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NEW ASYMPTOTIC EXPANSION AND ERROR BOUND FOR STIRLING FORMULA OF RECIPROCAL GAMMA FUNCTION

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Abstract. Studying the problem about if certain probability measures are determinate by its moments [4, 8, 10] is useful to know the asymptotic behavior of the probability densities for large values of argument. This requires, previously, the knowledge of the asymptotic expansion of reciprocal Gamma function $1/\Gamma(z)$ when $\Re z$ is large and $\Im z$ is fixed [8]. Then, the well known Stirling formula for large $|z|$ of the Gamma function $\Gamma(z)$ or its reciprocal $1/\Gamma(z)$ is not appropriate for this problem. So, the main aim of this paper is to obtain a new asymptotic expansion for reciprocal Gamma function valid for large $\Re z$ and establish a new explicit error bound for the first term of this expansion, that is, the Stirling formula.

1. Introduction

It's well known that, when $z \to \infty$ in the sector $|arg\, z| < \pi$, the reciprocal Gamma function has the following asymptotic expansion

$$\frac{1}{\Gamma(z)} \sim \frac{1}{\sqrt{2\pi}} z^{-z+\frac{1}{2}}\, e^{z} \sum_{n=0}^{\infty} \frac{\gamma_n}{z^n}, \tag{1}$$

where γ's are the Stirling coefficients. This expansion is frequently used in standard textbooks to illustrate the Saddle Point method (see for example [9, p. 69]). The first term of this expansion is the well known Stirling formula for reciprocal Gamma function:

$$\frac{1}{\Gamma(z)} \sim \frac{1}{\sqrt{2\pi}} z^{-z+\frac{1}{2}}\, e^{z}.$$

In [3], Boyd derived error bounds for the remainder of expansion (1) by using a resurgence formula for the Gamma function arising in a general theory for complex Laplace-type integrals developed by Berry and Howls [2]. More recently in [7], G.Nemes has derived a better error bound for the remainder in (1) when $\Re z > 0$ that modifies and improves Boyd's formula:

$$\frac{1}{\Gamma(z)} = \frac{e^{z} z^{-z+1/2}}{\sqrt{2\pi}} \left(\sum_{n=0}^{N-1} \frac{\gamma_n}{z^n} + \tilde{R}_N(z) \right), \tag{2}$$

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with

$$|\tilde{R}_N(z)| \leqslant \left(\frac{|\gamma_N|}{|z|^N} + \frac{|\gamma_{N+1}|}{|z|^{N+1}}\right) \begin{cases} \csc(2\theta) & \text{if } \frac{\pi}{4} < |\theta| < \frac{\pi}{2}, \\ 1 & \text{if } \quad |\theta| \leqslant \frac{\pi}{4}. \end{cases}$$

On the other hand, in the study of the problem of whether the random variable

$$A_t := \int_0^t e^{2B_s}\, ds$$

is determinate by its moments [8, 10] ((B_s , $0 \leqslant s$) denotes a real valued Brownian motion starting from 0) is convenient to know the asymptotic behavior of the probability densities $f_t(x)$ of A_t

$$f_t(x) = P(A_t \in dx) = \frac{e^{\frac{\pi^2}{8t}}}{2\pi\sqrt{t\,x^3}} \int_{-\infty}^{\infty} e^{-\frac{\cosh^2(u)}{2x}}\, e^{-\frac{u^2}{2t}} \cos\left(\frac{\pi u}{2t}\right) \cosh(u)\, du, \qquad x, t > 0, \tag{3}$$

for large positive x. The knowledge of the exact behavior of these densities when $x \to \infty$ is essential to decide if the measures $d\tau_t = f_t(x)\,dx$ are determinate by its moments. It is shown in [8] that $f_t(x)$ may be written in terms of

$$\Phi_n(x) := \int_{-\infty}^{\infty} \frac{g(y)}{\Gamma\left(n + \frac{1}{2} + x + iy\right)}\, dy \tag{4}$$

where $n \in \mathbb{N}$ and $g(y)$ verifies certain conditions to allow the integral converges. Therefore, we need to analyze the asymptotic behavior of $\Phi_n(x)$ when $x \to \infty$. To this end we can replace, in the right side of (4), the reciprocal Gamma function by it asymptotic expansion (or at least the first term) and interchange sum and integral. In order to make the calculations as simple as possible, we need an asymptotic expansion of reciprocal Gamma function when real part of argument $x \to \infty$ in inverse powers of x (here, formula (2) does not work). In order to be controlled the error, we also need to know an explicit expression for the remainder of this asymptotic expansion.

