Mathematical
Inequalities
& Applications
Volume 21, Number 4 (2018), 945–956 doi:10.7153/mia-2018-21-64

ON THE PTOLEMY CONSTANT OF SOME CONCRETE BANACH SPACES

ZHAN-FEI ZUO

(*Communicated by S. Varošanec*)

Abstract. In this paper, we firstly consider the relations involving the Ptolemy constant of the norms $\|.\|_\psi$ and $\|.\|_\phi$, where the convex functions and ψ and ϕ are comparable. Secondly, we determine this constant when norm is a mean of two norms. Finally, the constant was calculated for some concrete Banach spaces.

1. Introduction

Let X be a Banach space, and $S_X = \{x \in X : \|x\| = 1\}$, $B_X = \{x \in X : \|x\| \leqslant 1\}$ be the unit sphere and unit ball of X, respectively. Many geometric constants for a Banach space X have been investigated, such as the James constant $J(X)$ [3], von Neumann-Jordan constants $C_{NJ}(X)$ [9] and Ptolemy constant $C_p(X)$ (cf. [10, 18, 23–24]). These constants are important due to its strong connection with some useful geometric properties, such as uniformly nonsquareness and uniform normal structure (cf. [3, 5, 9–10, 19, 22, 25]). Moreover, the Ptolemy constant $C_p(X)$ turns out to be useful in the study of the equivalence of Green's functions of second-order linear elliptic operators [17]. It is also a useful tool in the study of the existence of positive solutions of certain nonlinear equations [6]. As mentioned above, it is thus meaningful to calculate the exact value of some constants in some concrete spaces (cf. [3–5, 7–9, 12–14, 20–21, 23–24]).

So far, the exact value of the von Neumann-Jordan constants $C_{NJ}(X)$ and James constant $J(X)$ have been calculated for many classical spaces, such as the Lebesgue space, the Lebesgue-Bochner spaces [7], the Lortenze sequence space [8] and the Bynum space [5]. Naturally, one hope to know the exact value of the Ptolemy constant $C_p(X)$ for these spaces. E. Llorens-Fuster [10] study the relationships between the Ptolemy constant $C_p(X)$ and several other geometric properties of normed spaces X. They also use renorming to calculate the precise values of the Ptolemy constant $C_p(X)$ for several normed spaces. In [23], we give a simple method to determine the Ptolemy constant $C_p(X)$ of absolute normalized norms on $\mathbb{R}^2$, which are complementary to the Llorens's results in [10]. Moreover, the exact values of the Ptolemy constant $C_p(X)$ were calculated in some classical Banach spaces, such as the space ℓ_p, Cesàro space $ces_p^{(2)}$, Lorentz sequence spaces $d^{(2)}(\omega,2)$ etc (cf. [23–24]). However, the exactly values for the Ptolemy constant $C_p(X)$ remain undiscovered in some concrete

Mathematics subject classification (2010): Primary 46B20, Secondary 46B25.
Keywords and phrases: The Ptolemy constant, Absolute normalized norm, Convex function.

Paper MIA-21-64

Banach spaces. In the present paper, we are interested in determining the Ptolemy constant $C_p(X)$ for some more concrete Banach spaces.

The remaining part of this paper is organized as follows. In Section 2, we recall some definitions and well known results, which shall be used in an essential way in the proofs of the main results. In Section 3, the relations involving the Ptolemy constant of the norms $\|.\|_\psi$ and $\|.\|_\phi$ were considered, where the convex functions and ψ and ϕ are comparable. As an application, in Section 4, we can compute the value of the Ptolemy constant $C_p(X)$ for more concrete Banach spaces. The new results which not only contain some previous results in [23], but also give the exactly values for the Ptolemy constant $C_p(X)$ in some more concrete Banach spaces.

2. Preliminaries

Let us first recall some definitions of some canstants:

$$J(X) = \sup\{\min\{\|x+y\|, \|x-y\|\} : x,y \in S_X\}$$

$$C_p(X) := \sup\left\{ \frac{\|x-y\|\|z\|}{\|x-z\|\|y\| + \|z-y\|\|x\|} : x,y,z \in X\setminus\{0\},\ x \neq y \neq z \neq x \right\}$$

It is well known that $1 \leqslant C_p(X) \leqslant 2$ for all normed spaces X. The Ptolemy inequality shows that $C_p(H) = 1$, whenever $(H, \|.\|)$ is an inner product space. In fact, the Ptolemy inequality and the Ptolemy constant $C_p(X)$ are meaningful in metric spaces too (see [10]). Recall that a norm on $\mathbb{R}^2$ is called absolute if $\|(z,w)\| = \|(|z|,|w|)\|$ for all $z,w \in \mathbb{R}$ and normalized if $\|(1,0)\| = \|(0,1)\| = 1$. Let N_α denote the family of all absolute normalized norms on $\mathbb{R}^2$, and let Ψ denote the family of all continuous convex functions on $[0,1]$ such that $\psi(1) = \psi(0) = 1$ and $\max\{1-t,t\} \leqslant \psi(t) \leqslant 1$ $(0 \leqslant t \leqslant 1)$. It has been shown that N_α and Ψ are one-to-one correspondence in view of the following theorem (see [2]).

