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REARRANGEMENTS OF GENERAL MONOTONE FUNCTIONS AND OF THEIR FOURIER TRANSFORMS

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Abstract. We extend further Boas' conjecture concerning functions nonincreasing on $(0,\infty)$ and their Fourier transforms by considering rearrangements of general monotone functions and of their Fourier transforms. These results are similar those of Sagher in proving Boas' conjecture, and follow up on recent work of Liflyand and Tikhonov in this area.

1. Notation

Let μ denote the Lebesgue measure. Let $E \subset \mathbf{C}$ be measurable. For $f(x)$, $\omega(x)$ measurable on E, with $\omega(x) > 0$ for $x \in E$, for $0 < q < \infty$, we denote

$$\|f\|_{L^q_\omega(E)} = \left(\int_E (\omega(x)|f(x)|)^q \, dx\right)^{\frac{1}{q}}$$

and for $q = \infty$, we denote

$$\|f\|_{L^\infty_\omega(E)} = \operatorname*{ess\,sup}_{x\in E} \omega(x)|f(x)|.$$

For $1 \leqslant q \leqslant \infty$, these define norms. For $0 < q < 1$, these define quasinorms; by abuse of language, we refer to them as norms. The weighted L^q-spaces are defined for $0 < q \leqslant \infty$:

$$L^q_\omega(E) = \{f : \|f\|_{L^q_\omega(E)} < \infty\}.$$

For $E \subset (0,\infty)$, we are particularly interested in

$$\omega(x) = x^{\frac{1}{p}-\frac{1}{q}}$$

and with this weight, we denote for $0 < p < \infty$, $0 < q < \infty$:

$$\|f\|_{L^q_{\omega(p,q)}(E)} = \left(\int_E \left(x^{\frac{1}{p}-\frac{1}{q}}|f(x)|\right)^q dx\right)^{\frac{1}{q}} = \left(\int_E \left(x^{\frac{1}{p}}|f(x)|\right)^q \frac{dx}{x}\right)^{\frac{1}{q}}$$

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while for $0<p<\infty$, $q=\infty$, we denote

$$\|f\|_{L^{\infty}_{\omega(p,\infty)}(E)}=\operatorname*{ess\,sup}_{x\in E} x^{\frac{1}{p}}|f(x)|.$$

For $p=q=\infty$, we let

$$\|f\|_{L^{\infty}_{\omega(\infty,\infty)}(E)}$$

denote the usual L^{∞}-norm of f on E. The $L^{q}_{\omega(p,q)}$-spaces are defined for $0<p<\infty$, $0<q\leqslant\infty$, or $p=q=\infty$:

$$L^{q}_{\omega(p,q)}(E)=\{f:\|f\|_{L^{q}_{\omega(p,q)}(E)}<\infty\}.$$

For a function f measurable and finite a.e. on $E\subset\mathbf{C}$, let f^* denote the nonincreasing rearrangement of $|f|$; that is, f^* is nonincreasing on $(0,\mu(E))$, and for all $\alpha>0$,

$$\mu\{f^*>\alpha\}=\mu\{|f|>\alpha\}.$$

Denote for $0<p<\infty$, $0<q<\infty$:

$$\|f\|_{L(p,q)(E)}=\left(\int_0^{\mu(E)}\left(x^{\frac{1}{p}-\frac{1}{q}}f^*(x)\right)^q dx\right)^{\frac{1}{q}}=\left(\int_0^{\mu(E)}\left(x^{\frac{1}{p}}f^*(x)\right)^q\frac{dx}{x}\right)^{\frac{1}{q}}$$

and for $0<p<\infty$, $q=\infty$:

$$\|f\|_{L(p,\infty)(E)}=\operatorname*{ess\,sup}_{0<x<\mu(E)} x^{\frac{1}{p}}f^*(x).$$

In fact, it can be shown that

$$\|f\|_{L(p,\infty)(E)}=\sup_{0<x<\mu(E)} x^{\frac{1}{p}}f^*(x).$$

For $p=q=\infty$, we let

$$\|f\|_{L(\infty,\infty)(E)}$$

denote the usual L^{∞}-norm of f on E. For $0<p<\infty$, $0<q\leqslant\infty$, these define quasi-norms; by abuse of language, we refer to them as norms. The Lorentz spaces $L(p,q)$ are defined for $0<p<\infty$, $0<q\leqslant\infty$, or $p=q=\infty$:

$$L(p,q)(E)=\{f:\|f\|_{L(p,q)(E)}<\infty\}.$$

In the sequel, for all spaces and norms above, if E does not appear in the notation, then it is assumed that $E=(0,\infty)$.