So, this work has two main purposes: on the one hand, in section 2, by using a modified Saddle Point method [6], we derive a new asymptotic expansion of $1/\Gamma(z)$ for large positive $\Re z > 0$ with an extra property: the explicit formula for its coefficients. Although for our purpose (to know the behavior of Φ_n) is enough with the first term, we calculate the complete expansion because is new in the literature. On the other hand, in section 3, we establish a new error estimate for the first term of this expansion, that is, the Stirling formula for large positive $\Re z > 0$

2. An asymptotic expansion of $1/\Gamma(z)$ for large $\Re z$

We start considering the contour integral representation of reciprocal Gamma function [1, eq. 5.9.2]:

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} e^s s^{-z} ds, \tag{5}$$

where the integration contour $\mathscr{C}$ begins at $-\infty$, circles the origin once in the positive direction, and returns to $-\infty$ as is shown in [1, fig. 5.9.2]. We write $z=x+\alpha+iy$ and change the integration variable $s=xt$; we obtain:

$$\frac{1}{\Gamma(\alpha+x+iy)}=\frac{e^x x^{1-\alpha-x-iy}}{2\pi i}\int_{-\infty}^{(0+)} e^{xf(t)}g(t)dt, \tag{6}$$

with $f(t)=t-\log(t)-1$ and $g(t)=t^{-\alpha-iy}$. In order to derive an asymptotic expansion of (6) for large positive x we apply the modified Saddle Point method (see [6, Theorem 1] for more details) instead the standard Saddle Point: unlike the standard method, the modified provides explicit formulas for the coefficients.

We have that the unique saddle point of the phase function $f(t)$ is $t_0=1$ and $f(1)=0$, $f'(1)=0$, $f''(1)=1$ and $f'''(1)\neq 0$; also $g(1)=1$. Then, following the idea introduced in [6], we rewrite the phase function $f(t)$ in the form of a Taylor polynomial at the saddle point $t_0=1$

$$f(t)=\frac{(t-1)^2}{2}+f_1(t).$$

From [6], we know that it is not necessary to compute the steepest descent paths of $f(t)$ at t_0, but only the steepest descent paths of the quadratic part of $f(t)$ at those points, that are simpler: they are nothing but straight lines. In this case, the steepest descent path of the quadratic part of $f(t)$ at t_0 is the straight line $\widetilde{\Gamma}=\{1+iz|z\in(-\infty,\infty)\}$.

From [6] we also know that it is not necessary to deform the path $\mathscr{C}$ into $\widetilde{\Gamma}$. It is enough to deform $\mathscr{C}$ into a new integration Γ path that contains a portion of $\widetilde{\Gamma}$ that includes t_0. In this case we define $\Gamma=\Gamma_\varepsilon\cup\Gamma_{t_0}$ with

$$\Gamma_\varepsilon=\{z\pm i\pi|z\in(-\infty,1]\}\qquad\text{and}\qquad\Gamma_{t_0}=\{1+iz|z\in[-\pi,\pi]\}$$

In order to apply the method introduced in [6], we need to show that the contribution of the integral on the paths Γ_ε is negligible compared with the contribution of Γ_{t_0}. We have that $\Re(f(t))=t-\frac{1}{2}\ln(t^2+\pi^2)-1$ is increasing on $(-\infty,1]$ and then attains its maximum at $t=1$, that is, $\Re(f(t))\leqslant -M_0$ with $M_0=\frac{1}{2}\ln(1+\pi^2)>0$. Therefore

$$\int_{\Gamma_\varepsilon} e^{xf(t)}g(t)dt=\mathscr{O}(e^{-xM_0}). \tag{7}$$

On the other hand, and using the standar Saddle Point method, it's also easy to prove that

$$\int_{\Gamma_{t_0}} e^{xf(t)}g(t)dt\sim g(t_0)\sqrt{\frac{2\pi}{-f''(t_0)x}}e^{xf(t_0)}=\mathscr{O}\left(\frac{1}{\sqrt{x}}\right). \tag{8}$$