THEOREM 1. *If $\|.\| \in N_\alpha$, then $\psi(t) = \|(1-t,t)\| \in \Psi$. On the other hand, if $\psi(t) \in \Psi$, defined a norm $\|.\|_\psi$ as*

$$\|(z,\omega)\|_\psi := \begin{cases} (|z|+|\omega|)\psi\left(\frac{|\omega|}{|z|+|\omega|}\right), & (z,\omega) \neq (0,0); \\ 0, & (z,\omega) = (0,0). \end{cases}$$

then the norm $\|.\|_\psi \in N_\alpha$.

A simple example of absolute normalized norm is usual l_p $(1 \leqslant p \leqslant \infty)$ norm. From Theorem 1, one can easily get the corresponding function of the l_p norm:

$$\psi_p(t) = \begin{cases} \{(1-t)^p + t^p\}^{1/p}, & 1 \leqslant p < \infty, \\ \max\{1-t,t\}, & p = \infty. \end{cases}$$

Also, the above correspondence enable us to get many non-l_p norms on $\mathbb{R}^2$, such as the below Examples 1–4. One of the properties of these norms is stated in the following result.

THEOREM 2. *Let* $\psi, \phi \in \Psi$ *and* $\phi \leqslant \psi$. *Put* $M = \max_{0 \leqslant t \leqslant 1} \frac{\psi(t)}{\phi(t)}$, *then*

$$\|.\|_\phi \leqslant \|.\|_\psi \leqslant M\|.\|_\phi.$$

In the paper [12], Mitani and Saito showed a practical calculation method for James constant $J(X)$ of an absolute norm on $\mathbb{R}^2$ and gave the following theorem.

THEOREM 3. *Let* $\psi \in \Psi$, *then*

$$J(\|.\|_\psi) = \max_{0 \leqslant t \leqslant 1/2} \frac{2-2t}{\psi(t)} \psi\left(\frac{1}{2-2t}\right). \tag{1}$$

Throughout this paper, we will use notation $f_\psi(t)$ for the function on the right-hand side of (1) i.e.

$$f_\psi(t) = \frac{2-2t}{\psi(t)} \psi\left(\frac{1}{2-2t}\right)$$

As a consequence of the above Theorem, they obtained the exact value of $J(\|.\|_\psi)$ for function ψ which is comparable with ψ_2. In [23], we consider the Ptolemy constant $C_p(X)$ of absolute normalized norms on $\mathbb{R}^2$ and get the following Lemma.

LEMMA 1. *Let* $\|.\|$ *and* $|.|$ *be two equivalent norms on* X, *namely for* $b \geqslant a > 0$, $a|.| \leqslant \|.\| \leqslant b|.|$, *then*

$$\frac{a^2 C_p(|.|)}{b^2} \leqslant C_p(\|.\|) \leqslant \frac{b^2 C_p(|.|)}{a^2}$$

Moreover, if $\|x\| = a|x|$, *then* $C_p(\|.\|) = C_p(|.|)$.

3. Main results

In this section, we obtain some Theorems which give the relations involving the Ptolemy constant in the case when $\psi \leqslant \phi$ or $\psi \geqslant \phi$. Now, let us put

$$M_1 = \max_{0 \leqslant t \leqslant 1} \frac{\phi(t)}{\psi(t)} \text{ and } M_2 = \max_{0 \leqslant t \leqslant 1} \frac{\psi(t)}{\phi(t)}.$$

For a norm $\|.\|$ on $\mathbb{R}^2$, we write $C_p(\|.\|)$ for $C_p(\mathbb{R}^2, \|.\|)$.

THEOREM 4. *Let* $\psi, \phi \in \Psi$ *and* $\psi \leqslant \phi$, *if the function* $\frac{\phi(t)}{\psi(t)}$ *attains its maximum at* $t = \frac{1}{2}$ *and* $C_p(\|.\|_\phi) = \frac{1}{2\phi^2(\frac{1}{2})}$, *then*

$$C_p(\|.\|_\psi) = \frac{1}{2\psi^2(\frac{1}{2})}.$$

Proof. By Theorem 2, we have $\frac{1}{M_1}\|.\|_\phi \leqslant \|.\|_\psi \leqslant \|.\|_\phi$. Using Lemma 1 with $a = \frac{1}{M_1}$ and $b = 1$, we get the following inequality.

$$C_p(\|.\|_\psi) \leqslant M_1^2 C_p(\|.\|_\phi).$$

Note that the function $\frac{\phi(t)}{\psi(t)}$ attains its maximum at $t = \frac{1}{2}$, i.e., $M_1 = \frac{\phi(\frac{1}{2})}{\psi(\frac{1}{2})}$ and $C_p(\|.\|_\phi) = \frac{1}{2\phi^2(\frac{1}{2})}$, then

$$C_p(\|.\|_\psi) \leqslant M_1^2 C_p(\|.\|_\phi) = \frac{1}{2\psi^2(\frac{1}{2})}. \tag{2}$$

