Mathematical
Inequalities
& Applications
Volume 21, Number 3 (2018), 779–793 doi:10.7153/mia-2018-21-56

TOPICAL FUNCTIONS: HERMITE–HADAMARD TYPE INEQUALITIES AND KANTOROVICH DUALITY

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(*Communicated by C. P. Niculescu*)

Abstract. For a certain class of elementary functions consisting of min-type functions, we apply techniques from abstract convex analysis to study Hermite-Hadamard type inequalities for increasing and plus-homogeneous (topical) functions. Some examples of such inequalities for functions with the special domains are given as well. In the next part, we study Kantorovich duality for the optimal mass transportation problems whenever the cost function is a min-type function. In this case, some pricing criteria are established as well.

1. Introduction

Let f be a convex function defined on the segment $[a,b]$ of the real line. Then the following inequality holds:

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant \frac{1}{2}(f(a)+f(b)). \tag{1}$$

These inequalities are well known as the Hermite-Hadamard inequalities [10]. There are many generalizations of these inequalities for classes of non-convex functions such as quasiconvex functions [13, 14], p-functions [13], ICAR (Increasing and Convex-Along-Rays) functions [6], IR (Increasing and Radiant) functions [19], and IPH (Increasing and Positively Homogeneous) functions [1]. More results regarding Hadamard type inequalities have been presented in [7].

For instance [14], if $f:[0,1]\longrightarrow\mathbb{R}$ is an arbitrary nonnegative quasiconvex function, then for every $u\in(0,1)$ one has

$$f(u) \leqslant \frac{1}{\min(u,1-u)}\int_0^1 f(x)\,dx. \tag{2}$$

The inequality (2) is sharp, which means that there is a quasiconvex function for which the equality holds.

If

$$D=\left\{(x,y)\in\mathbb{R}^2_+ : 0\leqslant x\leqslant a,\ 0\leqslant \frac{y}{x}\leqslant v\right\},$$

Mathematics subject classification (2010): 26D15, 26A51, 90C08, 90C46.
Keywords and phrases: Abstract convexity, Hermite-Hadamard type inequalities, topical function, optimal transportation, Kantorovich duality.

where $a > 0$ and $\nu > 0$, then for every ICAR function f we have:

$$f\left(\frac{a}{3}, \frac{\nu a}{3}\right) \leqslant \frac{1}{A(D)} \int_D f(x,y)\, dxdy,$$

where $A(D)$ is the area of D and this inequality is sharp [6].

The class of topical functions is one of the interesting classes of abstract convex functions (see [12, 15]). These functions have found many applications in various fields, such as economics, dynamical systems, mathematical optimization and etc. See for instances [5, 9, 11, 15] and references therein.

Applying techniques from abstract convex analysis, we present some Hermite-Hadamard type inequalities for the topical functions defined on $\mathbb{R}^n$. Some examples for particular domains are also presented.

The theory of optimal mass transportation has found many applications in applied sciences such as asymmetric information, incentive compatibility, multidimensional screening and in principal-agent paradigms, see [4, 8, 16, 17, 20] and references therein. The main key in this theory is the well-known Kantorovich duality problem, which we recall briefly here.

Let (X, μ) and (Y, ν) be Polish probability measure spaces. Let $\Pi(\mu, \nu)$ be the set of all positive measures π on the product space $X \times Y$ such that $\pi(A \times Y) = \mu(A)$ and $\pi(X \times B) = \nu(B)$, for all measurable sets $A \subseteq X$ and $B \subseteq Y$. Consider that the cost function $c : X \times Y \to \mathbb{R}$, which gives the cost of transporting one unit of mass at the point $x \in X$ to one unit of mass at the point $y \in Y$, is lower semi continuous. Therefore, the Monge-Kantorovich's mass transportation problem is stated as the following minimization problem:

$$\text{minimize} \int_{X \times Y} c(x,y) d\pi(x,y) \qquad \text{subject to} \quad \pi \in \Pi(\mu, \nu). \tag{3}$$

Assume that there are upper semi continuous functions $f_0 \in L^1(\mu)$ and $g_0 \in L^1(\nu)$ in such a way that for all $x \in X$ and $y \in Y$

$$f_0(x) + g_0(y) \leqslant c(x,y).$$

Therefore Kantorovich duality necessitates that

$$\min_{\pi \in \Pi(\mu,\nu)} \int_{X \times Y} c(x,y) d\pi(x,y) = \sup_{f \in L^1(\mu), g \in L^1(\nu), g - f \leqslant c} \Big(\int_Y g(y) d\nu(y) - \int_X f(x) d\mu(x) \Big). \tag{4}$$

To have an elaborated description of (4), we refer the reader to [20] (see also [3] Theorem 1.4). The right-hand side of the Kantorovich duality may be interpreted as a pricing problem: Suppose that a distributor buys a unit of mass at the source x for the price of $f(x)$ and he/she sells a unit of mass at the target y for the price of $g(y)$. Obviously, $g(y) \leqslant f(x) + c(x,y)$, otherwise the consumer at the point y buys the mass directly from the producer at the point x and analogously, producer at the point x may sell the mass directly at the point y. Moreover, the total price that the distributer should pay is $\int_X f(x) d\mu(x)$ and the total price for selling the total mass is $\int_Y g(y) d\nu(y)$. On the other hand, the left-hand side of (4) means that in the case of a direct trading, the traders intend to minimize the total cost of transportation, while the distributor looks forward to maximizing the profit.