We also have the following lemma; see p. 278 in [5]:

LEMMA 1. *Let $E\subset\mathbf{C}$ be measurable, and let f,g be measurable on E. Then*

$$\int_E|fg|\leqslant\int_0^{\mu(E)}f^*g^*.$$

As a consequence,
$$\|f\|_{L^q_{\omega(p,q)}(E)} \leqslant \|f\|_{L(p,q)(E)}$$
for $q \leqslant p$, and
$$\|f\|_{L(p,q)(E)} \leqslant \|f\|_{L^q_{\omega(p,q)}(E)}$$
for $q \geqslant p$. Of course, if $p = q$, then
$$\|f\|_{L^q_{\omega(p,q)}(E)} = \|f\|_{L(p,q)(E)}.$$

We let C denote a generic constant. Also, we shall say that two variable quantities D_1, D_2 are equivalent, and denote
$$D_1 \sim D_2$$
if there exists a constant $C > 0$, independent of the variables of D_1 and D_2, such that
$$\frac{1}{C} D_2 \leqslant D_1 \leqslant C D_2.$$

Finally, for $1 < p < \infty$, define p' so that $\frac{1}{p} + \frac{1}{p'} = 1$.

2. Introduction

Consider a function $f \in L^1(\mathbf{R})$. The odd and even parts of f may then be considered with their domains restricted to $(0,\infty)$. Thus, we define the Fourier sine transform of f as
$$\hat{f}_s(\xi) = \int_0^\infty f(t) \sin \xi t \, dt$$
and the Fourier cosine transform of f as
$$\hat{f}_c(\xi) = \int_0^\infty f(t) \cos \xi t \, dt.$$

Throughout this paper we shall let $\hat{f}(\xi)$ denote the Fourier sine or cosine transform, as appropriate.

Boas' conjecture (see [2]) can be stated using the above notation as follows:

THEOREM 1. *Let $f \in L^1$, and let $\hat{f}$ be the Fourier sine or cosine transform of f, with $f \geqslant 0$ and $f \searrow$, or $\hat{f} \geqslant 0$ and $\hat{f} \searrow$, on $(0,\infty)$. Let $1 < p < \infty$ and $1 < q < \infty$. Then $\hat{f} \in L^q_{\omega(p',q)}$ if and only if $f \in L^q_{\omega(p,q)}$.*

This was eventually proven by Sagher in [11] using interpolation theory. Specifically, he showed that for such f and $\hat{f}$, with $1 < p < \infty$ and $0 < q \leqslant \infty$,
$$\hat{f} \in L(p',q) \Leftrightarrow f \in L(p,q)$$
and if these conditions hold, then
$$\|\hat{f}\|_{L(p',q)} \sim \|f\|_{L(p,q)}.$$

He also showed using the Stein-Weiss interpolation theorem that for such f and $\hat{f}$, with $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$,

$$\hat{f} \in L^q_{\omega(p',q)} \Leftrightarrow f \in L^q_{\omega(p,q)}$$

and if these conditions hold, then

$$\|\hat{f}\|_{L^q_{\omega(p',q)}} \sim \|f\|_{L^q_{\omega(p,q)}}.$$

The general monotone sequences were introduced by Tikhonov in [12]; related sequences were considered earlier by Belov in [1]. Liflyand and Tikhonov extended the concept of general monotonicity to functions in [9] in multiple ways. In [7], [8], [9], and [10], Liflyand and Tikhonov examined extensively integrability conditions for the Fourier transforms of functions general monotone in some sense. Of particular interest was extending Boas' conjecture on the Fourier transform $\hat{f}$ of a function $f \geqslant 0$ nonincreasing on $(0,\infty)$. In [8] Liflyand and Tikhonov proved a generalization of Boas's conjecture to functions defined to be general monotone in the following way: a function f defined on $(0,\infty)$ is said to be general monotone if f is locally of bounded variation on $(0,\infty)$, $\lim_{x\to\infty} f(x) = 0$, and there exist constants $c > 1$ and C such that for $x \in (0,\infty)$,

$$\int_x^{2x} |df(t)| \leqslant C \int_{\frac{x}{c}}^{cx} |f(t)| \frac{dt}{t}. \tag{1}$$

In [10] Liflyand and Tikhonov considered weights other than just power functions on these general monotone functions and their Fourier transforms, including weights satisfying a Muckenhoupt condition. Thus, the usefulness in harmonic analysis of the general monotone functions is well-established.

In this paper, we use the definition of a general monotone function that appears in (1). We use the notation GM to denote this class of general monotone functions. In [8] Liflyand and Tikhonov described essential properties of general monotone functions. One of these is that if $f \in GM$, then the following two conditions are satisfied:

1. There exist constants $c > 1$ and A such that for $x \in (0,\infty)$,

$$|f(x)| \leqslant A \int_{\frac{x}{c}}^{cx} |f(t)| \frac{dt}{t}.$$

2. There exist constants $c > 1$ and B such that for $x, y \in (0,\infty)$, with $x < y$,

$$\int_x^{y} |df(t)| \leqslant B \int_{\frac{x}{c}}^{cy} |f(t)| \frac{dt}{t}.$$

This is the property of which we take advantage to obtain the results of this paper. In the sequel, these conditions will be referred to as Condition 1 and Condition 2, and the constants obtained from assuming these conditions are satisfied will be referred to as A and c, and B and c, respectively. We shall assume throughout that any function referred to in the hypothesis of any result is not identically equal to 0; otherwise, such result becomes trivial. Such an assumption leads to $A > 0$ or $B > 0$, as appropriate.

