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A SHARPENING OF A PROBLEM ON BERNSTEIN POLYNOMIALS AND CONVEX FUNCTIONS

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Abstract. We present an elementary proof of a conjecture by I. Raşa which is an inequality involving Bernstein basis polynomials and convex functions. It was affirmed in positive very recently by the use of stochastic convex orderings.

1. Introduction

The classical Bernstein polynomials, defined for $f \in C[0,1]$ by

$$(B_n f)(x) = \sum_{\nu=0}^{n} p_{n,\nu}(x) f\left(\frac{\nu}{n}\right) \qquad (x \in [0,1]),$$

with the basis polynomials

$$p_{n,\nu}(x) = \binom{n}{\nu} x^{\nu} (1-x)^{n-\nu} \qquad (\nu = 0,1,2,\ldots),$$

are the most prominent positive linear approximation operators (see [9]). If $f \in C[0,1]$ is convex the inequality

$$\sum_{i=0}^{n} \sum_{j=0}^{n} \left[p_{n,i}(x) p_{n,j}(x) + p_{n,i}(y) p_{n,j}(y) - 2p_{n,i}(x) p_{n,j}(y)\right] f\left(\frac{i+j}{2n}\right) \geqslant 0 \qquad (1)$$

is valid, for $x, y \in [0,1]$.

This inequality involving Bernstein basis polynomials and convex functions was stated as an open problem 25 years ago by Ioan Raşa. During the Conference on Ulam's Type Stability (Rytro, Poland, 2014), Raşa [11] recalled his problem.

Inequalities of type (1) have important applications. They are useful when studying whether the Bernstein-Schnabl operators preserve convexity (see [3, 4, 5]).

Very recently, J. Mrowiec, T. Rajba and S. Wąsowicz [10] affirmed the conjecture in positive. Their proof makes heavy use of probability theory. As a tool they applied stochastic convex orderings (which they proved for binomial distributions) as well as

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the so-called concentration inequality. After that one of the authors gave a short elementary proof [1] of inequality (1). The other author remarked in [12] that (1) is equivalent to

$$(B_{2n}f)(x)+(B_{2n}f)(y)\geqslant 2\sum_{i=0}^{n}\sum_{j=0}^{n}p_{n,i}(x)\,p_{n,j}(y)\,f\left(\frac{i+j}{2n}\right). \tag{2}$$

If f is convex on $[0,1]$ also $B_{2n}f$ is convex on $[0,1]$ (see [2, Corollary 6.3.8]). Therefore, we have

$$(B_{2n}f)(x)+(B_{2n}f)(y)\geqslant 2\,(B_{2n}f)\left(\frac{x+y}{2}\right). \tag{3}$$

Thus the following problem seems to be a natural one: Prove that

$$(B_{2n}f)\left(\frac{x+y}{2}\right)\geqslant \sum_{i=0}^{n}\sum_{j=0}^{n}p_{n,i}(x)\,p_{n,j}(y)\,f\left(\frac{i+j}{2n}\right), \tag{4}$$

for all convex $f\in C[0,1]$ and $x,y\in[0,1]$.

If (4) is valid, then (2) – and hence (1) – is a consequence of (3) and (4). Starting from these remarks, the second author presented the inequality (4) as an open problem in [12]. A probabilistic solution was found by A. Komisarski and T. Rajba [7] using the methods developed in [10] and [8].

The purpose of this short note is to give an analytic proof of the following theorem.

THEOREM 1. *Let $n,m\in\mathbb{N}$. If $f\in C[0,1]$ is a convex function, then the inequality*

$$(B_{mn}f)\left(\frac{1}{m}\sum_{\nu=1}^{m}x_\nu\right)\geqslant\sum_{i_1=0}^{n}\cdots\sum_{i_m=0}^{n}\left(\prod_{\nu=1}^{m}p_{n,i_\nu}(x_\nu)\right)f\left(\frac{1}{mn}\sum_{\nu=1}^{m}i_\nu\right),$$

is valid for all $x_1,\ldots,x_m\in[0,1]$.

Obviously, the special case $m=2$ is inequality (4).

2. An elementary proof of Theorem 1

Using the obvious identity

$$p_{n,i}(x)=\frac{1}{i!}\left(\frac{\partial}{\partial z}\right)^{i}\left[(1+xz)^n\right]\Big|_{z=-1}$$

we obtain

$$\begin{aligned}
&\sum_{i_1=0}^{n}\cdots\sum_{i_m=0}^{n}\left(\prod_{\nu=1}^{m}p_{n,i_\nu}(x_\nu)\right)f\left(\frac{1}{mn}\sum_{\nu=1}^{m}i_\nu\right)\\
&=\sum_{k=0}^{mn}f\left(\frac{k}{mn}\right)\sum_{i_1+\cdots+i_m=k}\prod_{\nu=1}^{m}\left(\frac{1}{i_\nu!}\left(\frac{\partial}{\partial z}\right)^{i_\nu}(1+x_\nu z)^n\right)\Bigg|_{z=-1}\\
&=\sum_{k=0}^{mn}f\left(\frac{k}{mn}\right)\frac{1}{k!}\left[\left(\frac{\partial}{\partial z}\right)^{k}\prod_{\nu=1}^{m}(1+x_\nu z)^n\right]\Bigg|_{z=-1},
\end{aligned}$$

