{[q(1-\beta)-1]\alpha} \tilde{g}^q(y)) \right\}^{\frac{1}{q}} \varepsilon^\alpha \\
&= \left\{ {}_1I_\infty^\alpha(x^{-(1+\varepsilon)\alpha}) \right\}^{\frac{1}{p}} \left\{ {}_0I_1^\alpha(y^{-(1-\varepsilon)\alpha} (dy)^\alpha \right\}^{\frac{1}{q}} \varepsilon^\alpha \\
&= \frac{1}{\Gamma(1+\alpha)}.
\end{aligned}$$

Since $H(u) = u^{(\beta+1)\alpha} E_\alpha(-\lambda^\alpha (xy)^\alpha)$ is continuous in $[x,\infty)$ and $\lim\limits_{u \to +\infty} H(u) = 0$, therefore exists a constant $M > 0$, satisfying $H(u) \leqslant M$. In addition, let $u^{-1+\frac{\varepsilon}{q}} = t$, then $u^{(-2+\frac{\varepsilon}{q})\alpha}(du)^\alpha = -\frac{1}{(1-\frac{\varepsilon}{q})^\alpha}(dt)^\alpha$, and we can write

$$
\begin{aligned}
{}_1I_\infty^\alpha & \left(x^{-(1+\varepsilon)\alpha}\right) {}_xI_\infty^\alpha \left(E_\alpha(-\lambda^\alpha u^\alpha) u^{(\beta-1+\frac{\varepsilon}{q})\alpha}\right) \\
&= \frac{1}{\Gamma^2(1+\alpha)} \int_1^\infty x^{-(1+\varepsilon)\alpha}(dx)^\alpha int_x^\infty E_\alpha(-\lambda^\alpha u^\alpha) u^{(\beta-1+\frac{\varepsilon}{q})\alpha}(du)^\alpha \\
&< \frac{M}{\Gamma^2(1+\alpha)} \int_1^\infty x^{-\alpha}(dx)^\alpha \int_x^\infty u^{(-2+\frac{\varepsilon}{q})\alpha}(du)^\alpha \\
&= \frac{-M}{(1-\frac{\varepsilon}{q})^\alpha \Gamma^2(1+\alpha)} \int_1^\infty x^{-\alpha}(dx)^\alpha \int_{x^{-1+\frac{\varepsilon}{q}}}^0 (dt)^\alpha \\
&= \frac{M}{(1-\frac{\varepsilon}{q})^\alpha \Gamma^2(1+\alpha)} \int_1^\infty x^{-(2-\frac{\varepsilon}{q})\alpha}(dx)^\alpha \\
&= \frac{-M}{(1-\frac{\varepsilon}{q})^{2\alpha} \Gamma^2(1+\alpha)} \int_1^0 (ds)^\alpha \\
&= \frac{M}{(1-\frac{\varepsilon}{q})^{2\alpha} \Gamma^2(1+\alpha)}
\end{aligned}
$$