On the other hand, let us put $x = (\frac{1}{2}, \frac{1}{2}), y = (\frac{1}{2}, \frac{-1}{2}), z = (1, 0)$, then

$$\begin{aligned}
\frac{\|x-y\|_\psi \|z\|_\psi}{\|x-z\|_\psi \|y\|_\psi + \|z-y\|_\psi \|x\|_\psi} &= \frac{\|(0,1)\|_\psi \|(1,0)\|_\psi}{\|(\frac{-1}{2}, \frac{1}{2})\|_\psi \|(\frac{1}{2}, \frac{-1}{2})\|_\psi + \|(\frac{1}{2}, \frac{1}{2})\|_\psi \|(\frac{1}{2}, \frac{1}{2})\|_\psi} \\
&= \frac{1}{2\psi^2(\frac{1}{2})} = M_1^2 C_p(\|.\|_\phi).
\end{aligned}$$

From (2) and the above equality, we have

$$C_p(\|.\|_\psi) = M_1^2 C_p(\|.\|_\phi) = \frac{1}{2\psi^2(\frac{1}{2})}. \quad \square$$

REMARK 1. If $\phi = \psi_2$, then we can get the value of $C_p(\|.\|_\psi)$ in [23]. In fact, we can also get the general result in the following.

COROLLARY 1. *Let $\psi \in \Psi$ and $\psi \leqslant \psi_p$ $(2 \leqslant p < \infty)$, if the function $\frac{\psi_p(t)}{\psi(t)}$ attains its maximum at $t = \frac{1}{2}$, then*

$$C_p(\|.\|_\psi) = \frac{1}{2\psi^2(\frac{1}{2})}.$$

Proof. In the case of $2 \leqslant p < \infty$, it is well known that $C_p(\|.\|_p) = 2^{\frac{2}{q}-1} = \frac{1}{2\psi_p^2(\frac{1}{2})}$, where p and q are conjugate exponents. Therefore, we have $C_p(\|.\|_\psi) = \frac{1}{2\psi^2(\frac{1}{2})}$ by Theorem 4. $\square$

THEOREM 5. *Let $\psi, \phi \in \Psi$ and $\psi \geqslant \phi$, if the function $\frac{\psi(t)}{\phi(t)}$ attains its maximum at $t = \frac{1}{2}$ and $C_p(\|.\|_\phi) = 2\phi^2(\frac{1}{2})$, then*

$$C_p(\|.\|_\psi) = 2\psi^2\left(\frac{1}{2}\right).$$

Proof. By Theorem 2, we have $\|.\|_\phi \leqslant \|.\|_\psi \leqslant M_2\|.\|_\phi$. Using Lemma 1 with $a=1$ and $b=M_2$, we get the following inequality.

$$C_p(\|.\|_\psi) \leqslant M_2^2 C_p(\|.\|_\phi).$$

Note that the function $\frac{\psi(t)}{\phi(t)}$ attains its maximum at $t=\frac{1}{2}$, i.e., $M_2 = \frac{\psi(\frac{1}{2})}{\phi(\frac{1}{2})}$ and $C_p(\|.\|_\phi) = 2\phi^2(\frac{1}{2})$, then

$$C_p(\|.\|_\psi) \leqslant M_2^2 C_p(\|.\|_\phi) = 2\psi^2(\frac{1}{2}). \tag{3}$$

On the other hand, let us put $x=(1,0)$, $y=(0,1)$, $z=(1,1)$, then

$$\begin{aligned}\frac{\|x-y\|_\psi\|z\|_\psi}{\|x-z\|_\psi\|y\|_\psi+\|z-y\|_\psi\|x\|_\psi} &= \frac{\|(1,-1)\|_\psi\|(1,1)\|_\psi}{\|(0,-1)\|_\psi\|(0,1)\|_\psi+\|(1,0)\|_\psi\|(1,0)\|_\psi}\\ &= 2\psi^2(\frac{1}{2}) = M_2^2 C_p(\|.\|_\phi)\end{aligned}$$

From (3) and the above equality, we have

$$C_p(\|.\|_\psi) = M_2^2 C_p(\|.\|_\phi) = 2\psi^2\left(\frac{1}{2}\right). \quad \square$$

Using the same idea, we obtain the corresponding results, stated in the following Remark 2 and Corollary 2, we omit the proofs.

REMARK 2. If $\phi = \psi_2$, then we can get the value of $C_p(\|.\|_\psi)$ in [23].

COROLLARY 2. *Let $\psi \in \Psi$ and $\psi \geqslant \psi_p$ $(1 \leqslant p \leqslant 2)$, if the function $\frac{\psi(t)}{\psi_p(t)}$ attains its maximum at $t=\frac{1}{2}$, then*

$$C_p(\|.\|_\psi) = 2\psi^2\left(\frac{1}{2}\right).$$

In fact, from Theorem 4 and Theorem 5, we can get some results related to the general mean. Firstly, we give the definition of general mean. If a and b are real numbers, then any number $m(a,b)$ is called a mean of numbers a and b if it satisfies

$$a \leqslant m(a,b) \leqslant b.$$

One of the most known mean is the weighted mean of order s defined as

$$M^{[s]}(a,b;\omega,1-\omega) = \begin{cases} (\omega a^s + (1-\omega)b^s)^{1/s}, & s \neq 0, +\infty, -\infty \\ a^\omega b^{1-\omega}, & s=0 \\ \max\{a,b\}, & s=\infty \\ \min\{a,b\}, & s=-\infty \end{cases}$$

where a,b are positive real numbers, $\omega \in (0,1)$. Of course, if s is positive, then a,b can be non-negative numbers.