In the second part of this paper we are going to study Kantorovich duality whenever the cost function is a specific minimum type function. We exploit a similar discussion from [3] in order to establish some pricing criteria whenever the primary pricing function is topical.

This article has the following structure. In section 2, we provide some preliminaries, definitions and results relative to topical functions. In section 3, we consider Hermite-Hadamard type inequalities for the class of topical functions. Some examples of such inequalities for functions defined on $\mathbb{R}^2$ are given in section 4. Finally we study mass transportation problem in the framework of Kantorovich duality for topical cost function in Section 5.

2. Preliminaries

We assume that $\mathbb{R}^n$ is equipped with the coordinate-wise order relation. A function $f:\mathbb{R}^n \longrightarrow \bar{\mathbb{R}} = [-\infty,+\infty]$ is said to be *increasing* if $x \leqslant y$ implies $f(x) \leqslant f(y)$ for all $x,\ y \in \mathbb{R}^n$. f is called *plus-homogeneous* if $f(x+\lambda\mathbf{1}) = f(x)+\lambda$ for all $x \in \mathbb{R}^n$ and all $\lambda \in \mathbb{R}$, where $\mathbf{1} = (1,\dots,1) \in \mathbb{R}^n$.

DEFINITION 1. A function $f:\mathbb{R}^n \longrightarrow \bar{\mathbb{R}}$ is called *topical* if it is increasing and plus-homogeneous.

LEMMA 1. [15] *Let $f:\mathbb{R}^n \longrightarrow \bar{\mathbb{R}}$ be a topical function.*
(i) If there exists $x \in \mathbb{R}^n$ such that $f(x) = +\infty$, then $f \equiv +\infty$.
(ii) If there exists $x \in \mathbb{R}^n$ such that $f(x) = -\infty$, then $f \equiv -\infty$.

It follows from Lemma 1 that a topical function is either finite (i.e., finite-valued at each $x \in \mathbb{R}^n$) or identically $+\infty$ or identically $-\infty$.

Now, we present the following simple examples of topical functions.

EXAMPLE 1. Every positive linear function $f:\mathbb{R}^n \longrightarrow \mathbb{R}$ (i.e., $f(x) \geqslant 0$ for all $x \geqslant 0$) such that $f(\mathbf{1}) = 1$ is topical.

EXAMPLE 2. Functions of the form

$$f(x) := \min_{1\leqslant i\leqslant n}(x_i + c_i) \text{ and } f(x) := \max_{1\leqslant i\leqslant n}(x_i + c_i)$$

are topical, where $c := (c_1,\dots,c_n) \in \mathbb{R}^n$.

EXAMPLE 3. The Log-sum-exp function of the form

$$f(x) = \frac{1}{p}\ln\Big(\sum_{i=1}^{n} e^{px_i}\Big), \quad (x \in \mathbb{R}^n),$$

where $0 < p < \infty$, is topical.

Let us mention some properties of the set Γ of all topical functions $f:\mathbb{R}^n \longrightarrow \bar{\mathbb{R}}$.
(1) We have $\Gamma+\mathbb{R}=\Gamma$, that is, if $f\in\Gamma$ and $c\in\mathbb{R}$, then $f+c\in\Gamma$.
(2) Γ is a convex set.
(3) Γ is a complete lattice, that is, if $\{f_\beta\}_{\beta\in B}$ is an arbitrary family of Γ and

$$f(x)=\sup_{\beta\in B} f_\beta(x), \quad (x\in\mathbb{R}^n),$$

then the function f belongs to Γ.
(4) Γ is closed with respect to the point-wise convergence of functions.

REMARK 1. Every finite topical function f is continuous on $\mathbb{R}^n$. Indeed, let $x,\ x_k\in\mathbb{R}^n$, $x_k\longrightarrow x$ and $\varepsilon>0$. Then $x-\varepsilon\mathbf{1}\leqslant x_k\leqslant x+\varepsilon\mathbf{1}$ for sufficiently large k. Since f is increasing and plus-homogeneous, one has

$$f(x)-\varepsilon=f(x-\varepsilon\mathbf{1})\leqslant f(x_k)\leqslant f(x+\varepsilon\mathbf{1})=f(x)+\varepsilon.$$

These inequalities imply the continuity of f at x.

Now, we recall some definitions from abstract convex analysis. Consider a set X and a set $\mathscr{H}$ of functions $h:X\longrightarrow\bar{\mathbb{R}}$. The function $f:X\longrightarrow\bar{\mathbb{R}}$ is called *abstract convex* with respect to $\mathscr{H}$ (or $\mathscr{H}$*-convex*) if there exists a subset U of $\mathscr{H}$ such that

$$f(x)=\sup_{h\in U} h(x), \quad (x\in X).$$

The set $\mathscr{H}$ is called the set of *elementary* functions. Consider the coupling function $\varphi:\mathbb{R}^n\times\mathbb{R}^n\longrightarrow\mathbb{R}$ defined by

$$\varphi(x,y)=\min_{1\leqslant i\leqslant n}(x_i+y_i), \quad (x,\ y\in\mathbb{R}^n). \tag{5}$$

It is shown in [15] that the function $f:\mathbb{R}^n\longrightarrow\bar{\mathbb{R}}$ is topical if and only if

$$f(x)\geqslant\varphi(x,-y)+f(y), \quad \forall x,\ y\in\mathbb{R}^n. \tag{6}$$

Formula (6) implies the following result.

