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IMPROVED HARDY AND RELLICH TYPE INEQUALITIES WITH TWO WEIGHT FUNCTIONS

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Abstract. In this work, we obtain several improved versions of two weight Hardy and Rellich type inequalities on the sub-Riemannian manifold $\mathbb{R}^{2n+1}$ defined by the vector fields

$$X_j = \frac{\partial}{\partial x_j} + 2ky_j|z|^{2k-2}\frac{\partial}{\partial l}\,, \quad Y_j = \frac{\partial}{\partial y_j} - 2kx_j|z|^{2k-2}\frac{\partial}{\partial l}\,, \qquad j = 1,2,\dots,n$$

where $(z,l) = (x,y,l) \in \mathbb{R}^{2n+1}$, $|z| = (|x|^2+|y|^2)^{1/2}$ and $k \geqslant 1$.

1. Introduction

In this work, we continue on our previous study of Hardy and Rellich type inequalities with two weight functions given in [6]. We prove the new improved versions of these inequalities on the sub-Riemannian manifold $\mathbb{R}^{2n+1}$ defined by the vector fields

$$X_j = \frac{\partial}{\partial x_j} + 2ky_j|z|^{2k-2}\frac{\partial}{\partial l}\,, \quad Y_j = \frac{\partial}{\partial y_j} - 2kx_j|z|^{2k-2}\frac{\partial}{\partial l}\,, \qquad j = 1,2,\dots,n \tag{1}$$

where $(z,l) = (x,y,l) \in \mathbb{R}^{2n+1}$, $|z| = (|x|^2+|y|^2)^{1/2}$ and $k \geqslant 1$.

It is well known that the constants $(n-2)^2/4$ and $n^2(n-4)^2/16$ are the best constants for the classical Hardy and Rellich type inequalities for $\phi \in C_0^\infty(\mathbb{R}^n)$:

$$\int_{\mathbb{R}^n} |\nabla\phi|^2 dx \geqslant \frac{(n-2)^2}{4}\int_{\mathbb{R}^n} \frac{|\phi|^2}{|x|^2}dx\,, \quad n \geqslant 3\,, \tag{2}$$

$$\int_{\mathbb{R}^n} |\Delta\phi|^2 dx \geqslant \frac{n^2(n-4)^2}{16}\int_{\mathbb{R}^n} \frac{|\phi|^2}{|x|^4}dx\,, \quad n \geqslant 5\,, \tag{3}$$

respectively, [17], [26]. Here, $|x|$ is the distance from the point x to the origin, and the coefficients $(n-2)^2/4$ and $n^2(n-4)^2/16$ are never achieved. The following inequality

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on a bounded domain $0 \in \Omega \subset \mathbb{R}^n$, $n \geqslant 2$, is known as Hardy-Sobolev inequality or L^p Hardy inequality

$$\int_\Omega |\nabla\phi|^p dx \geqslant \left(\frac{n-p}{p}\right)^p \int_\Omega \frac{|\phi|^p}{|x|^p} dx\,, \quad n \geqslant 2 \tag{4}$$

for any $1 < p < n$ and $\phi \in W_0^{1,p}(\Omega)$, where $W_0^{1,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$. The constant $(n-p)^p/p^p$ is the best constant for the equation above, and it is also never achieved. This fact has offered a new direction to researchers who had expected to improve these inequalities by adding nonnegative correction terms to the right hand side of the inequalities. In this way, Hardy inequality in (2) and Hardy-Sobolev inequality in (4) have been generalized and modified in many different ways, and the literature concerning such inequalities has begun to increase, see, e.g. [1], [2], [3], [4], [8], [11], [12], [13], [21], [22], [23], [25], [28] and references therein. In [11], Brezis and Vazquez improved the inequality (2) for any bounded domain Ω in $\mathbb{R}^n$, $n \geqslant 2$ and for every $\phi \in H_0^1(\Omega)$;

$$\int_\Omega |\nabla\phi|^2 dx \geqslant \frac{(n-2)2}{4} \int_\Omega \frac{\phi^2}{|x|^2} dx + H_2 \int_\Omega \phi^2 dx\,, \tag{5}$$

where H_2 is the first eigenvalue of the Laplacian in the unit disk in $\mathbb{R}^2$ and H_2 is the optimal constants independent of the dimension n. Inspired by the inequality (5), several improved versions of the inequalities (2) and (4) with different weight functions have been established in [1], [2], [3], [4], [13], [14], [15], [18], [21], [22], [28]. In [2], the Hardy-Sobolev inequality is improved by adding a singular weight term of the type $\log(R/|x|))^{-\gamma}$ for any $\phi \in W_0^{1,p}(\Omega)$ on the bounded domain Ω;

$$\int_\Omega |\nabla\phi|^p dx \geqslant \left(\frac{n-p}{p}\right)^p \int_\Omega \frac{|\phi|^p}{|x|^p} dx + C \int_B \frac{|\phi|^p}{|x|^p} \left(\log\frac{R}{|x|}\right)^{-\gamma} dx\,, \tag{6}$$

where $R \geqslant \sup_\Omega(|x|e^{2/p})$, $1 < p < n$, $\gamma \geqslant 2$ and $C > 0$ depends on n, p, R. One of the other extensions of (4) can also be seen [13].

