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# EXTENSIONS AND REFINEMENTS OF FEJER AND HERMITE–HADAMARD TYPE INEQUALITIES

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*Abstract.* In this paper extensions and refinements of Hermite-Hadamard and Fejer type inequalities are derived including monotonicity of some functions related to the Fejer inequality and extensions for functions, which are $1$-quasiconvex and for function with bounded second derivative. We deal also with Fejer inequalities in cases that $p$, the weight function in Fejer inequality, is not symmetric but monotone on $[a,b]$.

## 1. Introduction

The Hermite-Hadamard inequality says that for any convex function $f : I \to \mathbb{R}$, $I$ an interval, and for $a,b \in I$

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(t)\,dt \leqslant \frac{f(a)+f(b)}{2} \tag{1.1}$$

holds, and the Fejer inequality reads

$$f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx \leqslant \int_a^b f(t)\,p(t)dt \leqslant \frac{f(a)+f(b)}{2}\int_a^b p(x)\,dx \tag{1.2}$$

when $f$ is convex and $p : [a,b] \to \mathbb{R}$ is non-negative, integrable and symmetric around $x = \frac{a+b}{2}$.

In this paper extensions and refinements of Hermite-Hadamard and Fejer type inequalities, are discussed including monotonicity of some functions related to Fejer inequality and extensions for functions which are $1$-quasiconvex and for function with bounded second derivative. We deal also with Fejer inequalities in cases that $p$, the weight function in the Fejer inequality, is not symmetric but monotone on $[a,b]$.

This paper may be regarded as a complement and continuation of [3], where we dealt with Hermite-Hadamard and Fejer type inequalities for $N$-quasiconvex functions.

DEFINITION 1. Let $N \in \mathbb{N}$. A real-valued function $\psi_N$ defined on an interval $[a,b)$ with $0 \leqslant a < b \leqslant \infty$ is called $N$-quasiconvex if it can be represented as the product of a convex function $\varphi$ and the function $g(x) = x^N$. For $N = 0$, $\psi_0 = \varphi$ and for $N = 1$ the function $\psi_1(x) = x\varphi(x)$ is called $1$-quasiconvex function.

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We quote from [3] some refined Hermite-Hadamard and Fejer type inequalities for $N$-quasiconvex functions that we use in the sequel, in particular for $N = 1$:

LEMMA 1. [3, Theorem 1 and Corollary 1] *Let* $\varphi : [a,b] \to \mathbb{R},\ a \geqslant 0$ *be differentiable, convex and* $\psi_N(x) = x^N \varphi(x),\ N = 1,2,\ldots.$ *Let* $p : [a,b] \to \mathbb{R}$ *be non-negative, integrable and symmetric around* $x = \frac{a+b}{2}$.

*Then,*

$$\begin{aligned}
&\int_a^b \psi_N(x)\, p(x)\, dx \\
\geqslant\ & \psi_N\left(\frac{a+b}{2}\right)\int_a^b p(x)\, dx + \int_a^b \left(x - \frac{a+b}{2}\right)^2 \sum_{k=1}^{N} x^{k-1} \psi'_{N-k}\left(\frac{a+b}{2}\right) p(x)\, dx \\
=\ & \psi_N\left(\frac{a+b}{2}\right)\int_a^b p(x)\, dx + \int_a^b \left(x - \frac{a+b}{2}\right)^2 \left(\frac{\partial}{\partial \overline{x}}\left(\frac{x^N - \overline{x}^N}{x - \overline{x}}\varphi(\overline{x})\right)\bigg|_{\overline{x} = \frac{a+b}{2}}\right) p(x)\, dx,
\end{aligned}$$

*and*

$$\begin{aligned}
&\int_a^b \psi_N(x)\, p(x)\, dx \\
\leqslant\ & \frac{\psi_N(a) + \psi_N(b)}{2}\int_a^b p(x)\, dx \\
& - \frac{1}{(b-a)} \sum_{k=1}^{N} \int_a^b (x-a)(b-x)\, \psi'_{N-k}(x) \left((b-x)\, b^{k-1} + (x-a)\, a^{k-1}\right) p(x)\, dx. \\
=\ & \frac{\psi_N(a) + \psi_N(b)}{2}\int_a^b p(x)\, dx - \frac{1}{(b-a)} \int_a^b \left[(x-a)(b-x)^2 \frac{\partial}{\partial x}\left(\frac{b^N - x^N}{b - x}\varphi(x)\right)\right. \\
& \left. + (x-a)^2 (b-x) \frac{\partial}{\partial x}\left(\frac{x^N - a^N}{x - a}\varphi(x)\right)\right] p(x)\, dx.
\end{aligned}$$

*In particular if* $\varphi : [a,b] \to \mathbb{R},\ a \geqslant 0,$ *is a differentiable and convex function and* $\psi_1(x) = x\varphi(x),$ *then*

$$\begin{aligned}
& \psi_1\left(\frac{a+b}{2}\right)\int_a^b p(x)\, dx + \varphi'\left(\frac{a+b}{2}\right)\int_a^b \left(x - \frac{a+b}{2}\right)^2 p(x)\, dx \qquad (1.3) \\
\leqslant\ & \int_a^b \psi_1(x)\, p(x)\, dx \\
\leqslant\ & \frac{\psi_1(a) + \psi_1(b)}{2}\int_a^b p(x)\, dx - \int_a^b \varphi'(x)(b-x)(x-a)\, p(x)\, dx,
\end{aligned}$$

*where* $p : [a,b] \to \mathbb{R},$ *is non-negative, integrable and symmetric around* $x = \frac{a+b}{2}$.