PROPOSITION 1. Let f be an arbitrary function defined on $\mathbb{R}^n$ and $\Delta\subset\mathbb{R}^n$. Then the function

$$f_\Delta(x):=\sup_{y\in\Delta}(f(y)+\varphi(x,-y)), \quad (x\in\mathbb{R}^n)$$

is topical. Moreover if f is topical then:
(i) $f_\Delta(x)\leqslant f(x)$ for all $x\in\mathbb{R}^n$.
(ii) $f_\Delta(x)=f(x)$ for all $x\in\Delta$.

In the rest of this section and Section 3, we denote by φ_y the function defined on $\mathbb{R}^n$ by the formula $\varphi_y(x):=\varphi(x,y)$. Let $X_\varphi=\{\varphi_y : y\in\mathbb{R}^n\}$, then it is known that any function f defined on $\mathbb{R}^n$ is topical if and only if f is X_φ-convex, see [15].

DEFINITION 2. Let $D \subset \mathbb{R}^n$. A function $f : D \longrightarrow \bar{R}$ is called topical on D if there exists a topical function F defined on $\mathbb{R}^n$ such that $F(x) = f(x)$ for all $x \in D$.

PROPOSITION 2. Let f be a function defined on the set $D \subseteq \mathbb{R}^n$. The following assertions are equivalent:

(i) f is topical on D.

(ii) $\varphi(x,-y) + f(y) \leqslant f(x)$, for all $x,\ y \in D$.

(iii) f is abstract convex with respect to the set of functions $\varphi_{-y} + c : D \longrightarrow \mathbb{R}$ with $y \in D$ and $c \in \mathbb{R}$.

Proof. $(i) \Rightarrow (ii)$ Since f is topical on D, then there exists a topical function $F : \mathbb{R}^n \longrightarrow \bar{\mathbb{R}}$ such that $F(x) = f(x)$ for all $x \in D$. From Proposition 1, the function

$$F_D(x) = \sup_{y \in D}(\varphi_{-y}(x) + F(y)), \quad (x \in \mathbb{R}^n)$$

is topical and $F_D(x) = F(x)$ for all $x \in D$. It follows that

$$\sup_{y \in D}(\varphi_{-y}(x) + F(y)) = f(x)$$

for all $x \in D$. Therefore

$$\varphi(x,-y) + f(y) \leqslant f(x), \quad \forall x,\ y \in D.$$

$(ii) \Rightarrow (iii)$ Consider the function f_D defined on D as follows

$$f_D(x) = \sup_{y \in D}(\varphi_{-y}(x) + f(y)), \quad (x \in D).$$

It is clear that f_D is abstract convex with respect to the set $\{\varphi_{-y} + c : y \in D,\ c \in \mathbb{R}\}$. It follows from (ii) that for all $x \in D$

$$f_D(x) \leqslant f(x) = f(x) + \varphi(x,-x) \leqslant \sup_{y \in D}(\varphi_{-y}(x) + f(y)) = f_D(x).$$

So, $f_D(x) = f(x)$ for all $x \in D$ and we have the desired statement (iii).

$(iii) \Rightarrow (i)$ We have that there exists a set $\Delta \subset D \times \mathbb{R}$ such that

$$f(x) = \sup_{(y,c) \in \Delta}(\varphi_{-y}(x) + c), \quad (x \in D).$$

Now, consider the function F defined on $\mathbb{R}^n$ as follows

$$F(x) := \sup_{(y,c) \in \Delta}(\varphi_{-y}(x) + c), \quad (x \in \mathbb{R}^n).$$

Since the function $\varphi_{-y} + c$ is topical for all $y \in D$ and $c \in \mathbb{R}$, then F is topical and $F(x) = f(x)$ for all $x \in D$. Hence, f is topical on D. $\square$

3. Hermite-Hadamard type inequalities

Let $D\subset\mathbb{R}^n$ be a closed domain, that is, D is a bounded set such that $\mathrm{cl}(\mathrm{int}D)=D$. Let $Q(D)$ be the set of all points $\bar{x}\in D$ such that

$$\frac{1}{A(D)}\int_D \varphi_{-\bar{x}}(x)\,dx=1,$$

where $A(D)=\int_D dx$.

PROPOSITION 3. The set $Q(D)$ is compact.

Proof. Since D is compact, we only prove that $Q(D)$ is closed. Let $\{\bar{x}_n\}\subset Q(D)$ and $\bar{x}_n\longrightarrow\bar{x}$. Since $\varphi_x(x_n)=\varphi_{x_n}(x)$, φ_x is continuous and D is compact, the sequence $\{\varphi_{-\bar{x}_n}\}$ converges uniformly to $\varphi_{-\bar{x}}$ on D. On the other hand, $\frac{1}{A(D)}\int_D\varphi_{-\bar{x}_n}(x)\,dx=1$ for all $n\geqslant 1$, Therefore, $\frac{1}{A(D)}\int_D\varphi_{-\bar{x}}(x)\,dx=1$. Hence, $\bar{x}\in Q(D)$. □

PROPOSITION 4. Let $Q(D)$ be nonempty and f be a topical function defined on D. Then the following inequality holds:

$$\sup_{\bar{x}\in Q(D)} f(\bar{x})\leqslant\frac{1}{A(D)}\int_D f(x)\,dx-1. \tag{7}$$

Proof. Since f is topical on D, it follows from Proposition 2 that

$$\varphi_{-\bar{x}}(x)+f(\bar{x})\leqslant f(x),\quad \forall x,\ y\in D.$$

Let $\bar{x}\in Q(D)$. It follows from the definition of $Q(D)$ that

$$A(D)(1+f(\bar{x}))=\int_D(\varphi_{-\bar{x}}(x)+f(\bar{x}))\,dx\leqslant\int_D f(x)\,dx.$$

Therefore

$$f(\bar{x})\leqslant\frac{1}{A(D)}\int_D f(x)\,dx-1.$$

This completes the proof. □

Since any topical function is continuous and $Q(D)$ is compact, it follows that the supremum in (7) is attained.