An improved weighted L^2 Hardy inequality involving two weight functions modulated on distance functions from a point, ρ, and distance to the boundary of a domain Ω, δ, with smooth boundary, is also proved on a complete noncompact sub-Riemannian manifold of dimension $n > 1$ for $\phi \in C_0^\infty(\Omega \backslash \rho^{-1}\{0\})$ in [22]:

$$\int_\Omega \rho^\alpha |\nabla\phi|^2 dV \geqslant \frac{(C+\alpha-1)^2}{4} \int_\Omega \rho^\alpha \frac{|\phi|^2}{\rho^2} dV + \frac{1}{4} \int_\Omega \rho^\alpha \frac{|\nabla\delta|^2}{\delta^2} |\phi|^2 dV.$$

Here it is assumed that $\alpha \in \mathbb{R}$, $C+\alpha-1 > 0$, $|\nabla\rho| = 1$, $\Delta\rho \geqslant \frac{C}{\rho}$, $C > 1$ and $-\nabla\cdot(\rho^{1-C}\nabla\delta) \geqslant 0$ in the sense of distribution. Moreover, it is reasonable to consider that remainder terms on the right hand side of Rellich inequality (3) can also be added, and by this way, the various improved versions of Rellich type inequality have been established in [7], [13], [15], [21], [22], [27] and references therein.

On the other hand, there has been a growing interest in Hardy and Rellich type inequalities on the sub-Riemannian manifold $\mathbb{R}^{2n+1}$ defined by the vector fields (1) [5], [6], [24], [29] . L^p version of the inequality (2) has been established on the sub-Riemannian manifold in [29]. Then, a new weighted version of L^p Hardy inequality with a radial weight function ρ^{α} is obtained in [24];

$$\int_{\mathbb{R}^{2n+1}} \rho^{\alpha p}|\nabla_k \phi|^p dw \geqslant \frac{(Q-p+\alpha p)^p}{p^p} \int_{\mathbb{R}^{2n+1}} \rho^{\alpha p}\Big(\frac{|z|}{\rho}\Big)^{p(2k-1)} \frac{|\phi|^p}{\rho^p} dw$$

where $\phi \in C_0^{\infty}(\mathbb{R}^{2n+1} \setminus \{(0,0)\})$, $1<p<Q$, $Q \neq p$ and $Q-p+\alpha p>0$. Furthermore, the constant $(Q-p+\alpha p)^p/p^p$ is sharp. Another work in this direction is given in [6]. In this article, several new sharp weighted Hardy, Hardy-Poincaré and Rellich type inequalities for the sub-Riemannian manifold $\mathbb{R}^{2n+1}$ defined by the vector fields (1) are obtained by introducing a weight function given by the product of the radial functions ρ^{α} and $|z|^t$, where $\alpha, t \in \mathbb{R}$. Furthermore, we observed the influence of the new radial weight function $|z|^t$ on the sharp constant.

All these studies have motivated us to contribute in this direction. Since this paper relies on some results originally derived in [6] and [22], we extend the results in [6], [20], [21] and [22] to the sub-Riemannian manifold $\mathbb{R}^{2n+1}$ defined by the vector fields (1), and we also give some new improved versions of the weighted Hardy and Rellich type inequalities with two weight functions given by the product of the radial functions ρ^{α} and $|z|^t$, $\alpha, t \in \mathbb{R}$.

The outline of this paper is as follows: as we have provided a very brief survey of the inequalities that we studied in our paper, the next section is devoted to introduce fundamental notations, generalized Greiner vector fields, Greiner operator and basic facts about the operators ∇_k, Δ_k. Then, we establish an improved version of sharp weighted Hardy inequality derived in [6] in Section 3 and improved versions of two weighted Rellich type inequalities in Section 4.