Now, let us state a Theorem related to the general mean and then applied it to the weighted mean of order s, see Example 3.

THEOREM 6. *Let $\psi,\phi \in \Psi$ and $\psi \leqslant \phi$, $m(\psi,\phi)$ be a mean of functions ψ,ϕ and function $m(.)$ be a convex function, then*

(i) $\frac{m}{\psi}$ attains its maximum at $t=\frac{1}{2}$ and $C_p(\|.\|_\psi)=2\psi^2(\frac{1}{2})$, then

$$C_p(\|.\|_m)=2m^2\left(\frac{1}{2}\right).$$

(ii) $\frac{\phi}{m}$ attains its maximum at $t=\frac{1}{2}$ and $C_p(\|.\|_\phi)=\frac{1}{2\phi^2(\frac{1}{2})}$, then

$$C_p(\|.\|_m)=\frac{1}{2m^2(\frac{1}{2})}.$$

Proof. Any mean $m(\psi,\phi)$ has the property

$$\psi \leqslant m(\psi,\phi) \leqslant \phi.$$

Since $\psi,\varphi \in \Psi$ and after the assumption $m(.)$ is convex, it is easy to check that $m(.) \in \Psi$. Now, statements of this Theorem follow by results of Theorems 4 and Theorems 5. □

Moreover, when the function ψ is symmetric with respect to $\frac{1}{2}$, we have the following general results.

THEOREM 7. *Let $\psi,\phi \in \Psi$ and $\psi \leqslant \phi$, if $\psi(t)=\psi(1-t)$ for all $t\in[0,1]$ and $\sqrt{2}M_1\sqrt{C_p(\|.\|_\phi)} \in f_\psi([0,\frac{1}{2}])$, then $C_p(\|.\|_\psi)=M_1^2C_p(\|.\|_\phi)$.*

Proof. Let $t_0\in[0,\frac{1}{2}]$ be a number such that $\sqrt{2}M_1\sqrt{C_p(\|.\|_\phi)}=f_\psi(t_0)$, let us put $x=\frac{1}{\psi(t_0)}(1-t_0,t_0)$, $y=\frac{1}{\psi(t_0)}(-t_0,1-t_0)$, $z=\frac{1}{\psi(t_0)}(1-2t_0,1)$, note that ψ is symmetric with respect to $\frac{1}{2}$, then

$$\|x\|_\psi=\frac{1}{\psi(t_0)}\psi\left(\frac{\frac{t_0}{\psi(t_0)}}{\frac{1}{\psi(t_0)}}\right)=\frac{1}{\psi(t_0)}\psi(t_0)=1,$$

$$\|y\|_\psi=\frac{1}{\psi(t_0)}\psi\left(\frac{\frac{1-t_0}{\psi(t_0)}}{\frac{1}{\psi(t_0)}}\right)=\frac{1}{\psi(t_0)}\psi(1-t_0)=1,$$

$$\|z\|_\psi=\frac{2-2t_0}{\psi(t_0)}\psi\left(\frac{1}{2-2t_0}\right)=f_\psi(t_0),$$

$$\|x-y\|_\psi = \frac{2-2t_0}{\psi(t_0)}\psi\left(\frac{1-2t_0}{2-2t_0}\right) = \frac{2-2t_0}{\psi(t_0)}\psi\left(\frac{1}{2-2t_0}\right) = f_\psi(t_0).$$

Consequently, we have

$$\frac{\|x-y\|_\psi\|z\|_\psi}{\|x-z\|_\psi\|y\|_\psi+\|z-y\|_\psi\|x\|_\psi} = \frac{f_\psi^2(t_0)}{2} = M_1^2 C_p(\|.\|_\phi).$$

From the definition of $C_p(X)$, implies that

$$C_p(\|.\|_\psi) \geqslant M_1^2 C_p(\|.\|_\phi). \tag{4}$$

Combining this result (4) with the inequality from Theorem 4 that

$$C_p(\|.\|_\psi) \leqslant M_1^2 C_p(\|.\|_\phi),$$

we have the equality

$$C_p(\|.\|_\psi) = M_1^2 C_p(\|.\|_\phi). \quad \square$$

Similar proofs hold for the following Theorem 8, we omit it.

THEOREM 8. *Let $\psi,\phi \in \Psi$ and $\psi \geqslant \phi$, if $\psi(t)=\psi(1-t)$ for all $t\in[0,1]$ and $\sqrt{2}M_2\sqrt{C_p(\|.\|_\phi)} \in f_\psi([0,\frac{1}{2}])$, then $C_p(\|.\|_\psi) = M_2^2 C_p(\|.\|_\phi)$.*

4. Some Examples

As the application of the above Theorems, we will calculate the exactly values of $C_p(X)$ for some concrete Banach space. These results which not only contain the previous results in [23], but also give some new supplement results.