EXAMPLE 1. [3, Example 1] If $\varphi : [a,b] \to \mathbb{R},\ a \geqslant 0,$ is differentiable, convex and $\psi_1(x) = x\varphi(x)$, then

$$\begin{aligned}
&\psi_1\left(\frac{a+b}{2}\right) + \frac{1}{12}\varphi'\left(\frac{a+b}{2}\right)(b-a)^2 \\
&\leqslant \frac{1}{b-a}\int_a^b \psi_1(x)\,dx \\
&\leqslant \frac{\psi_1(a)+\psi_1(b)}{2} - \frac{1}{b-a}\int_a^b \varphi'(x)(b-x)(x-a)\,dx.
\end{aligned}$$

This is a refinement of the Hermite-Hadamard inequality (1.1) when $\varphi$ is increasing.

At the end of this paper we will prove that the following result from [3] generalizes and gives some simpler proofs of results from [6].

THEOREM 1. [3, Theorem 2] *Let $\varphi : [a,b] \to \mathbb{R},\ a \geqslant 0,$ be a differentiable, convex function and let $N = 1,2,3,\ldots,$. Then for $\psi_1(x) = x\varphi(x)$ we get that the inequalities*

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b \psi_1(x)\,dx \\
&\leqslant \frac{b-a}{6}\varphi(b) + \frac{b+2a}{3}\frac{1}{b-a}\int_a^b \varphi(x)\,dx \\
&\leqslant \frac{(b-a)(\psi_N(a)+\psi_N(b))}{6(b^N-a^N)} + \frac{(b^{N+1}-a^{N+1})+2ab(b^{N-1}-a^{N-1})}{3(b^N-a^N)}\frac{1}{b-a}\int_a^b \varphi(x)\,dx \\
&\leqslant \frac{\psi_1(a)+\psi_1(b)}{6} + \frac{(b+a)}{3}\frac{1}{b-a}\int_a^b \varphi(x)\,dx \\
&\leqslant \frac{\psi_1(a)+\psi_1(b)}{2} - \frac{(b-a)(\varphi(b)-\varphi(a))}{6}
\end{aligned} \tag{1.4}$$

*hold, which are Hermite-Hadamard refinements of (1.1) for $\psi_1$ when $\varphi(b)-\varphi(a) \geqslant 0$.*

REMARK 1. Using [3, Lemma 1], Theorem 1 is proved (see [3, Theorem 2]) by using Lemma 1 for each $n = 1,2,\ldots N$ from which the inequality

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b \psi_1(x)\,dx \\
&\leqslant \frac{(b-a)(\psi_N(a)+\psi_N(b))}{6(b^N-a^N)} + \frac{(b^{N+1}-a^{N+1})+2ab(b^{N-1}-a^{N-1})}{3(b^N-a^N)}\frac{1}{b-a}\int_a^b \varphi(x)\,dx
\end{aligned}$$

is derived.

As it is also proved in [3, Theorem 2] that the right hand-side of this inequality is monotone decreasing with $N$ and as it is also bounded below, the first and the third inequalities in (1.4) follow.

The paper is organized as follows: After this introductory section we discuss in Section 2 the monotonicity of some functions related to the Fejer inequality.

The third section is devoted to results involving Fejer and Hermite-Hadamard inequalities for functions with bounded second derivatives and demonstrate examples with relation to fractional integrals. In particular it is explained that our results imply generalized versions of some recent results by F. Chen [5]

Fejer's inequality for special convex functions obtained by replacing the non-negative symmetric function $p$ in (1.2) with monotone functions and some more results related to quasiconvexity are discussed in Section 4. In particular it is pointed out that our Theorem 1 directly implies some improved versions of results in [6].

## 2. Monotonicity of some functions related to the Fejer inequality

In this section we extend some of the results obtained in [3] and show that when $\psi$ is 1-quasiconvex, that is, $\psi(x) = x\varphi(x)$ and if $\varphi$ is convex, $\varphi' \geqslant 0$, then

$$P(t) = \int_a^b \psi\left(tx + (1-t)\frac{a+b}{2}\right) p(x)\,dx \tag{2.1}$$

and

$$Q(t) = \frac{1}{2}\int_a^b \left[\psi\left(\frac{1+t}{2}a + \frac{1-t}{2}x\right) p\left(\frac{x+a}{2}\right) + \psi\left(\frac{1+t}{2}b + \frac{1-t}{2}x\right) p\left(\frac{x+b}{2}\right)\right] dx \tag{2.2}$$

are non-decreasing in $t$, $0 \leqslant t \leqslant 1$ when $p = p(x)$ is non-negative, differentiable and symmetric around $x = \frac{a+b}{2}$.