2. Preliminary and notations

A generic point in $\mathbb{R}^{2n+1}$, $n \geqslant 1$ is defined by $w=(z,l)=(x,y,l) \in \mathbb{R}^{2n+1}$ where $x, y \in \mathbb{R}^n$, $z = x+\sqrt{-1}y$. The sub-elliptic gradient is the $2n$ dimensional vector field given by

$$\nabla_k := (X_1, \ldots, X_n, Y_1, \ldots, Y_n)$$

where X_j and Y_j are the smooth vector fields which are defined by

$$X_j = \frac{\partial}{\partial x_j} + 2ky_j|z|^{2k-2}\frac{\partial}{\partial l}, \quad Y_j = \frac{\partial}{\partial y_j} - 2kx_j|z|^{2k-2}\frac{\partial}{\partial l}, \qquad j=1,2,\ldots,n. \tag{7}$$

The generalized Greiner operator on $\mathbb{R}^{2n+1}$ is defined by

$$\Delta_k = \sum_{j=1}^{n}(X_j^2+Y_j^2) = \Delta_z + 4k^2|z|^{4k-2}\frac{\partial^2}{\partial l^2} + 4k|z|^{2k-2}\frac{\partial}{\partial l}T$$

where $\Delta_z = \sum_{j=1}^n (\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2})$ is the Laplacian in the variable $z = (x,y) \in \mathbb{R}^{2n}$ and T denotes the vector field $T = \sum_{j=1}^n (y_j \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial y_j})$. Δ_k possesses a natural family of dilations, namely,

$$\delta_\tau(z,l) = (\tau z, \tau^{2k} l), \quad \tau > 0 \tag{8}$$

associated with the vector fields in (7). It is easy to see that Δ_k is the homogeneous of degree two with respect to the dilation, i.e, $\Delta_k(\delta_\tau u) = \tau^2 \delta_\tau(\Delta_k u)$ where $\delta_\tau u(z,l) = u(\tau z, \tau^{2k} l)$. It is easy to verify that $d\delta_\tau(z) = \tau^Q dw = \tau^Q dzdl$ where

$$Q = 2(k+n)$$

is the homogeneous dimension with respect to dilation δ_τ and $dw = dzdl$ denotes the Lebesgue measure on $\mathbb{R}^{2n+1}$. The natural norm function is defined as

$$\rho(z,l) := (|z|^{4k} + l^2)^{1/4k} = (x^2+y^2)^{2k} + l^2)^{1/4k}. \tag{9}$$

It is known that ρ is related to the fundamental solution of sub-Laplacian Δ_k with singularity at the origin (see, [9], [10], [16], [29]). It is easy to show that ρ is homogeneous of degree one with respect to the dilations (8); $\rho(\delta_\tau(w)) = \tau\rho(w)$ for every τ. The gradient of the function ρ satisfies

$$\nabla_k \rho = (X_1\rho, .., X_j\rho, .., X_n\rho, Y_1\rho, ..., Y_j\rho, ..., Y_n\rho)$$

where

$$X_j\rho = \frac{|z|^{2(k-1)}}{\rho^{4k-1}}[x_j|z|^{2k} + y_j l]\,, \quad Y_j\rho = \frac{|z|^{2(k-1)}}{\rho^{4k-1}}[y_j|z|^{2k} - x_j l]$$

and its module is $|\nabla_k\rho| = \frac{|z|^{2k-1}}{\rho^{2k-1}}$, and

$$\Delta_k\rho = (Q-1)\frac{|\nabla_k\rho|^2}{\rho}. \tag{10}$$

Let us mention some expressions and identities, which we shall frequently encounter throughout the following calculations. Using the above formula we obtained

$$\nabla_k(|\nabla_k\rho|^2) \cdot \nabla_k\rho = 0$$

which shows that the norm function (9) is infinite harmonic in $\mathbb{R}^{2n+1} \setminus \{0\}$. An immediate consequence of the equation (10) is the following formula:

$$\nabla_k\Big(\frac{\rho}{|\nabla_k\rho|^2}\Big) \cdot \nabla_k\rho = 1.$$

A straightforward computation shows

$$\nabla_k \cdot \Big(\frac{\rho}{|\nabla_k\rho|^2}\nabla_k\rho\Big) = \nabla_k\Big(\frac{\rho}{|\nabla_k\rho|^2}\Big) \cdot \nabla_k\rho + \frac{\rho}{|\nabla_k\rho|^2}\Delta_k\rho = Q.$$

Besides, $\nabla_k \rho$ and $\nabla_k(|z|)$ satisfy the following relation

$$\nabla_k(|z|^t)\cdot\nabla_k\rho^\alpha = \alpha t|z|^t\rho^{\alpha-2}|\nabla_k\rho|^2. \tag{11}$$