REMARK 2. For each $\bar{x}\in Q(D)$ we have the following inequality:

$$f(\bar{x})\leqslant\frac{1}{A(D)}\int_D f(x)\,dx-1. \tag{8}$$

Note that the inequality (8) is sharp. For example, if $f(x)=\varphi_{-\bar{x}}(x)$, then (8) becomes an equality.

Now, we consider generalization of the inequality from the right-hand side of (1.1). Let f be a topical function defined on a closed domain $D \subseteq \mathbb{R}^n$. By Proposition 2.2, we have $\varphi(y,-x)+f(x) \leqslant f(y)$ for all $x,\ y \in D$. So

$$f(x) \leqslant f(y) + \max_{1\leqslant i\leqslant n}(x_i - y_i) \quad \forall x,\ y \in D. \tag{9}$$

Now, let $y \in D$ be a minimal element of the set D (note that the point $y \in D$ is called a *minimal* point of the set D, if $x \in D$ and $x \leqslant y$ implies that $x = y$). It follows from (9) that

$$\int_D f(x)\,dx \leqslant f(y)A(D) + \int_D \max_{1\leqslant i\leqslant n}(x_i - y_i)\,dx. \tag{10}$$

In the rest of this section, we describe the set $Q(D)$, where D is a subset of $\mathbb{R}^2$. Let $D \subset \mathbb{R}^2$ be a closed domain. We begin with points $(\bar{x},\bar{y}) \in Q(D)$, which does not belong to the interior of D. Let $t \in \mathbb{R}$ be a number such that

$$D \subset \{(x,y) \in \mathbb{R}^2 : y - x \leqslant t\}.$$

D is a subset of $\mathbb{R}^2$ contained in a half-space defined by the line $R_t = \{(x,y) \in \mathbb{R}^2 : y - x = t\}$. Assume that $D \cap R_t \neq \emptyset$. We are looking for a point $(\bar{x},\bar{y}) \in R_t$ that belongs to $Q(D)$, that is,

$$\frac{1}{A(D)}\int_D \min(x-\bar{x}, y-\bar{y})\,dxdy = 1, \tag{11}$$

where $A(D)$ is the area of D.

Since $\bar{y} - \bar{x} = t$ and $y - x \leqslant t$ for all $(x,y) \in D$, one has

$$\min(x-\bar{x}, y-\bar{y}) = y - \bar{y}.$$

Let

$$Y_D = \frac{1}{A(D)}\int_D y\,dxdy. \tag{12}$$

Then

$$\frac{1}{A(D)}\int_D \min(x-\bar{x}, y-\bar{y})\,dxdy = \frac{1}{A(D)}\int_D (y-\bar{y})\,dxdy = \frac{1}{A(D)}\int_D y\,dxdy - \bar{y}.$$

Thus (11) holds if $\bar{y} = Y_D - 1$. Since $(\bar{x},\bar{y}) \in R_t$, then we have $\bar{x} = Y_D - t - 1$.

We have proved the following result.

PROPOSITION 5. Let $D \subset \mathbb{R}^2$ be a closed domain and $t \in \mathbb{R}$ such that

$$D \subset \{(x,y) \in \mathbb{R}^2 : y - x \leqslant t\}.$$

Assume that $(Y_D - t - 1, Y_D - 1) \in D$. Then $(Y_D - t - 1, Y_D - 1) \in Q(D)$.

The following result is similar to Proposition 5. So we omit the proof.

PROPOSITION 6. Let $D \subset \mathbb{R}^2$ be a closed domain and $u \in \mathbb{R}$ such that

$$D \subset \{(x,y) \in \mathbb{R}^2 : y - x \geqslant u\}.$$

Let

$$X_D = \frac{1}{A(D)} \int_D x\,dxdy. \tag{13}$$

Assume that $(X_D - 1, X_D + u - 1) \in D$. Then $(X_D - 1, X_D + u - 1) \in Q(D)$.

We now describe points $(\bar{x}, \bar{y}) \in (intD) \cap Q(D)$. First we need some notations. Let $(\bar{x}, \bar{y}) \in intD$ and $\bar{y} - \bar{x} = t$. Consider the line $R_t = \{(x,y) \in \mathbb{R}^2 : y - x = t\}$. This line intersects $intD$ and divides D into two parts D_1 and D_2, which are located in different half-planes defined by the line R_t. We have $D = D_1 \cup D_2$ and $(intD_1) \cap (intD_1) = \emptyset$ ($intD_i \neq \emptyset$ for $i = 1,2$).

Let Y_{D_1} be the number defined by (12) for the domain D_1, X_{D_2} be the number defined by (13) for the domain D_2 and $\alpha = \frac{A(D_1)}{A(D)}$. It is clear that $0 < \alpha < 1$ and $1 - \alpha = \frac{A(D_2)}{A(D)}$.