To prove the theorems we use a similar technique as that used in [2], (there for superquadratic functions) and the following result that appears in the proof of [3, Theorem 1] and is related to $N$-quasiconvex functions:

LEMMA 2. *Let $\varphi : [a,b] \to \mathbb{R}$, $a \geqslant 0$, be a differentiable, convex function and $\psi_N(x) = x^N \varphi(x)$, $N = 1,2,3,\ldots,$. Let $p : [a,b] \to \mathbb{R}$ be non-negative, integrable and symmertic around $x = \frac{a+b}{2}$. Then,*

$$\begin{aligned}
&(\psi_N(b) + \psi_N(a))\,p(x) \\
\geqslant\; &\psi_N(x)\,p(x) + \psi_N(a+b-x)\,p(a+b-x) \\
&+ \frac{(x-a)(b-x)}{b-a}\sum_{k=1}^{N}\left((b-x)\,b^{k-1} + (x-a)\,a^{k-1}\right)\psi'_{N-k}(x)\,p(x) \\
&+ \frac{(x-a)(b-x)}{b-a}\sum_{k=1}^{N}\left((x-a)\,b^{k-1} + (b-x)\,a^{k-1}\right)\psi'_{N-k}(a+b-x)\,p(a+b-x).
\end{aligned} \tag{2.3}$$

We use in the following theorems 2 and 3, Inequality (2.3) for $N = 1$ and $p(x) = 1$.

Our first monotonicity result is a refinement of the known result which says that $P(s) \leqslant P(t)$ for a convex function $\psi$ (see [2, Theorem F]). It reads:

THEOREM 2. *Let $\psi$ be 1-quasiconvex function on $[a,b]$, $a \geqslant 0$, that is $\psi(x) = x\varphi(x)$. Let $\varphi$ be a differentiable convex function satisfying $\varphi' \geqslant 0$. Let $p = p(x)$ be non-negative integrable and symmetric around $x = \frac{a+b}{2}$. If $P(t)$ is defined by (2.1), then, for $0 \leqslant s \leqslant t \leqslant 1$, $t > 0$,*

$$\begin{aligned} P(s) &\leqslant P(t) - \left(t^2 - s^2\right) \int_a^b \left(x - \frac{a+b}{2}\right)^2 \varphi'\left(sx + (1-s)\frac{a+b}{2}\right) p(x)\,dx \\ &\leqslant P(t). \end{aligned} \tag{2.4}$$

*Proof.* For every $z$, $m$, $M$ on $[a,b]$, $a \leqslant m \leqslant z \leqslant M \leqslant b$ we get in the same way as we get for $N = 1$ and $p(x) = 1$ in (2.3) that:

$$\begin{aligned} &\psi(z) + \psi(M+m-z) \\ &\leqslant \psi(m) + \psi(M) - \left(\varphi'(z) + \varphi'(M+m-z)\right)(M-z)(z-m). \end{aligned} \tag{2.5}$$

Replacing in (2.5) $z$ by $sx + (1-s)\frac{a+b}{2}$, $M$ by $(a+b-x)t + (1-t)\frac{a+b}{2}$, and $m$ by $tx + (1-t)\frac{a+b}{2}$, we obtain that for $0 \leqslant s \leqslant t \leqslant 1$, $t > 0$, $a \leqslant x \leqslant \frac{a+b}{2}$,

$$\begin{aligned} &\psi\left(sx + (1-s)\frac{a+b}{2}\right) + \psi\left(s(a+b-x) + (1-s)\frac{a+b}{2}\right) \\ &\leqslant \psi\left(tx + (1-t)\frac{a+b}{2}\right) + \psi\left(t(a+b-x) + (1-t)\frac{a+b}{2}\right) \\ &\quad - \left(t^2 - s^2\right)\left(x - \frac{a+b}{2}\right)^2 \left(\varphi'\left(sx + (1-s)\frac{a+b}{2}\right)\right. \\ &\quad \left. + \varphi'\left(s(a+b-x) + (1-s)\frac{a+b}{2}\right)\right). \end{aligned} \tag{2.6}$$

Since $p(x)$ is non-negative and symmetric around $x = \frac{a+b}{2}$ we get from (2.6) that

$$\begin{aligned} &\int_a^b \psi\left(sx + (1-s)\frac{a+b}{2}\right) p(x)\,dx \\ &= \int_a^{\frac{a+b}{2}} \psi\left(sx + (1-s)\frac{a+b}{2}\right) p(x)\,dx \\ &\quad + \int_a^{\frac{a+b}{2}} \psi\left(s(a+b-x) + (1-s)\frac{a+b}{2}\right) p(a+b-x)\,dx \\ &\leqslant \int_a^{\frac{a+b}{2}} \psi\left(tx + (1-t)\frac{a+b}{2}\right) p(x)\,dx \\ &\quad + \int_a^{\frac{a+b}{2}} \psi\left(t(a+b-x) + (1-t)\frac{a+b}{2}\right) p(a+b-x)\,dx \\ &\quad - \int_a^{\frac{a+b}{2}} \left(t^2 - s^2\right)\left(x - \frac{a+b}{2}\right)^2 \varphi'\left(sx + (1-s)\frac{a+b}{2}\right) p(x)\,dx \end{aligned} \tag{2.7}$$

$$-\int_a^{\frac{a+b}{2}} \left(t^2-s^2\right)\left(x-\frac{a+b}{2}\right)^2 \varphi'\left(s\left(a+b-x\right)+\left(1-s\right)\frac{a+b}{2}\right) p\left(a+b-x\right)dx.$$