On the other hand, if $\phi = \phi(\rho)$ is a smooth radial function (i.e., ϕ only depends on the function ρ), then by a direct computation, it can be easily shown that

$$|\nabla_k\phi(\rho)| = \frac{|z|^{2k-1}}{\rho^{2k-1}}|\phi'(\rho)|.$$

The open ball with respect to ρ centered at the origin $(0,0)\in\mathbb{R}^{2n}\times\mathbb{R}$ with radius R will be denoted by

$$B_R(0) := \{(z,l)\in\mathbb{R}^{2n}\times\mathbb{R} : \rho < R\}.$$

Introducing the spherical coordinate transformation as in [19], the volume element satisfies the relation

$$dw = dzdl = \rho^{Q-1}d\rho(\sin\varphi)^{\frac{n-k}{k}}d\varphi\prod_{j=1}^{n-1}\left[\cos\psi_j(\sin\psi_j)^{2(n-j)}d\psi_j\right]\prod_{j=1}^{n}d\theta_j \tag{12}$$

where $0\leqslant\varphi\leqslant\pi$, $0\leqslant\psi_j\leqslant\pi/2$, $j=1,\ldots,n-1$ and $0\leqslant\theta_j\leqslant 2\pi$, $j=1,\ldots,n$, and

$$|z|^2 = \rho^2\sin^{\frac{1}{k}}\varphi.$$

3. Improved Hardy-type inequalities with two weight functions

The main result of this section is the following theorem which is an improved version of the sharp two weight Hardy inequality in [6]:

THEOREM 1. *Let Ω be a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^{2n+1}$. Let ρ and $|z|$ be nonnegative functions on Ω. Let $Q\geqslant 3$, $\alpha\in\mathbb{R}$, $t\in\mathbb{R}$, $Q+\alpha+t-2>0$ and $t+4k+2n-2>0$. Furthermore, let δ be a positive function such that $-\nabla_k\cdot(\rho^{2-t-Q}|z|^t\nabla_k\delta)\geqslant 0$ in the sense of distribution in Ω. Then the following inequality*

$$\begin{aligned}\int_\Omega \rho^\alpha|z|^t|\nabla_k\phi|^2dzdl \geqslant \left(\frac{Q+\alpha+t-2}{2}\right)^2\int_\Omega\rho^\alpha\frac{|z|^t\phi^2}{\rho^2}|\nabla_k\rho|^2dzdl\\ +\frac{1}{4}\int_\Omega\rho^{\alpha+t}\frac{|z|^t|\nabla_k\delta|^2}{\delta^2}\phi^2dzdl\end{aligned} \tag{13}$$

holds for all $\phi\in C_0^\infty(\Omega\setminus\{0\})$.

Proof. Let $\phi\in C_0^\infty(\Omega\setminus\{0\})$ and define $\psi=\rho^{-\gamma}\phi$, where $\gamma<0$. A direct calculation shows that

$$|\nabla_k\phi|^2 = \gamma^2\rho^{2\beta-2}|\nabla_k\rho|^2\psi^2 + 2\gamma\rho^{2\gamma-1}\psi\nabla_k\rho\cdot\nabla_k\psi + \rho^{2\gamma}|\nabla_k\psi|^2. \tag{14}$$

Multiplying both sides of (14) by the function $\rho^{\alpha}|z|^t$ and integrating over the region Ω, we have

$$\int_{\Omega}\rho^{\alpha}|z|^t|\nabla_k\phi|^2dzdl=\gamma^2\int_{\Omega}\rho^{2\gamma+\alpha-2}|z|^t|\nabla_k\rho|^2\psi^2dzdl \\ +2\gamma\int_{\Omega}\rho^{2\gamma+\alpha-1}|z|^t\psi\nabla_k\rho\cdot\nabla_k\psi dzdl+\int_{\Omega}\rho^{2\gamma+\alpha}|z|^t|\nabla_k\psi|^2dzdl. \tag{15}$$

Applying integration by parts to the second integral on the right hand side, we get

$$\int_{\Omega}\rho^{2\gamma+\alpha-1}|z|^t\psi\nabla_k\rho\cdot\nabla_k\psi dzdl=-\frac{1}{2}\int_{\Omega}\psi^2\nabla_k\cdot[\rho^{2\gamma+\alpha-1}|z|^t\nabla_k\rho]dzdl.$$

Next, using (10) and (12) yields

$$\nabla_k\cdot[\rho^{2\gamma+\alpha-1}|z|^t\nabla_k\rho]=(2\gamma+\alpha+Q+t-2)\rho^{2\gamma+\alpha-2}|z|^t|\nabla_k\rho|^2. \tag{16}$$

Substituting (16) into (15) and choosing $\gamma=(2-\alpha-t-Q)/2$, then substituting $\psi^2=\rho^{\alpha+t+Q-2}\phi^2$ yields

$$\int_{\Omega}\rho^{\alpha}|z|^t|\nabla_k\phi|^2dzdl=\frac{(Q+\alpha+t-2)^2}{4}\int_{\Omega}\rho^{\alpha-2}|z|^t|\nabla_k\rho|^2\phi^2dzdl \\ +\int_{\Omega}\rho^{2-t-Q}|z|^t|\nabla_k\psi|^2dzdl. \tag{17}$$