where the last equality follows by Leibniz rule for derivatives. In particular, we have

$$\sum_{i_1=0}^{n}\cdots\sum_{i_m=0}^{n}\left(\prod_{\nu=1}^{m}p_{n,i_\nu}(x)\right)f\left(\frac{1}{mn}\sum_{\nu=1}^{m}i_\nu\right)$$
$$=\sum_{k=0}^{mn}f\left(\frac{k}{mn}\right)\frac{1}{k!}\left[\left(\frac{\partial}{\partial z}\right)^k(1+xz)^{mn}\right]\Bigg|_{z=-1}=(B_{mn}f)(x).$$

Inserting

$$(B_{mn}f)\left(\frac{1}{m}\sum_{\nu=1}^{m}x_\nu\right)=\sum_{k=0}^{mn}f\left(\frac{k}{mn}\right)\frac{1}{k!}\left[\left(\frac{\partial}{\partial z}\right)^k\left(\frac{1}{m}\sum_{\nu=1}^{m}(1+x_\nu z)\right)^{mn}\right]\Bigg|_{z=-1}$$

we obtain

$$(B_{mn}f)\left(\frac{1}{m}\sum_{\nu=1}^{m}x_\nu\right)-\sum_{i_1=0}^{n}\cdots\sum_{i_m=0}^{n}\left(\prod_{\nu=1}^{m}p_{n,i_\nu}(x_\nu)\right)f\left(\frac{1}{mn}\sum_{\nu=1}^{m}i_\nu\right)$$
$$=\sum_{k=0}^{mn}f\left(\frac{k}{mn}\right)\frac{1}{k!}\left(\frac{\partial}{\partial z}\right)^k\left[\left(\frac{1}{m}\sum_{\nu=1}^{m}(1+x_\nu z)\right)^{mn}-\prod_{\nu=1}^{m}(1+x_\nu z)^n\right]\Bigg|_{z=-1}.$$

For fixed $n\in\mathbb{N}$ and $x_1,\ldots,x_m\in[0,1]$, we define

$$g(z)\equiv g_{m,n}(z;x_1,\ldots,x_m)=z^{-2}\left(\left(\frac{1}{m}\sum_{\nu=1}^{m}(1+x_\nu z)\right)^{mn}-\prod_{\nu=1}^{m}(1+x_\nu z)^n\right).$$

Note that g is a polynomial in z of degree at most $mn-2$.

LEMMA 1. *Fix $x_1,\ldots,x_m\in[0,1]$. Then, the function g satisfies $g^{(k)}(-1)\geqslant 0$, for $k=0,1,\ldots,mn-2$.*

Proof. For abbreviation, put $a_\nu=1+x_\nu z$. We have

$$\left(\frac{1}{m}\sum_{\nu=1}^{m}(1+x_\nu z)\right)^{mn}-\prod_{\nu=1}^{m}(1+x_\nu z)^n$$
$$=\left(\frac{1}{m}\sum_{\nu=1}^{m}a_\nu\right)^{mn}-\left(\prod_{\nu=1}^{m}a_\nu\right)^n$$
$$=\left(\left(\frac{1}{m}\sum_{\nu=1}^{m}a_\nu\right)^m-\prod_{\nu=1}^{m}a_\nu\right)\sum_{j=0}^{n-1}\left(\frac{1}{m}\sum_{\nu=1}^{m}a_\nu\right)^{n-1-j}\left(\prod_{\nu=1}^{m}a_\nu\right)^j.$$

We see that the second factor

$$\sum_{j=0}^{n-1}\left(\frac{1}{m}\sum_{\nu=1}^{m}a_\nu\right)^{n-1-j}\left(\prod_{\nu=1}^{m}a_\nu\right)^j$$

has non-negative derivatives of all orders at $z=-1$. This reduces the problem to the case $n=1$, i.e.,

$$\left(\sum_{\nu=1}^{m} a_\nu\right)^m - m^m \prod_{\nu=1}^{m} a_\nu$$

when multiplied by m^m. For $m=2$, we have

$$(a_1+a_2)^2 - 4a_1a_2 = (a_1-a_2)^2 = (x_1-x_2)^2 z^2$$

such that $g_{2,1}(z;x_1,x_2) = (x_1-x_2)^2/4$. For $m \geqslant 2$, Gusić [6, Theorem 1] (cf. [13, Eq. (2)]) proved the representation

$$\left(\sum_{\nu=1}^{m} a_\nu\right)^m - m^m \prod_{\nu=1}^{m} a_\nu = \sum_{1\leqslant i<j\leqslant m} (a_i-a_j)^2 P_{i,j}(a_1,\ldots,a_m),$$

for some homogeneous polynomials $P_{i,j}$ of degree $n-2$ with all non-negative coefficients. Applying this with $a_i = 1+x_i z$ we see that each $P_{i,j}(a_1,\ldots,a_m)$ has all the derivatives at $z=-1$ non-negative and so does $(a_i-a_j)^2/z^2$ by the $m=2$ case. Thus $g(z)$ has all the derivatives at $z=-1$ non-negative as well. $\square$

The key result is the next proposition. The proof follows the lines of [1, Prop. 1].