It has been shown that Boas' conjecture applies to a class of functions broader than originally described. In [8] Liflyand and Tikhonov considered functions general monotone on $(0,\infty)$ not necessarily integrable on $(0,\infty)$, but at least integrable in a neighborhood of zero. For such a function f, the Fourier sine and cosine transforms $\hat{f}_s$ and $\hat{f}_c$ are considered in at least a distributional sense. It follows from Theorem 2.1 in [8] that if $f \geqslant 0$, $f \in GM$, $1 < p < \infty$, $1 \leqslant q < \infty$, then $\hat{f} \in L^q_{\omega(p',q)}$ if and only if $f \in L^q_{\omega(p,q)}$, and

$$\|\hat{f}\|_{L^q_{\omega(p',q)}} \sim \|f\|_{L^q_{\omega(p,q)}}.$$

We shall see that more is true by considering the nonincreasing rearrangements of both f and $\hat{f}$. We assume that for all $\xi \in (0,\infty)$, $\hat{f}(\xi)$ exists as an improper integral. Theorem 4 shows that if $f \geqslant 0$ and satisfies Condition 2, $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, and $\hat{f}$ is the Fourier sine or cosine transform of f, then $f \in L^q_{\omega(p,q)} \cup L(p,q)$ implies $\hat{f} \in L^q_{\omega(p',q)} \cap L(p',q)$,

$$\|\hat{f}\|_{L^q_{\omega(p',q)}} \leqslant C(B,c,p)\|f\|_{L^q_{\omega(p,q)}} \sim \|f\|_{L(p,q)}$$

and

$$\|\hat{f}\|_{L(p',q)} \leqslant C(B,c,p)\|f\|_{L^q_{\omega(p,q)}} \sim \|f\|_{L(p,q)}.$$

Theorem 5 addresses the reverse inequality. For this result, a lower bound on the integral of $|\hat{f}|$ near zero is needed. We obtain this from Lemma 2.3 in [10], but more conditions on f and $\hat{f}$ are required. Here $f \geqslant 0$ on $(0,\infty)$, and for $a > 0$,

$$\int_a^\infty f(t)\frac{dt}{t} < \infty.$$

Also, for all $\xi \in (0,\infty)$, $\hat{f}(\xi)$ exists as an improper integral, with the convergence uniform on every compact subinterval of $(0,\infty)$, and $\hat{f}$ locally integrable on $(0,\infty)$. Finally, for $\hat{f}_s$, we require $tf(t)$ integrable in a neighborhood of zero, and for $\hat{f}_c$, we require $f(t)$ integrable in a such a neighborhood. When f and $\hat{f}$ satisfy these conditions, we shall say they satisfy the hypotheses of Lemma 2.3 in [10]. In this case, Theorem 5 then shows that if also f satisfies Condition 1, $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, then $\hat{f} \in L^q_{\omega(p',q)}$ implies $f \in L^q_{\omega(p,q)} \cap L(p,q)$, and

$$\|f\|_{L(p,q)} \sim \|f\|_{L^q_{\omega(p,q)}} \leqslant C(A,c,p)\|\hat{f}\|_{L^q_{\omega(p',q)}};$$

in addition, $\hat{f} \in L(p',q)$ implies $f \in L^q_{\omega(p,q)} \cap L(p,q)$, and

$$\|f\|_{L(p,q)} \sim \|f\|_{L^q_{\omega(p,q)}} \leqslant C(A,c,p)\|\hat{f}\|_{L(p',q)}.$$

Relatively simple modifications of the proofs of Liflyand and Tikhonov in [8] allow us to obtain the additional results.

Combining the results above, if $f \geqslant 0$, $f \in GM$, $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, and f and $\hat{f}$ satisfy the hypotheses of Lemma 2.3 in [10], then

$$\hat{f} \in L(p',q) \Leftrightarrow \hat{f} \in L^q_{\omega(p',q)} \Leftrightarrow f \in L^q_{\omega(p,q)} \Leftrightarrow f \in L(p,q)$$

and if these conditions hold, then

$$\|\hat{f}\|_{L(p',q)} \sim \|\hat{f}\|_{L^q_{\omega(p',q)}} \sim \|f\|_{L^q_{\omega(p,q)}} \sim \|f\|_{L(p,q)}$$

thus generalizing most of the results of Sagher in [11] to general monotone functions. For $f \in L^1$, many of these results appear in the work of Jurkat and Sampson [6].

Related results involving functions general monotone on $(0,\pi)$ can be found in [4]; the duals to those results involving general monotone sequences can be found in [3].