Now, let us define a new function $\varphi(z,l):=\frac{\psi(z,l)}{\sqrt{\delta(z,l)}}$ where $\delta(z,l)\in C_0^2(\Omega)$ and is a positive function and

$$|\nabla_k\psi|^2=\frac{1}{4}\frac{\varphi^2}{\delta}|\nabla_k\delta|^2+\varphi\nabla_k\delta\cdot\nabla_k\varphi+\delta|\nabla_k\varphi|^2.$$

Therefore, the last integral on the right hand side of (17) becomes

$$\int_{\Omega}\rho^{2-t-Q}|z|^t|\nabla_k\psi|^2dzdl\geqslant\frac{1}{4}\int_{\Omega}\rho^{2-Q}|z|^t\frac{|\nabla_k\delta|^2}{\delta^2}\varphi^2dzdl \\ +\int_{\Omega}\rho^{2-t-Q}|z|^t\varphi\nabla_k\delta\cdot\nabla_k\varphi dzdl.$$

By integration by parts, we get,

$$\int_{\Omega}\rho^{2-t-Q}|z|^t|\nabla_k\psi|^2dzdl\geqslant\frac{1}{4}\int_{\Omega}\rho^{2-Q}|z|^t\frac{|\nabla_k\delta|^2}{\delta^2}\psi^2dzdl \\ -\frac{1}{2}\int_{\Omega}\varphi^2\nabla_k\cdot(\rho^{2-t-Q}|z|^t\nabla_k\delta)dzdl.$$

Since

$$-\nabla_k\cdot(\rho^{2-t-Q}|z|^t\nabla_k\delta)\geqslant 0$$

and $\psi = \rho^{\frac{Q+\alpha+t-2}{2}}\phi$, then we get the desired inequality

$$\int_\Omega \rho^\alpha |z|^t |\nabla_k \phi|^2 dzdl \geqslant \left(\frac{Q+\alpha+t-2}{2}\right)^2 \int_\Omega \rho^{\alpha-2}|z|^t \phi^2 |\nabla_k \rho|^2 dzdl + \frac{1}{4}\int_\Omega \rho^{\alpha+t}|z|^t \frac{|\nabla_k \delta|^2}{\delta^2}\phi^2 dzdl. \quad \square$$

It is known that the improved versions of weighted Hardy type inequalities are used as main tools in order to obtain the improved version of weighted Rellich inequalities, thus the weigthed improved Hardy inequalities in (13) are used in the next section. Now, we choose model functions, which satisfy the assumptions of the above theorem as in [22]:

$$\begin{aligned} &a)\quad \delta_1 = \ln\left(\frac{R}{\rho}\right), \quad R > \sup_\Omega \rho\,, \\ &b)\quad \delta_2 = R-\rho \end{aligned} \tag{18}$$

where δ_2 is the distance function of a point $w=(z,l)\in B_R = \{w\in\mathbb{R}^{2n+1} | \rho < R\}$ to the boundary of B_R. It can be easily shown that both δ_1 and δ_2 satisfy the differential inequality in (3);

$$-\nabla_k \cdot (\rho^{2-t-Q}|z|^t \nabla_k \delta_1) = 0,$$

and

$$-\nabla_k \cdot (\rho^{2-t-Q}|z|^t \nabla_k \delta_2) = \rho^{1-t-Q}|z|^t |\nabla_k \rho|^2 \geqslant 0.$$

As consequences of Theorem 1 we have the following weighted L^2- Hardy-type inequalities with the different remainder terms:

COROLLARY 1. *Let Ω be a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^{2n+1}$ and $\delta := \ln\frac{R}{\rho}$, $R > \sup_\Omega \rho$. Let $\alpha\in\mathbb{R}$, $t\in\mathbb{R}$, $Q\geqslant 3$, $Q+\alpha+t-2>0$ and $Q+4k+2n-2>0$. Then the following inequality is valid for all $\phi\in C_0^\infty(\Omega\setminus\{0\})$;*

$$\int_\Omega \rho^\alpha |z|^t |\nabla_k \phi|^2 dzdl \geqslant \frac{(Q+\alpha+t-2)^2}{4}\int_\Omega \rho^{\alpha-2}|z|^t \phi^2 |\nabla_k \rho|^2 dzdl + \frac{1}{4}\int_\Omega \rho^{\alpha+t-2}|z|^t \left(\ln\frac{R}{\rho}\right)^{-2} |\nabla_k \rho|^2 \phi^2 dzdl. \tag{19}$$

REMARK 1. For $\gamma = 2$ and $p=2$, the inequality (19) is an extension of the inequality (6) with weight function $\rho^\alpha |z|^t$ to the Greiner vector fields.