EXAMPLE 1. Let $X=\mathbb{R}^2$ with the norm $\|.\|_{p,q,\lambda} = \max\{\|.\|_p, \lambda\|.\|_q\}$, where $1 \leqslant q \leqslant p \leqslant \infty$ and $\lambda \in [2^{\frac{1}{p}-\frac{1}{q}}, 1]$, then

$$C_p(\|.\|_{p,q,\lambda}) = \begin{cases} \lambda^2 2^{\frac{2}{q}-1} & \text{if } 1\leqslant q<p\leqslant 2 \\ \lambda^2 2^{\frac{2}{q}-1} & \text{if } 1\leqslant q<2<p\leqslant\infty \text{ and } \lambda\in(2^{\frac{1}{2}-\frac{1}{q}},1) \\ \frac{2^{1-\frac{2}{q}}}{\lambda^2} & \text{if } 2\leqslant q<p\leqslant\infty \end{cases}$$

Proof. It is very easy to check that $\|.\|_{p,q,\lambda} = \max\{\|.\|_p, \lambda\|.\|_q\} \in \mathbb{N}_\alpha$ and its corresponding function is

$$\psi(t) = \|(1-t,t)\| = \max\{\psi_p(t), \lambda\psi_q(t)\}.$$

Let $t_0\in[0,\frac{1}{2}]$ be a point such that $\psi_p(t_0)=\lambda\psi_q(t_0)$, then we have

$$\psi(t) = \begin{cases} \psi_p(t) & t\in[0,t_0] \\ \lambda\psi_q(t) & t\in\left[t_0,\frac{1}{2}\right] \end{cases}$$

In fact, $\psi(t)$ is symmetric with respect to $t=\frac{1}{2}$, which is expanded to the whole interval $[0,1]$.

(i) Suppose that $1 \leqslant q < p \leqslant 2$, from the definition it is obvious that $\psi(t) \geqslant \psi_p(t)$ and the function

$$\frac{\psi(t)}{\psi_p(t)} = \begin{cases} 1 & t \in [0,t_0] \cup [1-t_0,1] \\ \frac{\lambda \psi_q(t)}{\psi_p(t)} & t \in [t_0,1-t_0] \end{cases}$$

attains its maximum at $t = \frac{1}{2}$. Hence, from Corollary 2, we have

$$C_p(\|.\|) = \lambda^2 2^{\frac{2}{q}-1}.$$

(ii) Suppose that $1 \leqslant q < 2 < p \leqslant \infty$, note that $\psi(t) \geqslant \psi_2(t)$ if and only if $\lambda \in [2^{\frac{1}{2}-\frac{1}{q}},1)$, and it turns out that $\frac{\psi(t)}{\psi_2(t)}$ takes the maximum at $t = \frac{1}{2}$. By Corollary 2, we get that

$$C_p(\|.\|) = \lambda^2 2^{\frac{2}{q}-1}.$$

(iii) Suppose that $2 \leqslant q < p \leqslant \infty$, since $\psi_p(t) \leqslant \psi_q(t)$ and $\lambda \psi_q(t) \leqslant \psi_q(t)$, then $\psi(t) \leqslant \psi_q(t)$, it is easy check that the function

$$\frac{\psi_q(t)}{\psi(t)} = \begin{cases} \frac{\psi_q(t)}{\psi_p(t)} & t \in [0,t_0] \cup [1-t_0,1] \\ \frac{1}{\lambda} & t \in [t_0,1-t_0] \end{cases}$$

attains its maximum at $t = \frac{1}{2}$. By Corollary 1, we get that

$$C_p(\|.\|) = \frac{2^{1-\frac{2}{q}}}{\lambda^2}. \quad \square$$

REMARK 3. In fact, let us take $p = 2,\ q = 1$, we can get the results of Example 2.6 in [23]. Suppose that $p = \infty,\ q = 2$, we obtain the Example 2.7 in [23] by a simple transform. However, there are some problems which remain unsolved; the exact values of $C_p(\|.\|_{p,q,\lambda})$ for the case $1 \leqslant q < 2 < p \leqslant \infty$ and $\lambda \in (2^{\frac{1}{p}-\frac{1}{q}}, 2^{\frac{1}{2}-\frac{1}{q}})$.

In [23], the exact values of $C_p(d^{(2)}(\omega,2))$ were calculated. In the following, we can compute the Ptolemy constant of two-dimensional Lorentz sequence spaces $d^{(2)}(\omega,q)\,(2 \leqslant q < \infty)$ by Corollary 1.