THEOREM 1. *Let $(\bar{x}, \bar{y}) \in intD$ and $\bar{y} - \bar{x} = t$. Then $(\bar{x}, \bar{y}) \in Q(D)$ if and only if*

$$\bar{x} = \alpha(Y_{D_1} - t) + (1 - \alpha)X_{D_2} - 1, \quad \bar{y} = \bar{x} + t. \tag{14}$$

Proof. We have

$$\min(x - \bar{x}, y - \bar{y}) = y - \bar{y}, \quad \forall (x,y) \in D_1,$$

and

$$\min(x - \bar{x}, y - \bar{y}) = x - \bar{x}, \quad \forall (x,y) \in D_2.$$

So

$$\begin{aligned}
\frac{1}{A(D)} \int_D \min(x - \bar{x}, y - \bar{y})\,dxdy &= \frac{1}{A(D)} \Big(\int_{D_1} (y - \bar{y})\,dxdy + \int_{D_2} (x - \bar{x})\,dxdy \Big) \\
&= \frac{1}{A(D)} \Big(\int_{D_1} y\,dxdy - A(D_1)\bar{y} + \int_{D_2} x\,dxdy - A(D_2)\bar{x} \Big) \\
&= \frac{1}{A(D)} \big(A(D_1)(Y_{D_1} - \bar{y}) + (A(D_2)(X_{D_2} - \bar{x}) \big) \\
&= \alpha(Y_{D_1} - \bar{x} - t) + (1 - \alpha)(X_{D_2} - \bar{x}) \\
&= \alpha(Y_{D_1} - t) + (1 - \alpha)X_{D_2} - \bar{x}.
\end{aligned}$$

Assume that $(\bar{x}, \bar{y}) \in Q(D)$. Then $\alpha(Y_{D_1} - t) + (1 - \alpha)X_{D_2} - \bar{x} = 1$. Hence (14) holds. On the other hand, if (14) holds then

$$\frac{1}{A(D)} \int_D \min(x - \bar{x}, y - \bar{y})\,dxdy = 1,$$

whence $(\bar{x}, \bar{y}) \in Q(D)$. $\square$

4. Examples

To illustrate the results obtained in Section 3, we present some examples in the sequel.

EXAMPLE 4. Let $D \subset \mathbb{R}^2$ be the triangle with vertices $(a,0)$, $(a+\delta,0)$ and $(a+\delta,\delta)$, that is

$$D = \{(x,y) \in \mathbb{R}^2 : a \leqslant x \leqslant a+\delta,\ 0 \leqslant y \leqslant x-a\},$$

where $\delta \geqslant 3$. So

$$D \subset \{(x,y) \in \mathbb{R}^2 : y \leqslant x-a\}.$$

We are looking for a point $(\bar{x},\bar{y}) \in Q(D)$ that lies on the side of D with endpoints $(a,0)$ and $(a+\delta,\delta)$. To do this, we need to calculate Y_D, using Proposition 6. It is clear that $A(D) = \frac{\delta^2}{2}$. We have

$$Y_D = \frac{1}{A(D)} \int_D y\,dxdy = \frac{2}{\delta^2} \int_a^{a+\delta} (\int_0^{x-a} y\,dy)dx = \frac{1}{3}\delta.$$

Thus, $\bar{y} = Y_D - 1 = \frac{1}{3}\delta - 1$ and $\bar{x} = Y_D + a - 1 = \frac{1}{3}\delta + a - 1$. Therefore, if $\delta \geqslant 3$, then $(\bar{x},\bar{y}) \in D$.

It follows from Remark 2 that the following inequality holds for each topical function f.

$$f(\frac{1}{3}\delta + a - 1, \frac{1}{3}\delta - 1) \leqslant \frac{2}{\delta^2} \int_D f(x,y)\,dxdy - 1,$$

and so (since f is topical)

$$f\Big(\frac{1}{3}\delta + a, \frac{1}{3}\delta\Big) \leqslant \frac{2}{\delta^2} \int_D f(x,y)\,dxdy.$$

On the other hand, $(a,0)$ is the unique minimal point of the set D. Since the function f is increasing, $f(a,0) \leqslant f(x,y)$ for all $(x,y) \in D$. By (10), one concludes that

$$\int_D f(x,y)\,dxdy \leqslant \frac{\delta^2}{2} f(a,0) + \frac{\delta^3}{3}.$$

EXAMPLE 5. Consider the square in $\mathbb{R}^2$ formed by the points $(-a,0)$, $(0,a)$, $(a,0)$ and $(0,-a)$ as its vertices, which we denote by D (we assume that $a > 4$). Consider the line $R_t = \{(x,y) \in \mathbb{R}^2 : y - x = t\}$ that $|\,x\,| < \sqrt{a^2 - 4a}$ passing through the interior of D. This line divides D into two parts. Let the down-side part of $R_t \cap D$ be denoted by D_1 and the up-side part of $R_t \cap D$ be denoted by D_2 We are looking for a point $(\bar{x},\bar{y}) \in (intD) \cap Q(D)$. According to Proposition 1, we need to calculate Y_{D_1} and X_{D_2}. It is clear that $A(D) = 2a^2$, $A(D_1) = a(a+t)$ and $A(D_2) = a(a-t)$. We have

$$Y_{D_1} = \frac{1}{A(D_1)} \int_{D_1} y\,dxdy = \frac{1}{4}(t-a)$$

and

$$X_{D_2} = \frac{1}{A(D_2)} \int_{D_2} x\,dxdy = \frac{-1}{4}(t+a).$$

Then we get $(\bar{x}, \bar{y})$ as follows

$$\begin{aligned}
\bar{x} &= \frac{A(D_1)}{A(D)}\left(Y_{D_1} - t\right) + \frac{A(D_2)}{A(D)} X_{D_2} - 1 \\
&= \frac{a(a+t)}{2a^2}\left(\frac{1}{4}(t-a) - t\right) + \frac{a(a-t)}{2a^2}\left(\frac{-1}{4}(t+a)\right) - 1 \\
&= \frac{-1}{4a}(t+a)^2 - 1.
\end{aligned}$$

and $\bar{y} = \bar{x} + t = \frac{-1}{4a}(t+a)^2 + t - 1$. Note that $(\bar{x}, \bar{y}) \in intD$. This is equivalent to $\frac{-1}{2}(a+t) < \bar{x} < \frac{1}{2}(a-t)$. It is easy to see that if $|\bar{x}| < \sqrt{a^2 - 4a}$, then $(\bar{x}, \bar{y}) \in intD$.