COROLLARY 2. *Let B_R be a ball with center 0 and radius R and $\delta := R-\rho$. Let $\alpha\in\mathbb{R}$, $t\in\mathbb{R}$, $Q\geqslant 3$, $Q+\alpha+t-2>0$ and $Q+4k+2n-2>0$. Then we have*

$$\int_{B_R} \rho^\alpha |z|^t |\nabla_k \phi|^2 dzdl \geqslant \frac{(Q+\alpha+t-2)^2}{4}\int_{B_R} \rho^{\alpha-2}|z|^t \phi^2 |\nabla_k \rho|^2 dzdl + \frac{1}{4}\int_{B_R} \rho^{\alpha+t}|z|^t (R-\rho)^{-2} |\nabla_k \rho|^2 \phi^2 dzdl \tag{20}$$

for all $\phi\in C_0^\infty(B_R\setminus\{0\})$.

4. Improved versions of Rellich-type inequalities with two weight functions

In this section, the main results are the following theorems which are the improved versions of Theorem 3 and Theorem 4 in [6], respectively.

THEOREM 2. *Let Ω be a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^{2n+1}$. Let $|z|$ and ρ be nonnegative functions on Ω. Furthermore, let δ be a positive function such that $-\nabla_k \cdot (\rho^{2-t-Q}|z|^t \nabla_k \delta) \geqslant 0$ in Ω. Let $Q \geqslant 3$, $\alpha \in \mathbb{R}$, $t \in \mathbb{R}$, $t+Q+\alpha-2>0$ and $Q+t+2k-2>0$. Then the inequality*

$$\begin{aligned}\int_\Omega \frac{\rho^{\alpha+2}|z|^t}{|\nabla_k \rho|^2}|\Delta_k \phi|^2 dzdl \geqslant \zeta^2 \int_\Omega \rho^{\alpha-2}|z|^t |\nabla_k \rho|^2 \phi^2 dzdl \\ - \zeta t(t+2n-2)\int_\Omega \rho^\alpha |z|^{t-2}\phi^2 dzdl \\ + \frac{\zeta}{2}\int_\Omega \rho^{\alpha+t}|z|^t \frac{|\nabla_k \delta|^2}{\delta^2}\phi^2 dzdl\end{aligned} \tag{21}$$

holds for all $\phi \in C_0^2(\Omega \setminus \{0\})$ where $\zeta = \frac{(Q+t-2)^2-\alpha(\alpha+2t)}{4}$.

Proof. To prove the theorem, we use the identity (27) in [6]:

$$\begin{aligned}\int_\Omega |z|^t \rho^\alpha |\nabla_k \phi|^2 dzdl = \left(\frac{\xi}{2}+\alpha t\right)\int_\Omega |z|^t \rho^{\alpha-2}\phi^2 |\nabla_k \rho|^2 dzdl \\ + t\left(\frac{t}{2}-1+n\right)\int_\Omega \rho^\alpha |z|^{t-2}\phi^2 dzdl - \int_\Omega \rho^\alpha |z|^t \phi \Delta\phi dzdl\end{aligned} \tag{22}$$

where $\xi = \alpha(Q+\alpha-2)$. Applying the improved Hardy-type inequality in (13) on the right hand side of (22), we have,

$$\begin{aligned}&\left(\frac{\xi}{2}+\alpha t\right)\int_\Omega \rho^{\alpha-2}|z|^t \phi^2 |\nabla_k \rho|^2 dzdl + t\left(\frac{t}{2}+n-1\right)\int_\Omega \rho^\alpha |z|^{t-2}\phi^2 dzdl \\ &\quad - \int_\Omega \rho^\alpha |z|^t \phi \Delta\phi dzdl \\ &\geqslant \left(\frac{Q+\alpha+t-2}{2}\right)^2 \int_\Omega \rho^{\alpha-2}|z|^t |\nabla_k \rho|^2 \phi^2 dzdl + \frac{1}{4}\int_\Omega \frac{\rho^{\alpha+t}|z|^t \phi^2}{\delta^2}|\nabla_k \delta|^2 dzdl.\end{aligned}$$

After rearranging the above inequality we have,

$$\begin{aligned}-\int_\Omega \rho^\alpha |z|^t \phi \Delta_k \phi dzdl \geqslant \left[\frac{(Q+\alpha+t-2)^2}{4} - \left(\frac{\xi}{2}+\alpha t\right)\right]\int_\Omega \rho^{\alpha-2}|z|^t |\nabla_k \rho|^2 \phi^2 dzdl \\ - t\left(\frac{t}{2}+n-1\right)\int_\Omega \rho^\alpha |z|^{t-2}\phi^2 dzdl \\ + \frac{1}{4}\int_\Omega \frac{\rho^{\alpha+t}|z|^t}{\delta^2}|\nabla_k \delta|^2 \phi^2 dzdl.\end{aligned} \tag{23}$$