Further, setting $xy = u$, and by Lemma 1, we have

$$
\begin{aligned}
\tilde{I}\varepsilon^\alpha &= \varepsilon^\alpha \cdot {}_0I_\infty^\alpha \left[{}_0I_\infty^\alpha (E_\alpha(-\lambda^\alpha x^\alpha y^\alpha) \tilde{f}(x)\tilde{g}(y)) \right] \\
&= \frac{\varepsilon^\alpha}{\Gamma(1+\alpha)} \int_1^\infty x^{(\beta-1)\alpha-\frac{\varepsilon\alpha}{p}}(dx)^\alpha \left[\frac{1}{\Gamma(1+\alpha)} \int_0^1 E_\alpha(-\lambda^\alpha (xy)^\alpha) y^{(\beta-1)\alpha+\frac{\varepsilon\alpha}{q}}(dy)^\alpha \right] \\
&= \frac{\varepsilon^\alpha}{\Gamma(1+\alpha)} \int_1^\infty x^{-(1+\varepsilon)\alpha}(dx)^\alpha \left[\frac{1}{\Gamma(1+\alpha)} \int_0^x E_\alpha(-\lambda^\alpha u^\alpha) u^{(\beta-1+\frac{\varepsilon}{q})\alpha}(du)^\alpha \right] \\
&= \frac{1}{\Gamma^2(1+\alpha)} \int_0^\infty E_\alpha(-\lambda^\alpha u^\alpha) u^{(\beta-1+\frac{\varepsilon}{q})\alpha}(du)^\alpha \\
&\quad - \varepsilon^\alpha \cdot {}_1I_\infty^\alpha \left(x^{-(1+\varepsilon)\alpha}\right) {}_xI_\infty^\alpha \left(E_\alpha(-\lambda^\alpha u^\alpha) u^{(\beta-1+\frac{\varepsilon}{q})\alpha}\right) \\
&= \frac{\Gamma_\alpha(\beta+\frac{\varepsilon}{q})}{\lambda^{(\beta+\frac{\varepsilon}{q})\alpha}\Gamma^2(1+\alpha)} - \varepsilon^\alpha \cdot {}_1I_\infty^\alpha \left(x^{-(1+\varepsilon)\alpha}\right) {}_xI_\infty^\alpha \left(E_\alpha(-\lambda^\alpha u^\alpha) u^{(\beta-1+\frac{\varepsilon}{q})\alpha}\right) \\
&> \frac{\Gamma_\alpha(\beta+\frac{\varepsilon}{q})}{\lambda^{(\beta+\frac{\varepsilon}{q})\alpha}\Gamma(1+\alpha)} - \frac{M\varepsilon^\alpha}{(1-\frac{\varepsilon}{q})^{2\alpha}\Gamma^2(1+\alpha)} \\
&= \frac{\Gamma_\alpha(\beta)}{\lambda^{\beta\alpha}\Gamma(1+\alpha)}(1 - o(1)) \quad (\varepsilon \to 0^+). \quad \square
\end{aligned}
$$

3. Main results

For convenience, we use notation of the double fractional integral as (see [16]):

$$_0I_\infty^\alpha\left[_0I_\infty^\alpha F(x,y)\right] = \frac{1}{\Gamma^2(1+\alpha)}\int_0^\infty\int_0^\infty F(x,y)(dx)^\alpha(dy)^\alpha.$$

THEOREM 1. *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \alpha \leqslant 1$, $\beta, \lambda > 0$, $f, g (\geqslant 0) \in C_\alpha(0,\infty)$, $0 < {}_0I_\infty^\alpha\left(x^{[p(1-\beta)-1]\alpha}f^p(x)\right) < \infty$, $0 < {}_0I_\infty^\alpha\left(y^{[q(1-\beta)-1]\alpha}g^q(y)\right) < \infty$, *then we have*

$$_0I_\infty^\alpha\left[_0I_\infty^\alpha E_\alpha(-\lambda^\alpha(xy)^\alpha)f(x)g(y)\right] <$$
$$\frac{\Gamma_\alpha(\beta)}{\lambda^{\beta\alpha}}\left\{_0I_\infty^\alpha\left(x^{[p(1-\beta)-1]\alpha}f^p(x)\right)\right\}^{\frac{1}{p}}\left\{_0I_\infty^\alpha\left(y^{[q(1-\beta)-1]\alpha}g^q(y)\right)\right\}^{\frac{1}{q}}, \qquad (8)$$

where the constant factor $\frac{\Gamma_\alpha(\beta)}{\lambda^{\beta\alpha}}$ *is the best possible.*