Now, let (x', y') be a minimal point of D. It is clear that $x',\ y' \leqslant 0$ and $x' + y' = -a$. A simple calculation show that, (10) implies the following inequality.

$$\int_D f(x,y)\,dxdy \leqslant 2a^2 f(x', y') + 2a^3 - 2ax'y'.$$

5. Topical cost functions in mass transportation problems

In this section we are going to study Kantorovich duality whenever the cost function is a specific topical function defined (analogously) by the formula (5). More precisely we assume that

$$c(x,y) := -\phi(x,-y) := \max_{1 \leqslant i \leqslant n} (-x_i + y_i). \tag{15}$$

We denote the set of all topical function $f : \mathbb{R}^n \to \mathbb{R}$ by Ξ; and the abstract conjugate function of f with respect to $-c$ is denoted by f^{-c} ([18]), which is

$$f^{-c}(y) := \sup_{x \in \mathbb{R}^n} [-c(x,y) - f(x)].$$

THEOREM 2. *Let μ and ν be two positive finite Borel measures on $\mathbb{R}^n$. Assume that $\|.\|$ is a norm defined on $\mathbb{R}^n$ such that $\|.\| \in L^1(\mu) \cap L^1(\nu)$. Then*

$$\begin{aligned}
&\min_{\pi \in \Pi(\mu,\nu)} \int_{\mathbb{R}^n \times \mathbb{R}^n} \max_{1 \leqslant i \leqslant n} (-x_i + y_i) d\pi(x,y) \\
&\qquad = \sup_{f \in \Xi \cap L^1(\mu) \cap L^1(\nu)} \left(\int_{\mathbb{R}^n} f(y) d\nu(y) - \int_{\mathbb{R}^n} f(x) d\mu(x)\right).
\end{aligned}$$

Proof. Let $f_0(x) := -\max_{1 \leqslant i \leqslant n} x_i$ and $g_0(x) := \min_{1 \leqslant i \leqslant n} x_i$, for all $x \in \mathbb{R}^n$. Since $\|.\| \in L^1(\mu) \cap L^1(\nu)$, $f_0, g_0 \in L^1(\mu) \cap L^1(\nu)$. Clearly, $f_0(x) + g_0(x) \leqslant \max_{1 \leqslant i \leqslant n}(-x_i +$

y_i). Therefore, according to Kantorovich duality, the equality of (4) fulfills. Now we are going to show that if $g(y)-f(x)\leqslant c(x,y)$ for all $x,y\in\mathbb{R}^n$, then

$$g(y)-f(x)\leqslant -f^{-c}(y)-f(x)\leqslant -f^{-c}(y)-f^{(-c)(-c)}(x)\leqslant c(x,y). \qquad (16)$$

Since $-c(x,y)-f(x)\leqslant -g(y)$ for all $x,y\in\mathbb{R}^n$, $g(y)\leqslant -f^{-c}(y)$ for all $y\in\mathbb{R}^n$. On the other hand it is easy to see that $f^{(-c)(-c)}(x)\leqslant f(x)$ for all $x\in\mathbb{R}^n$. Indeed,

$$f^{(-c)(-c)}(x)=\sup_z\inf_w[\phi(x,-z)-\phi(w,-z)+f(w)]\leqslant f(x).$$

This shows that the inequalities of (16) hold. We need also to prove that $f^{(-c)(-c)}$ is topical and $f^{-c}(y)=-f(y)$. Since $f^{(-c)(-c)}(x)=\sup_{y\in\mathbb{R}^n}\phi(x,-y)-f^{-c}(y)$, it follows from Proposition 2 that $f^{(-c)(-c)}$ is topical. Now assume that f is a topical function and $\mathbf{1}$ is a vector whose components are all 1. Since $\phi(x,-y)\mathbf{1}\leqslant x-y$ for all $x,y\in\mathbb{R}^n$, one has

$$f(y)\leqslant f(x-\phi(x,-y)\mathbf{1})=f(x)-\phi(x,-y).$$

This implies that

$$f^{-c}(y)=\sup_{x\in\mathbb{R}^n}-c(x,y)-f(x)\leqslant -f(y).$$

On the other hand $-f(y)=-c(y,y)-f(y)\leqslant f^{-c}(y)$. Therefore, $-f^{-c}(y)=f(y)$. Moreover,

$$f^{(-c)(-c)(-c)}(y)=\sup_x\inf_w\sup_z[\phi(x,-y)-\phi(x,-w)+\phi(z,-w)-f(z)].$$

Letting $w:=y$, one has $f^{(-c)(-c)(-c)}(y)\leqslant f^{-c}(y)$, while letting $z:=x$, one has

$$f^{(-c)(-c)(-c)}(y)\geqslant f^{-c}(y).$$

This completes the proof.