Applying the Young's inequality to the term on the left hand side, we have,

$$-\int_{\Omega}\rho^{\alpha}\phi\Delta_k\phi dzdl \leqslant \varepsilon\int_{\Omega}\rho^{\alpha-2}|z|^t|\nabla_k\rho|^2\phi^2 dzdl+\frac{1}{4\varepsilon}\int_{\Omega}\frac{\rho^{\alpha+2}|z|^t}{|\nabla_k\rho|^2}|\Delta_k\phi|^2 dzdl, \quad (24)$$

where $\varepsilon>0$. Substituting (24) into (23), we obtain,

$$\int_{\Omega}\frac{\rho^{\alpha+2}|z|^t}{|\nabla_k\rho|^2}|\Delta_k\phi|^2 dzdl \geqslant f(\alpha,Q,t,\varepsilon)\int_{\Omega}\rho^{\alpha-2}|z|^t|\nabla_k\rho|^2\phi^2 dzdl$$
$$+g(t,n,\varepsilon)\int_{\Omega}\rho^{\alpha}|z|^{t-2}\phi^2 dzdl+\varepsilon\int_{\Omega}\frac{\rho^{\alpha+t}|z|^t}{\delta^2}\phi^2|\nabla_k\delta|^2 dzdl.$$

Here

$$f(\alpha,Q,t;\varepsilon)=4\varepsilon(\zeta-\varepsilon), \quad g(\xi,Q,\varepsilon)=-2\varepsilon t(t+2n-2)$$

where $\zeta=[(Q+t-2)^2-\alpha(\alpha+2t)]/4$. Note that the function $f(\alpha,Q,t,\varepsilon)$ attains its maximum for $\varepsilon=\zeta/2$ and this maximum is equal to $f(\alpha,Q,t,\varepsilon)=\zeta^2$. Therefore, we finally obtain the desired inequality:

$$\int_{\Omega}\frac{\rho^{\alpha+2}|z|^t}{|\nabla_k\rho|^2}|\Delta_k\phi|^2 dzdl \geqslant \zeta^2\int_{\Omega}\rho^{\alpha-2}|z|^t|\nabla_k\rho|^2\phi^2 dzdl$$
$$-\zeta t(t+2n-2)\int_{\Omega}\rho^{\alpha}|z|^{t-2}\phi^2 dzdl$$
$$+\frac{\zeta}{2}\int_{\Omega}\rho^{\alpha+t}|z|^t\frac{|\nabla_k\delta|^2}{\delta^2}\phi^2 dzdl. \quad \square$$

If different models are chosen for the distance function δ mentioned in the theorem above, of course under the condition $-\nabla_k\cdot(\rho^{2-t-Q}|z|^t\nabla_k\delta)\geqslant 0$, various weighted improved Rellich type inequalities can be obtained in accordance with the theorem. For example, the choices in (18) can be made for δ function.

COROLLARY 3. *Let Ω be a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^{2n+1}$ and $\delta:=\ln\frac{R}{\rho}$ where $R>\sup_{\Omega}\rho$. Let $\alpha\in\mathbb{R}$, $t\in\mathbb{R}$, $Q\geqslant 3$, $t+Q+\alpha-2>0$ and $Q+t+2k-2>0$. Then the following inequality is valid for all $\phi\in C_0^{\infty}(\Omega\setminus\{0\})$:*

$$\int_{\Omega}\frac{\rho^{\alpha+2}|z|^t}{|\nabla_k\rho|^2}|\Delta_k\phi|^2 dzdl \geqslant \zeta^2\int_{\Omega}\rho^{\alpha-2}|z|^t|\nabla_k\rho|^2\phi^2 dzdl$$
$$-\zeta t(t+2n-2)\int_{\Omega}\rho^{\alpha}|z|^{t-2}\phi^2 dzdl \quad (25)$$
$$+\frac{\zeta}{2}\int_{\Omega}\rho^{\alpha+t-2}|z|^t\left(\ln\frac{R}{\rho}\right)^{-2}|\nabla_k\rho|^2\phi^2 dzdl$$

where $\zeta=\frac{(Q+t-2)^2-\alpha(\alpha+2t)}{4}$.