Proof. By Hölder weighted inequality on fractal set and Lemma 3, we obtain

$$_0I_\infty^\alpha\left[_0I_\infty^\alpha E_\alpha(-\lambda^\alpha(xy)^\alpha)f(x)g(y)\right]$$

$$=_0I_\infty^\alpha\left[_0I_\infty^\alpha E_\alpha(-\lambda^\alpha(xy)^\alpha)\left[\frac{y^{\frac{(\beta-1)\alpha}{p}}}{x^{\frac{(\beta-1)\alpha}{q}}}\right]\left[\frac{x^{\frac{(\beta-1)\alpha}{q}}}{y^{\frac{(\beta-1)\alpha}{p}}}\right]f(x)g(y)\right]$$

$$\leqslant\left\{_0I_\infty^\alpha\left[_0I_\infty^\alpha E_\alpha(-\lambda^\alpha(xy)^\alpha)f^p(x)\frac{y^{(\beta-1)\alpha}}{x^{(p-1)(\beta-1)\alpha}}\right]\right\}^{\frac{1}{p}}$$

$$\times\left\{_0I_\infty^\alpha\left[_0I_\infty^\alpha E_\alpha(-\lambda^\alpha(xy)^\alpha)g^q(y)\frac{x^{(\beta-1)\alpha}}{y^{(q-1)(\beta-1)\alpha}}\right]\right\}^{\frac{1}{q}}$$

$$=\left\{_0I_\infty^\alpha\left[\omega(\alpha,\beta,\lambda,x)x^{(p(1-\beta)-1)\alpha}f^p(x)\right]\right\}^{\frac{1}{p}}\left\{_0I_\infty^\alpha\left[\omega(\alpha,\beta,\lambda,y)y^{(q(1-\beta)-1)\alpha}g^q(y)\right]\right\}^{\frac{1}{q}}$$

$$=\frac{\Gamma_\alpha(\beta)}{\lambda^{\beta\alpha}}\left\{_0I_\infty^\alpha\left(x^{[p(1-\beta)-1]\alpha}f^p(x)\right)\right\}^{\frac{1}{p}}\left\{_0I_\infty^\alpha\left(y^{[q(1-\beta)-1]\alpha}g^q(y)\right)\right\}^{\frac{1}{q}} \qquad (9)$$

If (9) keeps the form of equality for some $y \in (0,\infty)$, then, there exist constants A and B, such that they are not all zero, and

$$A\frac{y^{(\beta-1)\alpha}}{x^{(p-1)(\beta-1)\alpha}}f^p(x) = B\frac{x^{(\beta-1)\alpha}}{y^{(q-1)(\beta-1)\alpha}}g^q(y) \text{ a.e. in } (0,\infty)\times(0,\infty).$$

Assuming that $A \neq 0$, then there exist constant $C \neq 0$, such that $x^{[p(1-\beta)-1]\alpha}f^p(x) = \frac{C}{Ax^\alpha}$ a.e. *in* $x \in (0,\infty)$. We find that $\frac{1}{\Gamma(1+\alpha)}\int_0^\infty\frac{C}{Ax^\alpha}(dx)^\alpha = \frac{C}{A}Ln_\alpha x^\alpha|_0^\infty$ is diffuse, which contradicts the fact that $0 < {}_0I_\infty^\alpha\left(x^{[p(1-\beta)-1]\alpha}f^p(x)\right) < \infty$. Thus (9) takes the form of strict inequality. So we obtain (8).

If the constant factor $\frac{\Gamma_\alpha(\beta)}{\lambda^{\beta\alpha}}$ appearing on the right hand sides of (8) is not the best possible, then exists a positive constant $K < \frac{\Gamma_\alpha(\beta)}{\lambda^{\beta\alpha}}$, such that inequality (8) is still

valid when replacing $\frac{\Gamma_\alpha(\beta)}{\lambda^\beta\alpha}$ by K. Then by (6) and (7), we have $\frac{\Gamma_\alpha(\beta)}{\lambda^\beta\alpha}(1-o(1)) < K$. Letting $\varepsilon \to 0^+$, we get $K \geqslant \frac{\Gamma_\alpha(\beta)}{\lambda^\beta\alpha}$, which contradicts the fact that $K < \frac{\Gamma_\alpha(\beta)}{\lambda^\beta\alpha}$. Hence, the constant factor $\frac{\Gamma_\alpha(\beta)}{\lambda^\beta\alpha}$ of (8) is the best possible. The theorem is proved. $\quad\square$