From (2.7), by using again the symmetry of $p(x)$ around $\frac{a+b}{2}$, it follows, because $\varphi'(x) \geqslant 0$ and $0 \leqslant s \leqslant t \leqslant 1$, $t>0$, that (2.4) holds for $P(t)$ as defined in (2.1). The proof is complete. □

Next we prove the following further refinement of the Fejer inequality (1.2) for 1-quasiconvex functions:

COROLLARY 1. *Assume that the conditions of Theorem 2 on $\psi$ and $p$ hold. Then*

$$\begin{aligned}
&\psi\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx + s^2\varphi'\left(\frac{a+b}{2}\right)\int_a^b \left(x-\frac{a+b}{2}\right)^2 p(x)\,dx \\
&\leqslant \int_a^b \psi\left(sx+(1-s)\frac{a+b}{2}\right)p(x)\,dx \\
&\leqslant \int_a^b \psi(x)\,p(x)\,dx - \left(1-s^2\right)\int_a^b \left(x-\frac{a+b}{2}\right)^2 \varphi'\left(sx+(1-s)\frac{a+b}{2}\right)p(x)\,dx \\
&\leqslant \frac{\psi(a)+\psi(b)}{2}\int_a^b p(x)\,dx - \int_a^b \varphi'(x)\,(b-x)\,(x-a)\,p(x)\,dx \\
&\quad - \left(1-s^2\right)\int_a^b \left(x-\frac{a+b}{2}\right)^2 \varphi'\left(sx+(1-s)\frac{a+b}{2}\right)p(x)\,dx.
\end{aligned}$$

*Proof.* The left hand-side inequality follows from (2.4) by exchanging $s$ with $t$ so that $t \leqslant s$ and then taking $t=0$. The second inequality is obtained by taking $t=1$ in (2.4) The third inequality follows from the right hand-side of (1.3) in Lemma 1. □

The corresponding result for the function $Q(t)$ defined by (2.2) reads:

THEOREM 3. *Let $\psi$ and $p$ be defined as in Theorem 1, and let $Q(t)$ be defined by (2.2). If $0 \leqslant s \leqslant t \leqslant 1$, then*

$$Q(s) \leqslant Q(t) - \Delta(s,t), \tag{2.8}$$

*where*

$$\begin{aligned}
\Delta(s,t) = &\int_a^{\frac{a+b}{2}} \left(\varphi'\left((1-s)x+sa\right)+\varphi'\left((1-s)(a+b-x)+sb\right)\right) \\
&\times (t-s)\,(x-a)\,(a+b-2x+(t+s)\,(x-a))\,p(x)\,dx.
\end{aligned} \tag{2.9}$$

*Proof.* For $0 \leqslant s \leqslant t \leqslant 1$, $t>0$, $a \leqslant x \leqslant b$, we choose in (2.5)

$$z=(1-s)x+sa, \qquad m=(1-t)x+ta, \qquad M=(1-t)(a+b-x)+tb.$$

Evidently

$$\begin{aligned} a &\leqslant (1-t)x+ta \leqslant (1-s)x+sa \\ &\leqslant (1-s)(a+b-x)+sb \leqslant (1-t)(a+b-x)+tb \leqslant b, \end{aligned}$$

$$m+M=(1-t)x+ta+(1-t)(a+b-x)+tb=a+b,$$

$$z-m=(t-s)(x-a),$$

$$m+M-z=(1-s)(a+b-x)+sb.$$

Hence, from (2.5) we get that

$$\begin{aligned} &\psi((1-s)x+sa)+\psi((1-s)(a+b-x)+sb) \\ \leqslant\ &\psi((1-t)x+ta)+\psi((1-t)(a+b-x)+tb) \\ &-\left(\varphi'((1-s)x+sa)+\varphi'((1-s)(a+b-x)+sb)\right) \\ &\times(t-s)(x-a)(a+b-2x+(s+t)(x-a)). \end{aligned} \tag{2.10}$$

It follows from the symmetry of $p=p(x)$ that $Q(s)$ can be written as

$$Q(s)=\int_a^{\frac{a+b}{2}}(\psi((1-s)x+sa)+\psi((1-s)(a+b-x)+sb))\,p(x)\,dx \tag{2.11}$$

and therefore according to (2.10) we obtain that

$$\begin{aligned} Q(s) &= \int_a^{\frac{a+b}{2}}(\psi((1-s)x+sa)+\psi((1-s)(a+b-x)+sb))\,p(x)\,dx \\ &\leqslant \int_a^{\frac{a+b}{2}}(\psi((1-t)x+ta)+\psi((1-t)(a+b-x)+tb))\,p(x)\,dx \\ &\quad-\int_a^{\frac{a+b}{2}}\left(\left(\varphi'((1-s)x+sa)+\varphi'((1-s)(a+b-x)+sb)\right)\right. \\ &\quad\left.\times(t-s)(x-a)(a+b-2x+(s+t)(x-a))\right)p(x)\,dx. \end{aligned} \tag{2.12}$$