COROLLARY 4. *Let B_R be a ball domain with center 0 and radius R and $\delta := R-\rho$. Let $\alpha\in\mathbb{R}$, $t\in\mathbb{R}$, $Q\geqslant 3$, $t+Q+\alpha-2>0$ and $Q+t+2k-2>0$. Then, for all $\phi\in C_0^\infty(B_R\setminus\{0\})$ we have,*

$$\int_{B_R}\frac{\rho^{\alpha+2}|z|^t}{|\nabla_k\rho|^2}|\Delta_k\phi|^2dzdl \geqslant \zeta^2\int_{B_R}\rho^{\alpha-2}|z|^t|\nabla_k\rho|^2\phi^2dzdl - \zeta t(t+2n-2)\int_{B_R}\rho^{\alpha}|z|^{t-2}\phi^2dzdl + \frac{\zeta}{2}\int_{B_R}\rho^{\alpha+t-2}|z|^t\left(\ln\frac{R}{\rho}\right)^{-2}|\nabla_k\rho|^2\phi^2dzdl \tag{26}$$

where $\zeta=\frac{(Q+t-2)^2-\alpha(\alpha+2t)}{4}$.

THEOREM 3. *Let Ω be a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^{2n+1}$. Let ρ and $|z|$ be nonnegative functions on Ω . Furthermore, let $\delta\in C_0^2(\Omega)$ be positive function such that $-\nabla_k\cdot(\rho^{2-t-Q}|z|^t\nabla_k\delta)\geqslant 0$ in Ω. Let $Q\geqslant 3$, $Q+t+\alpha-2>0$ and $Q+t+2k-2>0$. Then the following inequality is valid $\phi\in C_0^\infty(\Omega\setminus\{0\})$:*

$$\int_{\Omega}\frac{\rho^{\alpha+2}|z|^t}{|\nabla_k\rho|^2}|\Delta_k\phi|^2dzdl \geqslant \frac{4\zeta^2}{(Q+\alpha+t-2)^2}\int_{\Omega}|z|^t\rho^{\alpha}|\nabla_k\phi|^2dzdl + \zeta\frac{(\alpha+2\alpha t+\zeta)}{(Q+\alpha+t-2)^2}\int_{\Omega}\frac{\rho^{\alpha+t}|z|^t|\nabla_k\delta|^2}{\delta^2}\phi^2dzdl - \zeta t(t+2n-2)\int_{\Omega}\rho^{\alpha}|z|^{t-2}\phi^2dzdl \tag{27}$$

where $\zeta=\frac{(Q+t-2)^2-\alpha(\alpha+2t)}{4}$.

Proof. Substituting the improved Hardy type inequality in (13) into the inequality (36) in [6], then rearranging the resulting inequality, we have,

$$\int_{\Omega}\frac{\rho^{\alpha+2}|z|^t}{|\nabla_k\rho|^2}|\Delta_k\phi|^2d \geqslant f(Q,\xi,\varepsilon)\int_{\Omega}\rho^{\alpha}|z|^t|\nabla_k\phi|^2dzdl + g(Q,\xi,\varepsilon)\int_{\Omega}\frac{\rho^{\alpha+t}|z|^t}{\delta^2}\phi^2|\nabla_k\delta|^2dzdl + h(Q,\xi,\varepsilon)\int_{\Omega}\rho^{\alpha}|z|^{t-2}\phi^2dzdl$$

where $f(Q,\xi,\varepsilon)=4\varepsilon[1-\frac{2(\xi+2\alpha t+2\varepsilon)}{(Q+\alpha+t-2)^2}]$, $g(Q,\xi,\varepsilon)=\frac{2\varepsilon(\xi+2\alpha t+2\varepsilon)}{(Q+\alpha+t-2)^2}$, $h(Q,\xi,\varepsilon)=-2\varepsilon t(t+2n-2)$. Here note that, the function f has the maximum of $\frac{4\zeta^2}{(Q+\alpha+t-2)^2}$ at

$\varepsilon = \frac{\zeta}{2}$. Thus, we obtain the desired inequality

$$\begin{aligned}\int_\Omega \frac{\rho^{\alpha+2}|z|^t}{|\nabla_k\rho|^2}|\Delta_k\phi|^2 dzdl \geqslant{}& \frac{4\zeta^2}{(Q+\alpha+t-2)^2}\int_\Omega |z|^t\rho^\alpha|\nabla_k\phi|^2 dzdl \\ &+\zeta\frac{(\alpha+2\alpha t+\zeta)}{(Q+\alpha+t-2)^2}\int_\Omega \frac{\rho^{\alpha+t}|z|^t|\nabla_k\delta|^2}{\delta^2}\phi^2 dzdl \\ &-\zeta t(t+2n-2)\int_\Omega \rho^\alpha|z|^{t-2}\phi^2 dzdl. \quad \square\end{aligned}$$

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