EXAMPLE 2. Let $0 < \omega < 1$ and $2 \leqslant q < \infty$. The two-dimensional Lorentz sequence space $d^{(2)}(\omega,q)$ is $\mathbb{R}^2$ with the norm

$$\|(x,y)\|_{\omega,q} = ((x^*)^q + \omega(y^*)^q)^{1/q},$$

where (x^*,y^*) is the rearrangement of $(|x|,|y|)$ satisfying $x^* \geqslant y^*$. Then

$$C_p(\|.\|) = 2(\frac{1}{1+\omega})^{2/q}.$$

Proof. It is well known that $\|(x,y)\|_{\omega,q}$ is a symmetric absolute normalized norm on $\mathbb{R}^2$, and the corresponding convex function is

$$\psi_{\omega,q}(t) = \begin{cases} ((1-t)^q + \omega t^q)^{1/q}, & \text{if } 0 \leqslant t \leqslant 1/2, \\ (t^q + \omega(1-t)^q)^{1/q}, & \text{if } 1/2 \leqslant t \leqslant 1. \end{cases}$$

It is check that $\psi_{\omega,q}(t) \leqslant \psi_q(t)$. Repeating the similar arguments in the proof of Example 2.8 in [23], we can get that $\frac{\psi_q(t)}{\psi_{\omega,q}(t)}$ attains its maximum at $t = \frac{1}{2}$. By Corollary 1, we have

$$C_p(\|.\|) = 2\left(\frac{1}{1+\omega}\right)^{2/q}. \quad \square$$

REMARK 4. In fact, since $\psi_{\omega,q}(t)$ is symmetric to $\frac{1}{2}$, take $\phi = \psi_2$ in Theorem 7, we can find a number $t_0 = \frac{1}{2}$ such that $\sqrt{2}M_1\sqrt{C_p(\|.\|_\phi)} = f_\psi(t_0)$, from Theorem 7, we can also get the above result. Similarly, we can also obtain the results in Example 2.9 in [23] by Theorem 7, and Example 2.6 in [23] by Theorem 8.

EXAMPLE 3. Let $1 \leqslant p < q \leqslant \infty$, $1 \leqslant s < \infty$ and $\lambda > 0$, the convex function $\psi_{\lambda,p,q,s}(t)$ is defined on $[0,1]$ as

$$\psi_{\lambda,p,q,s}(t) = (1+\lambda)^{-\frac{1}{s}}(\psi_p^s(t) + \lambda\psi_q^s(t))^{\frac{1}{s}}.$$

i.e. $\psi_{\lambda,p,q,s}(t)$ is a weighted mean of order s of functions ψ_p and ψ_q with weights $\frac{1}{1+\lambda}$ and $\frac{\lambda}{1+\lambda}$. The corresponding norm is

$$\|.\|_{\lambda,p,q,s} = (1+\lambda)^{-\frac{1}{s}}(\|.\|_p^s + \lambda\|.\|_q^s)^{\frac{1}{s}}.$$

Then

(i) If $1 \leqslant p < q \leqslant 2$, then $C_p(\|.\|_{\lambda,p,q,s}) = \frac{1}{2}(1+\lambda)^{\frac{-2}{s}}(2^{\frac{s}{p}} + \lambda 2^{\frac{s}{q}})^{\frac{2}{s}}$.

(ii) If $2 \leqslant p < q \leqslant \infty$, then $C_p(\|.\|_{\lambda,p,q,s}) = 2(1+\lambda)^{\frac{2}{s}}(2^{\frac{s}{q}} + \lambda 2^{\frac{s}{q}})^{\frac{-2}{s}}$.

Proof. Since $\psi_{\lambda,p,q,s}(t)$ is a weighted mean of order s of functions ψ_p and ψ_q, then

$$\psi_q(t) \leqslant \psi_{\lambda,p,q,s}(t) \leqslant \psi_p(t).$$

(i) Let first $1 \leqslant p < q \leqslant 2$, since $\psi_{\lambda,p,q,s}(t) \geqslant \psi_q(t)$ and $\frac{\psi^s_{\lambda,p,q,s}(t)}{\psi_q^s(t)}$ attains its maximum at the same point as $\frac{\psi_p(t)}{\psi_q(t)}$ attains its maximum at $t = \frac{1}{2}$ from a simple calculation. Take $\psi = \psi_q$ and $\phi = \psi_p$ in Theorem 6 (i), we have

$$C_p(\|.\|_{\lambda,p,q,s}) = 2\psi^2_{\lambda,p,q,s}\left(\frac{1}{2}\right) = 2(1+\lambda)^{\frac{-2}{s}}(2^{\frac{s}{p}} + \lambda 2^{\frac{s}{q}})^{\frac{2}{s}}.$$

(ii) Suppose that $2 \leqslant p < q \leqslant \infty$, since $\psi_{\lambda,p,q,s}(t) \leqslant \psi_p(t)$ and $\frac{\psi_p(t)}{\psi_{\lambda,p,q,s}(t)}$ attains its maximum at $t = \frac{1}{2}$. Similarly, take $\psi = \psi_q$ and $\phi = \psi_p$ in Theorem 6 (ii), we have

$$C_p(\|.\|_{\lambda,p,q,s}) = \frac{1}{2\psi^2_{\lambda,p,q,s}(\frac{1}{2})} = 2(1+\lambda)^{\frac{2}{s}}(2^{\frac{s}{q}} + \lambda 2^{\frac{s}{q}})^{\frac{-2}{s}}. \quad \square$$

REMARK 5. In particular, take $p = 2$, $q = \infty$, $s = 2$, we get results of Example 2.8 from [23]. If p,q are numbers from $[1,+\infty)$ and $s = 2$, the Ptolemy constants are calculated in the paper [24]. However, the exact values of $C_p(\|.\|_{p,q,\lambda})$ for the case $1 \leqslant p < 2 < q \leqslant \infty$ remain undiscovered.