3. Results

Hardy's inequalities can be expressed in the following forms:

THEOREM 2. *Let f be measurable on $(0,a)$, $0 < a \leqslant \infty$. Then for $\alpha > 0$, $1 \leqslant q < \infty$,*

$$\left(\int_0^a \left(x^{-\alpha} \int_0^x |f(t)| \frac{dt}{t}\right)^q \frac{dx}{x}\right)^{\frac{1}{q}} \leqslant \frac{1}{\alpha} \left(\int_0^a \left(x^{-\alpha}|f(x)|\right)^q \frac{dx}{x}\right)^{\frac{1}{q}}$$

and

$$\left(\int_0^a \left(x^{\alpha} \int_x^a |f(t)| \frac{dt}{t}\right)^q \frac{dx}{x}\right)^{\frac{1}{q}} \leqslant \frac{1}{\alpha} \left(\int_0^a \left(x^{\alpha}|f(x)|\right)^q \frac{dx}{x}\right)^{\frac{1}{q}}$$

The following theorem appears in [4] as two separate theorems.

THEOREM 3. *Let f be measurable on $(0,a)$, $0 < a \leqslant \infty$, and satisfy Condition 1 or Condition 2. For $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, or $p = q = \infty$, if $f \in L(p,q)(0,a)$, then $f \in L^q_{\omega(p,q)}(0,a)$, and*

$$\|f\|_{L^q_{\omega(p,q)}(0,a)} \leqslant C(D,c,p)\|f\|_{L(p,q)(0,a)}.$$

Likewise, for $0 < p < \infty$, $1 \leqslant q \leqslant \infty$, or $p = q = \infty$, if $f \in L^q_{\omega(p,q)}(0,a)$, then $f \in L(p,q)(0,a)$, and

$$\|f\|_{L(p,q)(0,a)} \leqslant C(D,c,p)\|f\|_{L^q_{\omega(p,q)}(0,a)}.$$

In both inequalities, $D = A$ if f satisfies Condition 1, and $D = B$ if f satisfies Condition 2.

We come to the first main result of this paper.

THEOREM 4. *Let $f \geqslant 0$ on $(0,\infty)$ and satisfy Condition 2. Let $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, and let $\hat{f}(\xi)$ be the Fourier sine or cosine transform of f, for all $\xi \in (0,\infty)$ existing as an improper integral. If $f \in L^q_{\omega(p,q)} \cup L(p,q)$, then $\hat{f} \in L^q_{\omega(p',q)} \cap L(p',q)$,*

$$\|\hat{f}\|_{L^q_{\omega(p',q)}} \leqslant C(B,c,p)\|f\|_{L^q_{\omega(p,q)}} \sim \|f\|_{L(p,q)} \tag{2}$$

and

$$\|\hat{f}\|_{L(p',q)} \leqslant C(B,c,p)\|f\|_{L^q_{\omega(p,q)}} \sim \|f\|_{L(p,q)}. \tag{3}$$

Proof. The equivalences in (2) and (3) follow from Theorem 3, so we show the remaining inequalities. We modify the proof of Theorem 2.1(A) in [8] as necessary. Assume first that $\hat{f}$ is the Fourier sine transform of f. For $x \in (0,\infty)$, $N > x$,

$$\left|\int_0^N f(t)\sin\xi t\,dt\right| \leqslant \int_0^x f(t)\,dt + \left|\int_x^N f(t)\sin\xi t\,dt\right|.$$

For $\xi \in (0,\infty)$, let $h_\xi(t) = -\frac{\cos\xi t}{\xi}$. Then

$$\left|\int_x^N f(t)\sin\xi t\,dt\right| = \left|\int_x^N f(t)h'_\xi(t)\,dt\right| = \left|\int_x^N f(t)dh_\xi(t)\right|.$$

Using integration by parts,

$$\begin{aligned}\left|\int_x^N f(t)dh_\xi(t)\right| &= \left|f(N)h_\xi(N) - f(x)h_\xi(x) - \int_x^N h_\xi(t)df(t)\right| \\ &\leqslant \frac{f(N)}{\xi} + \frac{f(x)}{\xi} + \frac{1}{\xi}\int_x^N |df(t)| \\ &\leqslant \frac{f(N)}{\xi} + \frac{f(x)}{\xi} + \frac{B}{\xi}\int_{\frac{x}{c}}^{cN} f(t)\frac{dt}{t}.\end{aligned}$$

Thus,

$$|\hat{f}(\xi)| = \lim_{N\to\infty}\left|\int_0^N f(t)\sin\xi t\,dt\right| \leqslant \int_0^x f(t)\,dt + \frac{f(x)}{\xi} + \frac{B}{\xi}\int_{\frac{x}{c}}^\infty f(t)\frac{dt}{t}. \tag{4}$$

Note that given $f \in L^q_{\omega(p,q)}$, $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, the above quantity is finite by Hölder's inequality. Also, the right-hand side of (4) is a nonincreasing function in ξ, so that also

$$\hat{f}^*(\xi) \leqslant \int_0^x f(t)\,dt + \frac{f(x)}{\xi} + \frac{B}{\xi}\int_{\frac{x}{c}}^\infty f(t)\frac{dt}{t}.$$