In other words we find by using (2.11) and (2.12) that (2.8) and (2.9) hold. The proof is complete. □

EXAMPLE 2. In the special case that $s=0,\ t=1$ we have that

$$\begin{aligned} Q(0) &= \int_a^b \psi(x)\,p(x)\,dx \leqslant Q(1)-\Delta(0,1) \\ &= \frac{\psi(a)+\psi(a)}{2}\int_a^b p(x)\,dx \\ &\quad-\int_a^{\frac{a+b}{2}}\left(\varphi'(x)+\varphi'(a+b-x)\right)(x-a)(b-x)\,p(x)\,dx \\ &= \frac{\psi(a)+\psi(a)}{2}\int_a^b p(x)\,dx-\int_a^b \varphi'(x)(x-a)(b-x)\,p(x)\,dx, \end{aligned}$$

which is the same as the right hand-side of (1.3). Hence, Theorem 3 implies in particular a further refinement of the Fejer inequality (1.2).

## 3. Fejer and Hermite-Hadamard inequalities for functions with bounded second derivatives

We prove now immediate results about Fejer and Hermite-Hadamard type inequalities for functions with bounded second derivatives.

THEOREM 4. *Let* $f:[a,b]\to\mathbb{R}$ *be a function such that* $m\leqslant f''(x)\leqslant M$, *and* $p:[a,b]\to\mathbb{R}$ *is integrable, non-negative and symmetric on* $[a,b]$. *Then*

$$\frac{f(a)+f(b)}{2}\int_a^b p(x)\,dx-\frac{M}{2}\int_a^b (x-a)(b-x)\,p(x)\,dx\leqslant\int_a^b f(x)\,p(x)\,dx, \tag{3.1}$$

$$f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx+\frac{m}{2}\int_a^b\left(x-\frac{a+b}{2}\right)^2 p(x)\,dx\leqslant\int_a^b f(x)\,p(x)\,dx, \tag{3.2}$$

$$\int_a^b f(x)\,p(x)\,dx\leqslant\frac{f(a)+f(b)}{2}\int_a^b p(x)\,dx-\frac{m}{2}\int_a^b (x-a)(b-x)\,p(x)\,dx, \tag{3.3}$$

*and*

$$\int_a^b f(x)\,p(x)\,dx\leqslant f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx+\frac{M}{2}\int_a^b\left(x-\frac{a+b}{2}\right)^2 p(x)\,dx. \tag{3.4}$$

*Proof.* Since $m\leqslant f''(x)\leqslant M$ we find that the functions

$$g_1(x)=\frac{M}{2}(x-a)(x-b)-f(x),$$

$$g_2(x)=f(x)-\frac{m}{2}\left(x-\frac{a+b}{2}\right)^2,$$

$$g_3(x)=f(x)-\frac{m}{2}(x-a)(x-b),$$

and

$$g_4(x)=\frac{M}{2}\left(x-\frac{a+b}{2}\right)^2-f(x),$$

are convex on $[a,b]$ and by using (1.2) for $g_i$, $i=1,\ldots,4$, we get the four inequalities (3.1), (3.2), (3.3) and (3.4), respectively. □

In the next statement we combine the results obtained for 1-quasiconvex functions with the results obtained for functions with bounded second derivative. This is obtained by using inequalities (1.3), (3.2) and (3.4).

COROLLARY 2. *Let $\varphi$ be a twice differentiable convex function on $[a,b]$, $a \geqslant 0$, and let $\psi(x) = x\varphi(x)$. Let $\psi(x)$ be such that $m \leqslant \psi''(x) \leqslant M$ and let $p(x)$ be non-negative, integrable and symmetric around $x = \frac{a+b}{2}$. Then*

$$\begin{aligned} & K\int_a^b \left(x-\frac{a+b}{2}\right)^2 p(x)\,dx \\ \leqslant & \int_a^b \psi(x)\,p(x)\,dx - \psi\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx \\ \leqslant & \frac{M}{2}\int_a^b \left(x-\frac{a+b}{2}\right)^2 p(x)\,dx \end{aligned} \tag{3.5}$$

*where* $K = \min\left(\frac{m}{2}, \varphi'\left(\frac{a+b}{2}\right)\right)$

REMARK 2. The function $\psi$ satisfies $\psi''(x) = (x\varphi(x))'' = 2\varphi'(x) + x\varphi''(x) \geqslant 2\varphi'(x)$. Therefore $\varphi'(x) \leqslant \frac{\psi''(x)}{2} \leqslant \frac{M}{2}$ and (3.5) gives a nice double inequality for $\int_a^b \psi(x)\,p(x)\,dx$ when $\psi$ is 1-quasiconvex function with bounded second derivative. This means that by replacing $\int_a^b \psi(x)\,p(x)\,dx$ with $\psi\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx$, we get that the maximum length or the error obtained is limited by

$$\left(\frac{M}{2} - K\right)\int_a^b \left(x-\frac{a+b}{2}\right)^2 p(x)\,dx,$$

where $K = \min\left(\frac{m}{2}, \varphi'\left(\frac{a+b}{2}\right)\right)$ and $p(x)$ is non-negative, integrable and symmetric on $[a,b]$.