In the following, we will calculate the exact values of $C_p(X)$ for the space X^p. We firstly recall the definition of X^p. Let $X = \mathbb{R}^2$ with absolute normalized norm $\|.\|_X$ and with a function $\psi_X \in \Psi$, corresponding to this norm. For any $p \in (1,+\infty)$, space X^p with the norm

$$\|x\| = \||x|^p\|_X^{1/p}.$$

COROLLARY 3. *Let X^p be a two-dimensional Banach spaces with the norm*

$$\|x\| = \||x|^p\|_X^{1/p}.$$

If the corresponding function ψ_X attains its minimum at the point $t = \frac{1}{2}$. For $2 \leqslant p < \infty$, then

$$C_p(\|.\|) = \frac{1}{2\psi^2_{X^p}(\frac{1}{2})}.$$

Proof. From the definition of the norm, it is clear that $\|x\| = \||x|^p\|_X^{1/p} \in \mathbb{N}_\alpha$ and its corresponding convex function is

$$\psi_{X^p}(t) = \|(1-t,t)\|_{X^p} = [(1-t)^p + t^p]^{\frac{1}{p}} \psi_X^{\frac{1}{p}}\left(\frac{t^p}{(1-t)^p + t^p}\right).$$

Since $\psi_X \leqslant 1$, we firstly have $\psi_{X^p} \leqslant \psi_p$, it is evident that

$$\frac{\psi_p(t)}{\psi_{X^p}(t)} = \psi_X^{\frac{-1}{p}}\left(\frac{t^p}{(1-t)^p + t^p}\right).$$

For arbitrary $t \in [0,1]$, the variable $s = \frac{t^p}{(1-t)^p+t^p}$ also belongs to $[0,1]$. Since the function ψ_X attains its minimum at the point $t = \frac{1}{2}$, then $\psi_X\left(\frac{t^p}{(1-t)^p+t^p}\right)$ attains its minimum at $t = \frac{1}{2}$, so the function $\psi_X^{\frac{-1}{p}}\left(\frac{t^p}{(1-t)^p+t^p}\right)$ attains its maximum at $\frac{1}{2}$. By Corollary 1, we get that

$$C_p(\|.\|) = \frac{1}{2\psi^2_{X^p}(\frac{1}{2})}. \quad \square$$

REMARK 6. It is proved that if X is a Banach lattice, then X^p space is a Banach lattice for $p \in (1, +\infty)$. Some results about X^p spaces can be found in [15], [16]. As a application, we give a concrete Example which satisfy the conditions in Corollary 3.

EXAMPLE 4. Let $X = \mathbb{R}^2$, the convex function $\psi(t)$ is defined on $[0,1]$ as

$$\psi_X(t) = (1 - t + t^2)^{\frac{1}{2}}.$$

The corresponding norm is

$$\|(x,y)\| = ((|x|^2 + |x||y|) + |y|^2)^{\frac{1}{2}},$$

It is obvious that $\|(x,y)\|$ is a absolute normalized norm on $\mathbb{R}^2$. By a standard discussion, it is easy to check that the corresponding function $\psi_X(t) = \sqrt{1 - t + t^2}$ attains its minimum at the point $\frac{1}{2}$. For $p \geqslant 2$, then the corresponding space X^p has the norm

$$\|(x,y)\| = ((|x|^{2p} + |x|^p|y|^p + |y|^{2p})^{\frac{1}{2p}}.$$

And the corresponding convex function is

$$\psi_{X^p}(t) = \|(1-t,t)\|_{X^p} = [(1-t)^p + t^p]^{\frac{1}{p}} \psi_X^{\frac{1}{p}} \left(\frac{t^p}{(1-t)^p + t^p} \right).$$

By Corollary 3, we have that

$$C_p(X^p) = \frac{1}{2\psi_{X^p}^2(1/2)} = \frac{2}{3^{\frac{1}{p}}}.$$

Acknowledgements. This research was partly supported by the Scientific and Technological Research Program of Chongqing Municipal Education Commission (Grant No. KJQN201801205), the Chongqing New-star Plan of Science and Technology(No. KJXX2017012), the Scientific Technological Research Program of the Chongqing Three Gorges University (No. 16PY11), the Chongqing Municipal Key Laboratory of Institutions of Higher Education (Grant No. [2017]3), the Program of Chongqing Development and Reform Commission (Grant No. 2017[1007]), the Key Laboratory for Nonlinear Science and System Structure, Chongqing Three Gorges University.

REFERENCES

[1] J. ALONSO AND P. MARTIN, *A counterexample for a conjecture of G. Zbăganu about the Neumann-Jordan constant*, Rev. Roum. Math. Pures Appl. **51**, (2006), 135–141.