PROPOSITION 1. *Fix $x_1,\ldots,x_m \in [0,1]$. Then, for any real numbers $a_0,\ldots,a_{mn}$, the identity*

$$\begin{aligned}&\sum_{i_1=0}^{n}\cdots\sum_{i_m=0}^{n}\left[\left(\prod_{\nu=1}^{m} p_{n,i_\nu}\left(\frac{1}{m}\sum_{\nu=1}^{m} x_\nu\right)\right) - \left(\prod_{\nu=1}^{m} p_{n,i_\nu}(x_\nu)\right)\right]\cdot a_{|i|}\\ &= \sum_{k=0}^{mn-2} \left(\Delta^2 a_k\right) \frac{1}{k!} g^{(k)}(-1)\end{aligned} \tag{5}$$

is valid.

Here we put $|i| = \sum_{\nu=1}^{m} i_\nu$ and Δ denotes the forward difference $\Delta a_k := a_{k+1}-a_k$ such that $\Delta^2 a_k = a_{k+2} - 2a_{k+1} + a_k$.

Because g is a polynomial in z of degree at most $mn-2$, it is obvious that $g^{(mn-1)}(-1) = g^{(mn)}(-1) = 0$.

Proof of Theorem 1. For $k=0,1,\ldots,mn-2$, we put

$$a_k = f\left(\frac{k}{mn}\right).$$

If $f \in C[0,1]$ is a convex function it follows that $\Delta^2 a_k \geqslant 0$, for $k=0,1,\ldots,mn-2$. Therefore, application of Proposition 1 proves Theorem 1. $\square$

REFERENCES

[1] U. ABEL, *An inequality involving Bernstein polynomials and convex functions*, J. Approx. Theory **222** (2017), 1–7, http://dx.doi.org/10.1016/j.jat.2017.05.006.

[2] F. ALTOMARE AND M. CAMPITI, *Korovkin-Type Approximation Theory and its Applications*, De Gruyter Studies in Mathematics **17**, Walter de Gruyter, Berlin-New York, 1994.

[3] F. ALTOMARE, M. CAPPELLETTI MONTANO, V. LEONESSA AND I. RAŞA, *Markov Operators, Positive Semigroups and Approximation Processes*, de Gruyter Studies in Mathematics **61**, Walter de Gruyter, Berlin/Boston, 2014.

[4] F. ALTOMARE, M. CAPPELLETTI MONTANO, V. LEONESSA AND I. RAŞA, *Elliptic differential operators and positive semigroups associated with generalized Kantorovich operators*, to appear.

[5] F. ALTOMARE, M. CAPPELLETTI MONTANO, V. LEONESSA AND I. RAŞA, *A generalization of Kantorovich operators for convex compact subsets*, to appear in Banach J. Math. Anal.

[6] I. GUSIĆ, *A purely algebraic proof of AG inequality*, Math. Inequal. Appl. **8** (2005), 191–198.

[7] A. KOMISARSKI, T. RAJBA, A sharpening of a problem on Bernstein polynomials and convex function and related results, arXiv:1801.02247v1

[8] A. KOMISARSKI, T. RAJBA, *Muirhead inequality for convex orders and a problem of I. Raşa on Bernstein polynomials*, J. Math. Anal. Appl. **458** (2018), 821–830.

[9] G. G. LORENTZ, *Bernstein polynomials (Mathematical Expositions*, no. 8., University of Toronto Press, Toronto 1953.

[10] J. MROWIEC, T. RAJBA AND S. WĄSOWICZ, *A solution to the problem of Raşa connected with Bernstein polynomials*, J. Math. Anal. Appl. **446** (2017), 864–878, http://doi.org/10.1016/j.jmaa.2016.09.009.

[11] I. RAŞA, *2. Problem*, p. 164, in: Report of Meeting Conference on Ulam's Type Stability, Rytro, Poland, June 2–6, 2014, Ann. Univ. Paedagog. Crac. Stud. Math. **13** (2014), 139–169, doi:10.2478/aupcsm-2014-0011.

[12] I. RASA, *Bernstein polynomials and convexity: recent probabilistic and analytic proofs*, The Workshop “Numerical Analysis, Approximation and Modeling”, T. Popoviciu Institute of Numerical Analysis, Cluj-Napoca, June 14, 2017, http://ictp.acad.ro/zilele-academice-clujene-2017/.

[13] T. TARARYKOVA, *An explicit representation as quasi-sum of squares of a polynomial generated by the AG inequality*, Math. Inequal. Appl. **9** (2006), 649–659.

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