THEOREM 2. *Under the conditions of Theorem 1, we have*

$$_0I_\infty^\alpha\left\{y^{\frac{(q(\beta-1)+1)\alpha}{q-1}}\left[_0I_\infty^\alpha E_\alpha(-\lambda^\alpha(xy)^\alpha)f(x)\right]^p\right\} < \left[\frac{\Gamma_\alpha(\beta)}{\lambda^\beta\alpha}\right]^p {}_0I_\infty^\alpha\left(x^{[p(1-\beta)-1]\alpha}f^p(x)\right),$$

(10)

where the constant factor $\left[\frac{\Gamma_\alpha(\beta)}{\lambda^\beta\alpha}\right]^p$ is the best possible, and inequality (10) is equivalent to inequality (8).

Proof. Letting $[f(x)]_n := \min\{n, f(x)\}$. Since $0 < {}_0I_\infty^\alpha\left(x^{[p(1-\beta)-1]\alpha}f^p(x)\right) < \infty$, there exists positive integer $n_0 \in \mathbb{N}$, such that $0 < {}_\frac{1}{n}I_n^\alpha\left(x^{[p(1-\beta)-1]\alpha}f^p(x)\right) < \infty$ $(n \geqslant n_0)$. Setting $g_n(y) := y^{\frac{(q(\beta-1)+1)\alpha}{q-1}}\left[{}_\frac{1}{n}I_n^\alpha(E_\alpha(-\lambda^\alpha(xy)^\alpha)[f(x)]_n)\right]^{\frac{p}{q}}$ $(\frac{1}{n} < y < n,\ n \geqslant n_0)$, when $n \geqslant n_0$, by (8) we find

$$0 < {}_\frac{1}{n}I_n^\alpha\left(y^{[q(1-\beta)-1]\alpha}g_n^q(y)\right)$$

$$= {}_\frac{1}{n}I_n^\alpha\left(y^{[q(1-\beta)-1]\alpha}g_n^{(q-1)}(y)g_n(y)\right)$$

$$= {}_\frac{1}{n}I_n^\alpha\left\{{}_\frac{1}{n}I_n^\alpha(E_\alpha(-\lambda^\alpha(xy)^\alpha)[f(x)]_n)y^{\frac{(q(\beta-1)+1)\alpha}{q-1}}\left[{}_\frac{1}{n}I_n^\alpha(E_\alpha(-\lambda^\alpha(xy)^\alpha)[f(x)]_n)\right]^{\frac{p}{q}}\right\}$$

$$= {}_\frac{1}{n}I_n^\alpha\left\{{}_\frac{1}{n}I_n^\alpha\left(E_\alpha(-\lambda^\alpha(xy)^\alpha)[f(x)]_ng_n(y)\right)\right\}$$

$$< \frac{\Gamma_\alpha(\beta)}{\lambda^\beta\alpha}\left\{{}_\frac{1}{n}I_n^\alpha\left(x^{[p(1-\beta)-1]\alpha}[f(x)]_n^p\right)\right\}^{\frac{1}{p}}\left\{{}_\frac{1}{n}I_n^\alpha\left(y^{[q(1-\beta)-1]\alpha}g_n^q(y)\right)\right\}^{\frac{1}{q}}.$$

(11)

Moreover, by (11) we have

$$0 < {}_\frac{1}{n}I_n^\alpha\left(y^{[q(1-\beta)-1]\alpha}g_n^q(y)\right) = {}_\frac{1}{n}I_n^\alpha\left\{y^{\frac{(q(\beta-1)+1)\alpha}{q-1}}\left[{}_\frac{1}{n}I_n^\alpha(E_\alpha(-\lambda^\alpha(xy)^\alpha)[f(x)]_n)\right]^p\right\}$$

$$< \left[\frac{\Gamma_\alpha(\beta)}{\lambda^\beta\alpha}\right]^p {}_\frac{1}{n}I_n^\alpha\left(x^{[p(1-\beta)-1]\alpha}[f(x)]_n^p\right) < \infty.$$

(12)

It follows that $0 < {}_0I_\infty^\alpha\left(y^{[q(1-\beta)-1]\alpha}g_\infty^q(y)\right) < \infty$. For $n \to \infty$, by (8), both (11) and (12) still keep the form of strict inequalities. Hence we have inequality (10).