[2] F. BONSALL AND J. DUNCAN, *Numerical Ranges II.*, London Mathematical Society Lecture Notes Series, **10**, Cambridge Univ. Press, New York, 1973.

[3] J. GAO AND K. S. LAU, *On two classes Banach spaces with uniform normal structure*, Studia Math. **99** (1991), 41–56.

[4] T. IKEDA AND M. KATO, *Notes on von Neumann-Jordan and James constants for absolute norms on* $\mathbb{R}^2$, Mediterr. J. Math. **11** (2014), 633–642.

[5] A. JIMÉNEZ-MELADO, E. LLORENS-FUSTER AND S. SAEJUNG, *The von Newman-Jordan constant, weak orthogonality and normal structure in Banach spaces*, Proc. Am. Math. Soc. **134** (2006), 355–364.

[6] N. KALTON AND I. VERBITSKY, *Nonlinear equations and weighted norm inequalities*, Trans. Amer. Math. Soc. **351** (1999), 3441–3497.

[7] M. KATO AND Y. TAKAHASHI, *On the von Neumann-Jordan constant for Banach spaces*, Proc. Amer. Math. Soc. **125** (1997), 1055–1062.

[8] M. KATO AND L. MALIGRANDA, *On James and von Neumann-Jordan constants of Lorentz sequence spaces*, J. Math. Anal. Appl. **258** (2001), 457–465.

[9] M. KATO, L. MALIGRANDA AND Y. TAKAHASHI, *On James and Jordan-von Neumann constants and normal structure coefficient of Banach spaces*, Studia Math. **144** (2001), 275–295.

[10] E. LLORENS-FUSTER, E. MAZCUÑÁN-NAVARRO AND S. REICH, *The Ptolemy and Zbăganu constants of normed spaces*, Nonlinear Analysis **72** (2010), 3984–3993.

[11] L. MALIGRANDA AND L. E. PERSSON, *Generalized duality of some Banach function spaces*, Indag. Math. **51** (1989), 323–338.

[12] K. MITANI AND K. SAITO, *The James constant of absolute norms on* $\mathbb{R}^2$, J. Nonlinear Convex Anal. **4** (2003), 399–410.

[13] H. MIZUGUCHI AND K. SAITO, *Some geometric constants of absolute normalized norms on* $\mathbb{R}^2$, Ann. Funct. Anal. **2** (2011), 22–33.

[14] H. MIZUGUCHI AND K. SAITO, *On the upper bound of some geometric constants of absolute normalized norms on* $\mathbb{R}^2$, Mediterr. J. Math. **13** (2016), 309–322.

[15] H. L. NIKOLOVA AND L. E. PERSSON, *Some properties of* X^p *spaces*, in: J. Musielak, H. Hudzik, R. Urbanski (eds.), Function spaces, Teubner-Texte zur Matematik **120** (1991), 174–185.

[16] L. PERSSON (Eds), *Some elementary inequalities in connection with* X^p *spaces*, Constructive theory of functions, Publishing House of the Bulgarian Academy of Sciences, (1988), 367–376.

[17] Y. PINCHOVER, *Maximum and anti-maximum principles and eigenfunctions estimates via perturbation theory of positive solutions of elliptic equations*, Math. Ann. **314** (1999), 555–590.

[18] Y. PINCHOVER, S. REICH AND I. SHAFRIR, *The Ptolemy constant of a normed space*, Amer. Math. Monthly. **108** (2001), 475–476.

[19] S. SAEJUNG, *On James and von Neumann-Jordan constants and sufficient conditions for the fixed point property*, J. Math. Anal. Appl. **323** (2006), 1018–1024.

[20] K. SAITO, M. KATO AND Y. TAKAHASHI, *Von Neumann-Jordan constant of absolute normalized norms on* C^2, J. Math. Anal. Appl. **244** (2000), 515–532.

[21] R. TANAKA, *Tingley's problem on symmetric absolute normalized norms on* R^2, Acta Math. Sin. **30** (2014), 1324–1340.

[22] Z. ZUO AND Y. CUI, *Some modulus and normal structure in Banach space*, Journal of Inequalities and Applications **2009** (2009), Article ID 676373.

[23] Z. ZUO, *The Ptolemy constant of absolute normalized norms on* R^2, Journal of Inequalities and Applications **2012** (2012), 107.

[24] Z. ZUO, *A reconsideration on the Ptolemy Constant of absolute normalized norms*, Acta Mathematica Sinica (Chinese Series) **58** (2015), 337–344.

[25] Z. ZUO AND C. TANG, *Jordan-von Neumann type constant and fixed points for multivalued nonexpansive mappings*, Journal of Mathematical Inequalities. **10** (2016), 649–657.

(Received March 2, 2017)

Zhan-Fei Zuo
Department of Mathematics and Statistics
Chongqing Three Gorges University
Chongqing, 404100
and
Key Laboratory of Intelligent Information
Processing and Control Chongqing Three Gorges University
Chongqing, 404100
e-mail: zuozhanfei@139.com