Let $g(\xi) = |\hat{f}(\xi)|$ or $\hat{f}^*(\xi)$. For $1 \leqslant q < \infty$, using Minkowski's inequality,

$$\begin{aligned}
\|g\|_{L^q_{\omega(p',q)}} &= \left(\int_0^\infty \left(\xi^{\frac{1}{p'}} g(\xi)\right)^q \frac{d\xi}{\xi}\right)^{\frac{1}{q}} \\
&= \left(1 - \frac{1}{2p'}\right)\left(\int_0^\infty \left(\xi^{1+\frac{1}{2p'}} \int_0^{\frac{1}{\xi}} x^{1-\frac{1}{2p'}} g(\xi) \frac{dx}{x}\right)^q \frac{d\xi}{\xi}\right)^{\frac{1}{q}} \\
&\leqslant \left(1 - \frac{1}{2p'}\right)\left(\int_0^\infty \left(\xi^{1+\frac{1}{2p'}} \int_0^{\frac{1}{\xi}} x^{1-\frac{1}{2p'}} \left(\int_0^x f(t)\,dt\right) \frac{dx}{x}\right)^q \frac{d\xi}{\xi}\right)^{\frac{1}{q}} \\
&\quad + \left(1 - \frac{1}{2p'}\right)\left(\int_0^\infty \left(\xi^{\frac{1}{2p'}} \int_0^{\frac{1}{\xi}} x^{1-\frac{1}{2p'}} f(x) \frac{dx}{x}\right)^q \frac{d\xi}{\xi}\right)^{\frac{1}{q}} \\
&\quad + B\left(1 - \frac{1}{2p'}\right)\left(\int_0^\infty \left(\xi^{\frac{1}{2p'}} \int_0^{\frac{1}{\xi}} x^{1-\frac{1}{2p'}} \left(\int_{\frac{x}{c}}^\infty f(t) \frac{dt}{t}\right) \frac{dx}{x}\right)^q \frac{d\xi}{\xi}\right)^{\frac{1}{q}}.
\end{aligned} \tag{5}$$

We estimate each term in (5) separately. For the first, changing variables on the outer integral and then using Hardy's inequality twice,

$$\begin{aligned}
&\left(\int_0^\infty \left(\xi^{1+\frac{1}{2p'}} \int_0^{\frac{1}{\xi}} x^{1-\frac{1}{2p'}} \left(\int_0^x f(t)\,dt\right) \frac{dx}{x}\right)^q \frac{d\xi}{\xi}\right)^{\frac{1}{q}} \\
&= \left(\int_0^\infty \left(u^{-1-\frac{1}{2p'}} \int_0^u x^{1-\frac{1}{2p'}} \left(\int_0^x t f(t) \frac{dt}{t}\right) \frac{dx}{x}\right)^q \frac{du}{u}\right)^{\frac{1}{q}} \\
&\leqslant \frac{2p'}{2p'+1}\left(\int_0^\infty \left(u^{-\frac{1}{p'}} \int_0^u t f(t) \frac{dt}{t}\right)^q \frac{du}{u}\right)^{\frac{1}{q}} \\
&\leqslant \frac{2(p')^2}{2p'+1}\left(\int_0^\infty \left(u^{\frac{1}{p}} f(u)\right)^q \frac{du}{u}\right)^{\frac{1}{q}} = \frac{2(p')^2}{2p'+1}\|f\|_{L^q_{\omega(p,q)}}.
\end{aligned}$$

For the second, changing variables on the outer integral and then using Hardy's inequality,

$$\begin{aligned}
&\left(\int_0^\infty \left(\xi^{\frac{1}{2p'}} \int_0^{\frac{1}{\xi}} x^{1-\frac{1}{2p'}} f(x) \frac{dx}{x}\right)^q \frac{d\xi}{\xi}\right)^{\frac{1}{q}} \\
&= \left(\int_0^\infty \left(u^{-\frac{1}{2p'}} \int_0^u x^{1-\frac{1}{2p'}} f(x) \frac{dx}{x}\right)^q \frac{du}{u}\right)^{\frac{1}{q}} \\
&\leqslant 2p'\left(\int_0^\infty \left(u^{\frac{1}{p}} f(u)\right)^q \frac{du}{u}\right)^{\frac{1}{q}} = 2p'\|f\|_{L^q_{\omega(p,q)}}.
\end{aligned}$$