On the other hand, by Hölder's inequality on fractal set and (10), we have

$$_0I_\infty^\alpha\left[_0I_\infty^\alpha E_\alpha(-\lambda^\alpha(xy)^\alpha)f(x)g(y)\right]$$

$$= {}_0I_\infty^\alpha\left\{\left[y^{\frac{(q(\beta-1)+1)\alpha}{p(q-1)}} {}_0I_\infty^\alpha(E_\alpha(-\lambda^\alpha(xy)^\alpha)f(x))\right]\left[y^{\frac{(q(1-\beta)-1)\alpha}{p(q-1)}}g(y)\right]\right\}$$

$$\leqslant \left\{{}_0I_\infty^\alpha\left(y^{\frac{(q(\beta-1)+1)\alpha}{(q-1)}}\left[{}_0I_\infty^\alpha(E_\alpha(-\lambda^\alpha(xy)^\alpha)f(x))\right]^p\right)\right\}^{\frac{1}{p}}\left\{{}_0I_\infty^\alpha\left(y^{[q(1-\beta)-1]\alpha}g^q(y)\right)\right\}^{\frac{1}{p}}$$

$$< \frac{\Gamma_\alpha(\beta)}{\lambda^\beta\alpha}\left\{{}_0I_\infty^\alpha\left(x^{[p(1-\beta)-1]\alpha}f^p(x)\right)\right\}^{\frac{1}{p}}\left\{{}_0I_\infty^\alpha\left(y^{[q(1-\beta)-1]\alpha}g^q(y)\right)\right\}^{\frac{1}{q}}.$$

The above inequality is (8), thus inequality (10) is equivalent to inequality (8).

If the constant factor $\left[\frac{\Gamma_\alpha(\beta)}{\lambda^{\beta\alpha}}\right]^p$ appearing on the right hand sides of (10) is not best possible, then by (10), we can get a contradiction that the constant factor $\frac{\Gamma_\alpha(\beta)}{\lambda^{\beta\alpha}}$ appearing on the right hand sides of (8) is not the best possible. Thus the constant factor $\left[\frac{\Gamma_\alpha(\beta)}{\lambda^{\beta\alpha}}\right]^p$ in (10) is the best possible.   □

4. Simple applications

Selecting α values and appropriate β, λ values in (8) and (10), by Lemma 3, some Hilbert-type fractional integral inequalities and their equivalent forms are obtained.

EXAMPLE 1. Letting $\alpha = \beta = \lambda = 1$, we obtain (3) and (4).

EXAMPLE 2. Letting $\alpha = 0.5$, $\beta = \lambda = 1$, $p = q = 2$. If f, g $(> 0) \in C_{0.5}(0, \infty)$, $0 < {}_0I_\infty^{0.5}\left(x^{-0.5}f^2(x)\right) < \infty$, $0 < {}_0I_\infty^{0.5}\left(y^{-0.5}g^2(y)\right) < \infty$, then we have the following equivalence inequalities:

$$
{}_0I_\infty^{0.5}\left[{}_0I_\infty^{0.5}\left((E_{0.5}(-(xy)^{0.5})f(x)g(y))\right)\right] < \left\{{}_0I_\infty^{0.5}\left(x^{-0.5}f^2(x)\right)\right\}^{\frac{1}{2}}\left\{{}_0I_\infty^{0.5}\left(y^{-0.5}g^2(y)\right)\right\}^{\frac{1}{2}},
$$
(13)

$$
{}_0I_\infty^{0.5}\left\{y^{0.5}\left[{}_0I_\infty^{0.5}\left(E_{0.5}(-(xy)^{0.5})f(x)\right)\right]^2\right\} < {}_0I_\infty^{0.5}\left(x^{-0.5}f^2(x)\right).
$$
(14)

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