COROLLARY 1. *Suppose that $\mu=\nu$ and $\|.\|\in L^1(\mu)\cap L^1(\nu)$ for some norm $\|.\|$ defined on $\mathbb{R}^n$. Then the optimal plan $\pi\in\Pi(\mu,\nu)$ is concentrated on the graph of the identity mapping $I:\mathbb{R}^n\to\mathbb{R}^n$ defined by $I(x):=x$ for all $x\in\mathbb{R}^n$. In this case*

$$\min_{\pi\in\Pi(\mu,\nu)}\int_{\mathbb{R}^n\times\mathbb{R}^n}\max_{1\leqslant i\leqslant n}(-x_i+y_i)d\pi(x,y)=0.$$

It is worthy saying that since μ and ν are finite measures on $\mathbb{R}^n$, their supports are both σ-compact. In addition, if the supports of μ and ν are compact, then the assumption $\|.\|\in L^1(\mu)\cap L^1(\nu)$ is redundant and may be eliminated from both Theorem 2 and Corollary 1.

In the sequel we present a perspective on establishing of pricing functions. Let $M:\mathbb{R}^n\rightrightarrows\mathbb{R}^n$ be a set valued mapping and $\emptyset\neq S\subseteq\mathrm{dom}(M):=\{x\in R^n|M(x)\neq\emptyset\}$. Assume that we have an optimal plan π concentrated on the $\mathrm{graph}(M):=\{(x,y)\in$

$\mathbb{R}^n \times \mathbb{R}^n | y \in M(x)\}$. Then it is well-known that M must be $-c$-cyclically monotone [20]. Indeed, let (f,g) be the the pair of pricing functions solving the right-hand side of (4). Assume that there are some $x,y \in \mathbb{R}^n$ for which $g(y) - f(x) < c(x,y)$. Then, if a nonzero mass is transported from x to y, the equality of (4) does not hold. Therefore, π is concentrated on the set

$$\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : g(y) - f(x) = c(x,y)\}.$$

Using a similar argument of the proof of Theorem 2, one could replace the function g by $-f^{-c}$. On the other hand $\partial_{-c} f$ defined by

$$\begin{aligned}\partial_{-c} f(x) &:= \{y \in \mathbb{R}^n | f^{-c}(y) + f(x) = -c(x,y)\} \\ &= \{y \in \mathbb{R}^n | c(x,y) - c(z,y) \leqslant f(z) - f(x),\ \forall z \in \mathbb{R}^n\}\end{aligned}$$

is $-c$-cyclically monotone. Since $\mathrm{graph}(M) \subseteq \mathrm{graph}(\partial_{-c} f)$, $\mathrm{graph}(M)$ is $-c$-cyclically monotone.

Now assume that we already have a pricing function f which solves the right-hand side of the equality (4) and an optimal plan π concentrated on a $-c$-cyclically monotone operator M is at hand. Suppose that we are going to adjust a new pricing function in such a way that the new function is $-c$-convex and coincides with the old pricing function f over a nonempty subset S of $\mathrm{dom}(M)$. The set of all such functions is denoted by $\mathscr{A}_{[-c,f|_S,S]}$. Recall that a function $h : \mathbb{R}^n \to (-\infty, +\infty]$ is called $-c$-convex, if there exist a function $g : \mathbb{R}^n \to (-\infty, +\infty]$ such that $h(x) = g^{-c}(x)$ for all $x \in \mathbb{R}^n$. We denote by $\mathscr{H}_{-c}$ the set of all $-c$-convex functions. Therefore,

$$\mathscr{A}_{[-c,f|_S,S]} := \{h \in \mathscr{H}_{-c} : \mathrm{graph}(M) \subset \mathrm{graph}(\partial_{-c} h),\ h|_S = f|_S\}.$$

The set $\mathscr{A}_{[-c,f|_S,S]}$ is well-studied in [2, 3]. Concerning the aforementioned pricing argument, a natural task is to seek the infimum and supremum of $\mathscr{A}$. Let

$$\alpha_{[-c,f|_S,M]}(x) := \inf_{h \in \mathscr{A}_{[-c,f|_S,M]}} h(x), \quad \gamma_{[-c,f|_S,M]}(x) := \sup_{h \in \mathscr{A}_{[-c,f|_S,M]}} h(x). \tag{17}$$

Taking in to account the above argument, $\alpha_{[-c,f|_S,M]}$ is the best new pricing function for the end customers while $\gamma_{[-c,f|_S,M]}$ is the best one for the producer in such a way that both of them coincide with the old pricing function f over S. We are going to characterize functions $\alpha_{[-c,f|_S,M]}$ and $\gamma_{[-c,f|_S,M]}$ whenever the cost function is defined by (15). To do this, analogously to Proposition 1, we present the following proposition.

PROPOSITION 7. Let f be an arbitrary function defined on $\mathbb{R}^n$ and $\Delta \subset \mathbb{R}^n$. Then the function

$$f^{\Delta}(x) := \inf_{y \in \Delta} (f(y) - \varphi(-x,y)), \quad (x \in \mathbb{R}^n)$$

is topical. Moreover if f is topical then:

(i) $f^{\Delta}(x) \geqslant f(x)$ for all $x \in \mathbb{R}^n$.
(ii) $f^{\Delta}(x) = f(x)$ for all $x \in \Delta$.