We show now examples of the use of the Fejer inequality (1.2) and Theorem 4 regarding fractional integrals. These examples appear in [5, Theorem 1.1 and Theorem 1.2]. We show a more general case where the proof is simpler and shorter than that in [5].

The following example is related to the Fejer inequality (1.2) when the non-negative integrable symmetric function $p(x)$ around $x = \frac{a+b}{2}$ is related to fractional integrals:

EXAMPLE 3. Let $f : [a,b] \to \mathbb{R}$ be an integrable convex function with $a < b$. Then the following inequalities for fractional integrals hold:

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left(J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right) \leqslant \frac{f(a)+f(b)}{2} \tag{3.6}$$

with $\alpha > 0$, where

$$J_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\,dt, \quad x > a,$$

$$J_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)\,dt, \quad x < b,$$

and $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{\alpha-1}dt$.

REMARK 3. It is easy to see that Inequality (3.6) is a particular case of the Fejer inequality (1.2), where

$$p(x)=\frac{\alpha}{2(b-a)^{\alpha}}\left((x-a)^{\alpha-1}+(b-x)^{\alpha-1}\right),\quad a<x<b,\quad \alpha>0, \tag{3.7}$$

$p(x)$ is symmetric around $x=\frac{a+b}{2}$, non-negative, integrable, $\int_a^b p(x)\,dx=1$ and $f$ is a positive convex function on $[a,b]$, and because

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left(J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right)\\
&=\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left(\frac{1}{\Gamma(\alpha)}\int_a^b(b-t)^{\alpha-1}f(t)\,dt+\frac{1}{\Gamma(\alpha)}\int_a^b(t-a)^{\alpha-1}f(t)\,dt\right)\\
&=\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left(\frac{1}{\Gamma(\alpha)}\int_a^b\left((b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right)f(t)\,dt\right)\\
&=\int_a^b f(t)\,p(t)\,dt.
\end{aligned}$$

In particular, we pronounce that Inequality (3.6) holds without the restriction that $f$ is non-negative as required in [5, Theorem 1.1].

Moreover, going back to Theorem 4, if $p(x)$ is defined by (3.7), then we have the following:

EXAMPLE 4. For a twice differentiable function $f$ with $m\leqslant f''(x)\leqslant M$ on $[a,b]$ and $p$ as above, we get that:

$$\begin{aligned}
&\frac{f(a)+f(b)}{2}-\frac{M\alpha}{4(b-a)^{\alpha}}\int_a^b(x-a)(b-x)\left((x-a)^{\alpha-1}+(b-x)^{\alpha-1}\right)dx\\
&\leqslant\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left(J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right),
\end{aligned}$$

$$\begin{aligned}
&f\left(\frac{a+b}{2}\right)+\frac{m\alpha}{4(b-a)^{\alpha}}\int_a^b\left(x-\frac{a+b}{2}\right)^2\left((x-a)^{\alpha-1}+(b-x)^{\alpha-1}\right)dx\\
&\leqslant\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left(J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right),
\end{aligned}$$

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left(J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right)\\
&\leqslant\frac{f(a)+f(b)}{2}-\frac{m\alpha}{4(b-a)^{\alpha}}\int_a^b(x-a)(b-x)\left((x-a)^{\alpha-1}+(b-x)^{\alpha-1}\right)dx,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left(J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right)\\
&\leqslant f\left(\frac{a+b}{2}\right)+\frac{M\alpha}{4(b-a)^{\alpha}}\int_a^b\left(x-\frac{a+b}{2}\right)^2\left((x-a)^{\alpha-1}+(b-x)^{\alpha-1}\right)dx.
\end{aligned}$$

REMARK 4. The inequalities in Example 4 appeared in [5, Theorem 1.2] and in our case it is just a particular case of Theorem 4.

## 4. Remarks about other Fejer type inequalities

We extend Fejer's inequality for special convex functions by replacing the non-negative symmetric function $p = p(x)$ in (1.2) with monotone functions.

THEOREM 5. *Let $\varphi : [a,b] \to \mathbb{R}$, be a differentiable and convex function. Let $p : [a,b] \to \mathbb{R}$ be a non-negative, integrable and monotone function.*

*a) Let $p'(x) \leqslant 0$, $a \leqslant x \leqslant b$ and $\varphi(a) \leqslant \varphi(b)$ (see Figure 1). Then*

$$\int_a^b \varphi(t)\, p(t) dt \leqslant \frac{\varphi(a)+\varphi(b)}{2} \int_a^b p(x) dx. \tag{4.1}$$

*b) Let $p'(x) \geqslant 0$, $a \leqslant x \leqslant b$ and $\varphi(a) \leqslant \varphi\left(\frac{a+b}{2}\right)$ (see Figure 2). Then*

$$\varphi\left(\frac{a+b}{2}\right) \int_a^b p(x) dx \leqslant \int_a^b \varphi(t)\, p(t) dt. \tag{4.2}$$

*c) If $p'(x) \geqslant 0$, $a \leqslant x \leqslant b$ and $\varphi(a) \geqslant \varphi(b)$, then (4.1) holds.*

*d) If $p'(x) \leqslant 0$, $a \leqslant x \leqslant b$ and $\varphi(a) \geqslant \varphi\left(\frac{a+b}{2}\right)$, then (4.2) holds.*

REMARK 5. In particular cases a) and b) hold when $\varphi$ is increasing and cases c) and d) hold when $\varphi$ is decreasing.