For the third, changing variables on the outer two integrals and then using Hardy's inequality twice,

$$
\begin{aligned}
&\left(\int_0^\infty \left(\xi^{\frac{1}{2p'}} \int_0^{\frac{1}{\xi}} x^{1-\frac{1}{2p'}} \left(\int_{\frac{x}{c}}^\infty f(t)\frac{dt}{t}\right)\frac{dx}{x}\right)^q \frac{d\xi}{\xi}\right)^{\frac{1}{q}} \\
&= c^{1-\frac{1}{2p'}} \left(\int_0^\infty \left(\xi^{\frac{1}{2p'}} \int_0^{\frac{1}{c\xi}} y^{1-\frac{1}{2p'}} \left(\int_y^\infty f(t)\frac{dt}{t}\right)\frac{dy}{y}\right)^q \frac{d\xi}{\xi}\right)^{\frac{1}{q}} \\
&= c^{\frac{1}{p}} \left(\int_0^\infty \left(u^{-\frac{1}{2p'}} \int_0^{u} y^{1-\frac{1}{2p'}} \left(\int_y^\infty f(t)\frac{dt}{t}\right)\frac{dy}{y}\right)^q \frac{du}{u}\right)^{\frac{1}{q}} \\
&\leqslant 2p'c^{\frac{1}{p}} \left(\int_0^\infty \left(u^{\frac{1}{p}} \int_u^\infty f(t)\frac{dt}{t}\right)^q \frac{du}{u}\right)^{\frac{1}{q}} \\
&\leqslant 2pp'c^{\frac{1}{p}} \left(\int_0^\infty \left(u^{\frac{1}{p}} f(u)\right)^q \frac{du}{u}\right)^{\frac{1}{q}} = 2pp'c^{\frac{1}{p}} \|f\|_{L^q_{\omega(p,q)}}.
\end{aligned}
$$

Therefore,

$$
\|g\|_{L^q_{\omega(p',q)}} \leqslant C(B,c,p)\|f\|_{L^q_{\omega(p,q)}}.
$$

The proof in the case $q=\infty$ is similar.

If $\hat{f}$ is the Fourier cosine transform of f, let $h_\xi(t) = \frac{\sin \xi t}{\xi}$, and the proof proceeds in an identical fashion. □

The second main result of this paper is Theorem 5. We first require a lemma, which appears as Lemma 2.3 in [10].

LEMMA 2. *Let $f \geqslant 0$ on $(0,\infty)$ be such that for $a>0$,*

$$
\int_a^\infty f(t)\frac{dt}{t} < \infty.
$$

For all $\xi \in (0,\infty)$, let $\hat{f}_s(\xi)$ and $\hat{f}_c(\xi)$ exist as improper integrals, converging uniformly on every compact set away from zero, and be locally integrable on $(0,\infty)$. Let $c>1$. If $tf(t)$ is integrable near zero, then there exists a constant $C_1(c)$ such that

$$
\int_{\frac{x}{c}}^{cx} f(t)\frac{dt}{t} \leqslant C_1(c)\int_0^{\frac{1}{x}} |\hat{f}_s(\xi)|\,d\xi
$$

and if $f(t)$ is integrable near zero, then there exists a constant $C_2(c)$ such that

$$
\int_{\frac{x}{c}}^{cx} f(t)\frac{dt}{t} \leqslant C_2(c)\int_0^{\frac{1}{x}} |\hat{f}_c(\xi)|\,d\xi
$$

THEOREM 5. *Let f satisfy Condition 1. Let $1<p<\infty$, $1 \leqslant q \leqslant \infty$, and let f and $\hat{f}$ satisfy the hypotheses of Lemma 2.3 in* [10]. *If $\hat{f} \in L^q_{\omega(p',q)}$, then $f \in L^q_{\omega(p,q)} \cap L(p,q)$, and*

$$\|f\|_{L(p,q)} \sim \|f\|_{L^q_{\omega(p,q)}} \leqslant C(A,c,p)\|\hat{f}\|_{L^q_{\omega(p',q)}}. \tag{6}$$

Similarly, if $\hat{f} \in L(p',q)$, then $f \in L^q_{\omega(p,q)} \cap L(p,q)$, and

$$\|f\|_{L(p,q)} \sim \|f\|_{L^q_{\omega(p,q)}} \leqslant C(A,c,p)\|\hat{f}\|_{L(p',q)}. \tag{7}$$

Proof. The equivalences in (6) and (7) follow from Theorem 3, so we show the remaining inequalities. We modify the proof of Theorem 2.1(B) in [8] as necessary. First let $\hat{f}$ denote the Fourier sine transform of f. From the proof of Lemma 2.3 in [10], for $x \in (0,\infty)$,

$$\int_0^{\frac{\pi}{cx}} \hat{f}(\xi)\,d\xi = 2\int_0^\infty f(t)\sin^2\frac{\pi t}{2cx}\frac{dt}{t}.$$

Thus,

$$\begin{aligned}\int_0^{\frac{\pi}{cx}} |\hat{f}(\xi)|\,d\xi &\geqslant \left|\int_0^{\frac{\pi}{cx}} \hat{f}(\xi)\,d\xi\right| = 2\int_0^\infty f(t)\sin^2\frac{\pi t}{2cx}\frac{dt}{t} \geqslant 2\int_{\frac{x}{c}}^{cx} f(t)\sin^2\frac{\pi t}{2cx}\frac{dt}{t}\\ &\geqslant 2\int_{\frac{x}{c}}^{cx} f(t)\frac{t^2}{c^2x^2}\frac{dt}{t} \geqslant \frac{2}{c^4}\int_{\frac{x}{c}}^{cx} f(t)\frac{dt}{t}.\end{aligned}$$