Proof. The results follow from the fact that

$$y - \phi(-x,y)\mathbf{1} \geqslant x, \quad \forall x,y \in \mathbb{R}^n,$$

where $\mathbf{1} \in \mathbb{R}^n$ is a vector whose components are all 1. $\square$

The following observation adjusts the new pricing function whenever the old pricing function is topical. For the set valued mapping M, we mean by $\mathrm{Im}(M)$, the well-known image space of M, i.e.

$$\mathrm{Im}(M) = \{y \in \mathbb{R}^n : (x,y) \in \mathrm{graph}(M) \text{ for some } x \in \mathrm{dom}(M)\}.$$

THEOREM 3. *Let $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set valued mapping and $\emptyset \neq S \subseteq \mathrm{dom}(M)$. Assume that $c(x,y) = -\phi(x,-y)$ and $f : \mathrm{dom}(M) \to (-\infty,+\infty]$ is a topical function on* $\mathrm{dom}(M)$ *such that* $\mathrm{graph}(M) \subseteq \mathrm{graph}(\partial_{-c} f)$. *Let $\alpha_{[-c,f|_S,M]}$ and $\gamma_{[-c,f|_S,M]}$ be defined by* (17). *Then:*

(i) $\alpha_{[-c,f|_{\mathrm{dom}(M)},M]}(x) = f_{\mathrm{Im}(M)}(x)$ *for all $x \in \mathbb{R}^n$.*

(ii) $\alpha_{[-c,f|_S,I_S]}(x) = f_S(x)$ *and* $\gamma_{[-c,f|_S,I_S]}(x) = f^S(x)$ *for all $x \in \mathbb{R}^n$, where I_S is the identity mapping defined on S.*

Proof. As seen from the proof of Theorem 2, $f^{-c}(y) = -f(y)$ for all $y \in \mathbb{R}^n$. Thus

$$\partial_{-c} f(x) = \{y \in \mathbb{R}^n : f(x) = \phi(x,-y) + f(y)\}.$$

(i): First we show that $f_{\mathrm{Im}(M)} \in \mathscr{A}_{[-c,f|_{\mathrm{dom}(M)},M]}$. Applying Proposition 1, $f(x) \geqslant f_{\mathrm{Im}(M)}(x)$ for all $x \in \mathbb{R}^n$. Let $x \in \mathrm{dom}(M)$. Thus there is $y \in \mathrm{Im}(M)$ such that $(x,y) \in \mathrm{graph}(\partial_{-c} f)$. Therefore, $f_{\mathrm{Im}(M)}(x) \geqslant f(y) + \phi(x,-y) = f(x)$. So, $f_{\mathrm{Im}(M)}(x) = f(x)$ for all $x \in \mathrm{dom}(M)$.

Now assume that $(x_0,y_0) \in \mathrm{graph}(M)$ and $x \in \mathbb{R}^n$. Then

$$\begin{aligned} f_{\mathrm{Im}(M)}(x_0) + \phi(x,-y_0) - \phi(x_0,-y_0) &= f(x_0) + \phi(x,-y_0) - \phi(x_0,-y_0) \\ &= f(y_0) + \phi(x,-y_0) \\ &\leqslant f_{\mathrm{Im}(M)}(x). \end{aligned}$$

This implies that $(x_0,y_0) \in \mathrm{graph}(\partial_{-c} f_{\mathrm{Im}(M)})$. Therefore, $f_{\mathrm{Im}(M)} \in \mathscr{A}_{[-c,f|_{\mathrm{dom}(M)},M]}$. To complete the proof, let $h \in \mathscr{A}_{[-c,f|_{\mathrm{dom}(M)},M]}$ and $x \in \mathbb{R}^n$ be arbitrary. Then one has

$$\begin{aligned} f_{\mathrm{Im}(M)}(x) &= \sup_{y_0 \in \mathrm{Im}(M)} f(y_0) + \phi(x,-y_0) \\ &= \sup_{(x_0,y_0) \in \mathrm{graph}(M)} f(x_0) - \phi(x_0,-y_0) + \phi(x,-y_0) \\ &= \sup_{(x_0,y_0) \in \mathrm{graph}(M)} h(x_0) - \phi(x_0,-y_0) + \phi(x,-y_0) \\ &\leqslant h(x), \end{aligned}$$

which the equalities come from the facts that $(x_0, y_0) \in \operatorname{graph} \partial_{-c} h$ and $h(x_0) = f(x_0)$. This completes the proof of (i).

(ii): The equality $\alpha_{[-c, f|_S, I_S]}(x) = f_S(x)$ is an immediate consequence of (i). Therefore we only prove the second equality. According to Proposition 7 part (ii), $f^S(x) = f(x)$ for all $x \in S$. Let $x \in S$. Using again Proposition 7 part (i), one has for all $w \in \mathbb{R}^n$

$$f^S(x) + \phi(w, -x) - \phi(x, -x) = f(x) + \phi(w, -x) = f(w) \leqslant f^S(w).$$

This implies that $(x, x) \in \operatorname{graph}(\partial_{-c} f^S)$ for all $x \in S$. Hence $f^S \in \mathscr{A}_{[-c, f|_S, I_S]}$. Assume now that $h \in \mathscr{A}_{[-c, f|_S, I_S]}$. Applying the fact that $h \in \mathscr{H}_{-c}$ and Proposition 2, h is topical. Therefore for all $w \in \mathbb{R}^n$

$$f^S(w) = \inf_{y \in S}(f(y) - \varphi(-w, y)) = \inf_{y \in S}(h(y) - \varphi(-w, y)) = h^S(w) \geqslant h(w).$$

Hence the proof is complete. $\square$

Acknowledgements. The authors are deeply grateful of the anonymous referee for his valuable remarks. Especially his comment regarding Kantorovich duality of optimal mass transportation problem led authors to add the section 5 into the present version of the paper.

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(Received March 21, 2017)

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