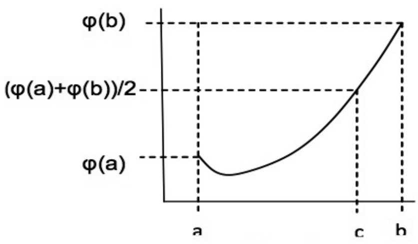

Figure 1.

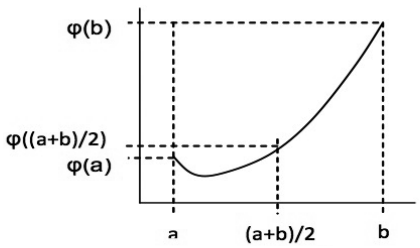

Figure 2.

*Proof.* a) From $\varphi(a) \leqslant \varphi(b)$ we get that $\varphi(a) \leqslant \frac{\varphi(a)+\varphi(b)}{2} \leqslant \varphi(b)$ and together with the convexity of $\varphi$ as $\varphi(t) - \frac{\varphi(a)+\varphi(b)}{2} \leqslant 0$ when $a \leqslant t \leqslant c$ where $\varphi(c) = \frac{\varphi(a)+\varphi(b)}{2}$, we get from Hermite-Hadamard inequalities

$$\int_a^b \left(\varphi(t) - \frac{\varphi(a)+\varphi(b)}{2}\right) dt \leqslant 0$$

and also,

$$\int_a^x \left(\varphi(t) - \frac{\varphi(a)+\varphi(b)}{2}\right) dt \leqslant 0$$

for all $x$ on $[a,b]$ (see Figure 1). Denoting

$$f(t) = \varphi(t) - \frac{\varphi(a)+\varphi(b)}{2}$$

we get that

$$\begin{aligned}\int_a^b f(t)\,p(t)\,dt &= \left[\left(\int_a^x f(t)\,dt\right)p(x)\right]_a^b - \int_a^b \left(\int_a^x f(t)\,dt\right)p'(x)\,dx \\ &= p(b)\int_a^b f(t)\,dt - \int_a^b \left(\int_a^x f(t)\,dt\right)p'(x)\,dx \leqslant 0,\end{aligned}$$

which means that (4.1) holds.

b) From $\varphi(a) \leqslant \varphi\left(\frac{a+b}{2}\right)$ and the convexity of $\varphi$ we get that $\varphi(a) \leqslant \varphi\left(\frac{a+b}{2}\right) \leqslant \varphi(b)$. Denoting $g(t) = \varphi(t) - \varphi\left(\frac{a+b}{2}\right)$, we find that $g(t) \geqslant 0$ when $\frac{a+b}{2} \leqslant t \leqslant b$, and since $\int_a^b g(t)\,dt \geqslant 0$ and because of the convexity of $\varphi$, $\varphi(t) \leqslant \varphi\left(\frac{a+b}{2}\right)$ when $a \leqslant t \leqslant \frac{a+b}{2}$ and therefore $\int_x^b g(t)\,dt \geqslant 0$ for all $a \leqslant x \leqslant b$, (see Figure 2).

Hence

$$\begin{aligned}\int_a^b g(t)\,p(t)\,dt &= \left[-\left(\int_x^b g(t)\,dt\right)p(x)\right]_a^b + \int_a^b \left(\int_x^b g(t)\,dt\right)p'(x)\,dx \\ &= p(a)\int_a^b g(t)\,dt + \int_a^b \left(\int_x^b g(t)\,dt\right)p'(x)\,dx \geqslant 0,\end{aligned}$$

in other words (4.2) is proved.

The cases c) and d) can be proved in a similar way so we omit the details. The proof is complete. $\square$

We finish by going back to Theorem 1 ([3, Theorem 2]), and pointing out that from this theorem we may get an upper bound for $\frac{1}{b-a}\int_a^b \varphi(x)\,x^2 dx = \frac{1}{b-a}\int_a^b \psi_2(x)\,dx$ where $\varphi$ is a convex increasing function. We show that our upper bound of $\frac{1}{b-a}\int_a^b \psi_2(x)\,dx$ and also the upper bound of $\frac{1}{b-a}\int_a^b \varphi(x)\,x dx = \frac{1}{b-a}\int_a^b \psi_1(x)\,dx$ are better than those derived from the following theorem in [6, Theorem 1].