By Lemma 1,

$$\int_0^{\frac{\pi}{cx}} \hat{f}^*(\xi)\,d\xi \geqslant \int_0^{\frac{\pi}{cx}} |\hat{f}(\xi)|\,d\xi$$

so that also

$$\int_0^{\frac{\pi}{cx}} \hat{f}^*(\xi)\,d\xi \geqslant \frac{2}{c^4}\int_{\frac{x}{c}}^{cx} f(t)\frac{dt}{t}.$$

Therefore, for $1 \leqslant q < \infty$, letting $g(\xi) = |\hat{f}(\xi)|$ or $\hat{f}^*(\xi)$, and using a change of variables and Hardy's inequality,

$$\begin{aligned}\|f\|_{L^q_{\omega(p,q)}} &= \left(\int_0^\infty \left(x^{\frac{1}{p}}|f(x)|\right)^q \frac{dx}{x}\right)^{\frac{1}{q}} \leqslant A\left(\int_0^\infty \left(x^{\frac{1}{p}}\int_{\frac{x}{c}}^{cx} f(t)\frac{dt}{t}\right)^q \frac{dx}{x}\right)^{\frac{1}{q}}\\ &\leqslant \frac{1}{2}Ac^4\left(\int_0^\infty \left(x^{\frac{1}{p}}\int_0^{\frac{\pi}{cx}} g(\xi)\,d\xi\right)^q \frac{dx}{x}\right)^{\frac{1}{q}}\\ &= \frac{1}{2}Ac^4\left(\frac{\pi}{c}\right)^{\frac{1}{p}}\left(\int_0^\infty \left(u^{-\frac{1}{p}}\int_0^u \xi g(\xi)\frac{d\xi}{\xi}\right)^q \frac{du}{u}\right)^{\frac{1}{q}}\\ &\leqslant \frac{\pi^{\frac{1}{p}}p}{2}Ac^{4-\frac{1}{p}}\left(\int_0^\infty \left(u^{\frac{1}{p'}}g(u)\right)^q \frac{du}{u}\right)^{\frac{1}{q}} = \frac{\pi^{\frac{1}{p}}p}{2}Ac^{4-\frac{1}{p}}\|g\|_{L^q_{\omega(p',q)}}.\end{aligned}$$

The proof in the case $q=\infty$ is similar.

Next let $\hat{f}$ denote the Fourier cosine transform of f. For $0<\varepsilon<u<\infty$, using the Uniform Convergence theorem,

$$\begin{aligned}\int_\varepsilon^u \hat{f}(\xi)\,d\xi &= \int_\varepsilon^u \left(\lim_{N\to\infty}\int_0^N f(t)\cos\xi t\,dt\right)d\xi\\ &= \lim_{N\to\infty}\int_\varepsilon^u \left(\int_0^N f(t)\cos\xi t\,dt\right)d\xi = \lim_{N\to\infty}\int_0^N \left(\int_\varepsilon^u \cos t\xi\,d\xi\right)f(t)\,dt\\ &= \lim_{N\to\infty}\int_0^N (\sin ut-\sin\varepsilon t)f(t)\frac{dt}{t} = \int_0^\infty (\sin ut-\sin\varepsilon t)f(t)\frac{dt}{t}.\end{aligned}$$

Letting $\varepsilon\to 0$, by the Dominated Convergence Theorem,

$$\int_0^u \hat{f}(\xi)\,d\xi = \int_0^\infty f(t)\sin ut\frac{dt}{t}.$$

Thus, for $x\in(0,\infty)$,

$$\begin{aligned}\int_0^{\frac{\pi}{cx}}\left(\int_0^u \hat{f}(\xi)\,d\xi\right)du &= \int_0^{\frac{\pi}{cx}}\left(\int_0^\infty f(t)\sin ut\frac{dt}{t}\right)du = \int_0^\infty f(t)\left(\int_0^{\frac{\pi}{cx}}\sin tu\,du\right)\frac{dt}{t}\\ &= \int_0^\infty f(t)\frac{1-\cos\frac{\pi t}{cx}}{t}\frac{dt}{t} = 2\int_0^\infty f(t)\frac{\sin^2\frac{\pi t}{2cx}}{t}\frac{dt}{t}.\end{aligned}$$

Thus,

$$\begin{aligned}\int_0^{\frac{\pi}{cx}}\left(\int_0^u |\hat{f}(\xi)|\,d\xi\right)du \geqslant \left|\int_0^{\frac{\pi}{cx}}\left(\int_0^u \hat{f}(\xi)\,d\xi\right)du\right| &= 2\int_0^\infty f(t)\frac{\sin^2\frac{\pi t}{2cx}}{t}\frac{dt}{t}\\ &\geqslant 2\int_{\frac{x}{c}}^{cx} f(t)\frac{\sin^2\frac{\pi t}{2cx}}{t}\frac{dt}{t} \geqslant 2\int_{\frac{x}{c}}^{cx} f(t)\frac{t}{c^2x^2}\frac{dt}{t}\\ &\geqslant \frac{2}{c^3x}\int_{\frac{x}{c}}^{cx} f(t)\frac{dt}{t}.\end{aligned}$$