THEOREM A. *Let $\varphi$ and $g$ be real-valued, non-negative and convex functions on $[a,b]$, then:*

$$\begin{aligned}&2\varphi\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) - \frac{\varphi(a)g(a)+\varphi(b)g(b)}{2} + \frac{(\varphi(b)-\varphi(a))(g(b)-g(a))}{3} \\ &\leqslant \frac{1}{b-a}\int_a^b \varphi(x)\,g(x)\,dx \\ &\leqslant \frac{\varphi(a)g(a)+\varphi(b)g(b)}{2} - \frac{(\varphi(b)-\varphi(a))(g(b)-g(a))}{6}.\end{aligned} \tag{4.3}$$

REMARK 6. We see that when $g(x) = x$ inequalities (4.3) in Theorem A for $\psi_1(x) = x\varphi(x)$ are

$$2\psi_1\left(\frac{a+b}{2}\right) - \frac{\psi_1(a)+\psi_1(b)}{2} + \frac{(\varphi(b)-\varphi(a))(b-a)}{3} \tag{4.4}$$
$$\leqslant \frac{1}{b-a}\int_a^b \psi_1(x)\,dx$$
$$\leqslant \frac{\psi_1(a)+\psi_1(b)}{2} - \frac{(\varphi(b)-\varphi(a))(b-a)}{6},$$

and when $g(x) = x^2$ inequalities (4.3) in Theorem A for $\psi_2(x) = x^2\varphi(x)$ are

$$2\psi_2\left(\frac{a+b}{2}\right) - \frac{\psi_2(a)+\psi_2(b)}{2} + \frac{(\varphi(b)-\varphi(a))\left(b^2-a^2\right)}{3} \tag{4.5}$$
$$\leqslant \frac{1}{b-a}\int_a^b \psi_2(x)\,dx$$
$$\leqslant \frac{\psi_2(a)+\psi_2(b)}{2} - \frac{(\varphi(b)-\varphi(a))\left(b^2-a^2\right)}{6}.$$

EXAMPLE 5. By comparing (4.4) with the inequalities (1.4) we realize that our result of the upper bound of Hermite-Hadamard inequality in (1.4):

$$\frac{1}{b-a}\int_a^b \psi_1(x)\,dx \leqslant \frac{b-a}{6}\varphi(b) + \frac{b+2a}{3}\frac{1}{b-a}\int_a^b \varphi(x)\,dx \tag{4.6}$$
$$\leqslant \frac{\psi_1(a)+\psi_1(b)}{2} - \frac{(\varphi(b)-\varphi(a))(b-a)}{6},$$

is a refinement of the upper bound obtained in the right hand-side of (4.4).

Moreover, if $\varphi$ is convex function on $[a,b]$, $0 \leqslant a < b$, our inequalities (1.4) and (4.6) are valid for convex functions that are also not non-negative, whereas Theorem A and therefore also (4.4) are proved only for convex non-negative $\varphi$.

We see that (4.5) holds for non-negative convex functions, as stated in Theorem A, (which is proved in [3, Theorem 1]), whereas (4.7) below uses twice the first inequaily of (4.6) and therefore $\psi_1$ has to be convex. For $\psi_1$ to be convex it is sufficient that $\varphi$ is convex and increasing on $[a,b]$, $a \geqslant 0$. Therefore (4.5) and (4.7) are compared below when $\varphi$ is convex, increasing and non-negative on $[a,b]$, $0 \leqslant a < b$ (in such cases $\psi_2$ is also convex).

Using twice (4.6) we get that

$$\frac{1}{b-a}\int_a^b \psi_2(x)\,dx \tag{4.7}$$
$$\leqslant \frac{b-a}{6}\psi_1(b) + \frac{b+2a}{3}\frac{1}{b-a}\int_a^b \psi_1(x)\,dx$$
$$\leqslant \frac{b-a}{6}b\varphi(b) + \frac{b+2a}{3}\left(\frac{b-a}{6}\varphi(b) + \frac{b+2a}{3}\frac{1}{b-a}\int_a^b \varphi(x)\,dx\right)$$
$$\leqslant \frac{b-a}{6}b\varphi(b) + \frac{b+2a}{3}\left(\frac{b-a}{6}\varphi(b) + \frac{b+2a}{6}(\varphi(a)+\varphi(b))\right).$$

The following simple calculation shows that in fact (4.7) is a better upper bound than (4.5) when $\varphi$ is differentiable non-negative, increasing and convex function on $[a,b]$, $a \geqslant 0$:

The right hand-side of (4.5) rewritten as

$$\varphi(b)\left(\frac{2}{6}b^2+\frac{1}{6}a^2\right)+\varphi(a)\left(\frac{2}{6}a^2+\frac{1}{6}b^2\right),$$

is greater than the right hand-side of (4.7) rewritten as

$$\varphi(b)\left(\frac{5}{18}b^2+\frac{4}{18}ab+\frac{2}{18}a^2\right)+\varphi(a)\left(\frac{1}{18}b^2+\frac{4}{18}ab+\frac{4}{18}a^2\right),$$

because for $\varphi(x) \geqslant 0$

$$\begin{aligned}
&\left[\varphi(b)\left(\frac{2}{6}b^2+\frac{1}{6}a^2\right)+\varphi(a)\left(\frac{2}{6}a^2+\frac{1}{6}b^2\right)\right]\\
&\quad-\left[\varphi(b)\left(\frac{5}{18}b^2+\frac{4}{18}ab+\frac{2}{18}a^2\right)+\varphi(a)\left(\frac{1}{18}b^2+\frac{4}{18}ab+\frac{4}{18}a^2\right)\right]\\
&=\frac{(b-a)^2}{18}\left(\varphi(b)+2\varphi(a)\right) \geqslant 0.
\end{aligned}$$

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