By Lemma 1,

$$\int_0^{\frac{\pi}{cx}}\left(\int_0^u \hat{f}^*(\xi)\,d\xi\right)du \geqslant \int_0^{\frac{\pi}{cx}}\left(\int_0^u |\hat{f}(\xi)|\,d\xi\right)du$$

so that also

$$\int_0^{\frac{\pi}{cx}}\left(\int_0^u \hat{f}^*(\xi)\,d\xi\right)du \geqslant \frac{2}{c^3x}\int_{\frac{x}{c}}^{cx} f(t)\frac{dt}{t}.$$

Therefore, for $1 \leqslant q < \infty$, letting $g(\xi) = |\hat{f}(\xi)|$ or $\hat{f}^*(\xi)$, and using a change of variables and Hardy's inequality twice,

$$\begin{aligned}
\|f\|_{L^q_{\omega(p,q)}} &= \left(\int_0^\infty \left(x^{\frac{1}{p}} f(x)\right)^q \frac{dx}{x}\right)^{\frac{1}{q}} \leqslant A \left(\int_0^\infty \left(x^{\frac{1}{p}} \int_{\frac{x}{c}}^{cx} f(t) \frac{dt}{t}\right)^q \frac{dx}{x}\right)^{\frac{1}{q}} \\
&\leqslant \frac{1}{2} A c^3 \left(\int_0^\infty \left(x^{1+\frac{1}{p}} \int_0^{\frac{\pi}{cx}} \left(\int_0^u g(\xi)\, d\xi\right) du\right)^q \frac{dx}{x}\right)^{\frac{1}{q}} \\
&= \frac{1}{2} A c^3 \left(\frac{\pi}{c}\right)^{1+\frac{1}{p}} \left(\int_0^\infty \left(v^{-1-\frac{1}{p}} \int_0^v u \left(\int_0^u \xi g(\xi) \frac{d\xi}{\xi}\right) \frac{du}{u}\right)^q \frac{dv}{v}\right)^{\frac{1}{q}} \\
&\leqslant \frac{\pi^{1+\frac{1}{p}} p}{2(p+1)} A c^{2-\frac{1}{p}} \left(\int_0^\infty \left(v^{-\frac{1}{p}} \int_0^v \xi g(\xi) \frac{d\xi}{\xi}\right)^q \frac{dv}{v}\right)^{\frac{1}{q}} \\
&\leqslant \frac{\pi^{1+\frac{1}{p}} p^2}{2(p+1)} A c^{2-\frac{1}{p}} \left(\int_0^\infty \left(v^{\frac{1}{p'}} g(v)\right)^q \frac{dv}{v}\right)^{\frac{1}{q}} = \frac{\pi^{1+\frac{1}{p}} p^2}{2(p+1)} A c^{2-\frac{1}{p}} \|g\|_{L^q_{\omega(p',q)}}.
\end{aligned}$$

The proof in the case $q = \infty$ is similar. $\square$

Putting together Theorems 4 and 5, we obtain:

COROLLARY 1. *Let $f \geqslant 0$ and $f \in GM$ on $(0,\infty)$. Let $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, and let f and $\hat{f}$ satisfy the hypotheses of Lemma 2.3 in* [10]. *Then the following statements are equivalent:*

1. $\hat{f} \in L(p',q)$.
2. $\hat{f} \in L^q_{\omega(p',q)}$.
3. $f \in L^q_{\omega(p,q)}$.
4. $f \in L(p,q)$.

Moreover,

$$\|\hat{f}\|_{L(p',q)} \sim \|\hat{f}\|_{L^q_{\omega(p',q)}} \sim \|f\|_{L^q_{\omega(p,q)}} \sim \|f\|_{L(p,q)}. \tag{8}$$

Proof. The equivalences of the last two statements and of the last two norms in (8) follow from Theorem 3. For the equivalences of the first and third statements and the first and third norms in (8), we obtain from Theorem 4 that $f \in L^q_{\omega(p,q)}$ implies $\hat{f} \in L(p',q)$, and

$$\|\hat{f}\|_{L(p',q)} \leqslant C(B,c,p) \|f\|_{L^q_{\omega(p,q)}}$$

and from Theorem 5 that $\hat{f} \in L(p',q)$ implies $f \in L^q_{\omega(p,q)}$, and

$$\|f\|_{L^q_{\omega(p,q)}} \leqslant C(A,c,p) \|\hat{f}\|_{L(p',q)}.$$

The equivalences of the second and third statements and the second and third norms in (8) are obtained similarly. $\square$

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