Therefore, apart from the exponentially negligible term (7), the integral (6) over the path $\mathscr{C}$ equals the integral over the path Γ_{t_0}:

$$\int_{-\infty}^{(0+)} e^{xf(t)}g(t)dt\sim\int_{\Gamma_{t_0}} e^{xf(t)}g(t)dt=\int_{-\pi}^{\pi} e^{xf(1+it)}g(1+it)dt$$

Now, we can split the integrand

$$e^{xf(t)}g(t) = e^{\frac{x}{2}(t-1)^2}e^{xf_1(t)}g(t)$$

with $f_1(t) = f(t) - \frac{(t-1)^2 x}{2}$. Putting the Taylor expansion of $e^{xf_1(t)}g(t)$ at t_0 into above integral and interchanging sum by integral we can obtain the desired asymptotic expansion (see [6] for more details):

$$\frac{1}{\Gamma(\alpha+x+iy)} \sim \frac{e^x x^{1/2-\alpha-x-iy}}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \sum_{k=2n-2}^{2n} (-1)^{k+n} a_{2(n+k)}(x) \frac{\Gamma(k+n+1/2)}{\Gamma(1/2)} \left(\frac{2}{x}\right)^{k+n} \tag{9}$$

with

$$a_{2n}(x) := \sum_{k=0}^{n} \frac{(-1)^k x^k}{k!\,2^k} \sum_{j=0}^{2n-2k} \binom{-x-\alpha-iy}{2n-2k-j} \frac{x^j}{j!}.$$

Remark that, in the second sumatory of (9), when k takes the value of $-n$ with n natural must be understood as zero ($k=0$).

The following tables illustrate the approximation of $1/\Gamma(\alpha+x+iy)$ supplied by the expansions given by formula (9) for diferents values of α, x and y

x	n			
	0	2	4	6
125	0.0162002	0.0002263	0.0000377	0.0000304
250	0.008133	0.0000284	$8.495 \cdot 10^{-7}$	$9.344 \cdot 10^{-8}$
500	0.004074	$3.567 \cdot 10^{-6}$	$2.236 \cdot 10^{-8}$	$3.935 \cdot 10^{-10}$
1000	0.002039	$4.468 \cdot 10^{-7}$	$6.414 \cdot 10^{-10}$	$2.417 \cdot 10^{-12}$
2000	0.001021	$5.591 \cdot 10^{-8}$	$2.136 \cdot 10^{-11}$	$2.690 \cdot 10^{-13}$

Table 1: *Relative error for $y=2$, $\alpha=1/2$ and several values of x and the number of terms n of the approximation given by* (9).

x	n			
	0	2	4	6
125	0.020026	0.000377	0.0000715	0.0000639
250	0.010056	0.0000454	$1.396 \cdot 10^{-6}$	$1.667 \cdot 10^{-7}$
500	0.005039	$5.591 \cdot 10^{-6}$	$3.337 \cdot 10^{-8}$	$6.177 \cdot 10^{-10}$
1000	0.002522	$6.936 \cdot 10^{-7}$	$9.053 \cdot 10^{-10}$	$3.517 \cdot 10^{-12}$
2000	0.001261	$8.639 \cdot 10^{-8}$	$2.728 \cdot 10^{-11}$	$1.152 \cdot 10^{-12}$

Table 2: *Relative error for $y=-2$, $\alpha=3/2$ and several values of x and the number of terms n of the approximation given by* (9).

3. Error bound for Stirling formula

We start with the following lemma such will be usefull later.

LEMMA 1. *Let* $f:\mathbb{R}\longrightarrow\mathbb{C}$ *be a twice diferentiable function in* $(a,b)\subseteq\mathbb{R}$. *Then there exist* $c_1,c_2\in(a,b)$ *such that*

$$|f(b)-f(a)|\leqslant(b-a)|f'(c_1)| \tag{10}$$

$$|f(b)-f(a)|\leqslant(b-a)|f'(a)|+\frac{(b-a)^2}{2}|f''(c_2)| \tag{11}$$

Proof. Put $z:=f(b)-f(a)$ and define

$$\varphi(t)=z\bullet f(t)\qquad(a\leqslant t\leqslant b)$$

where $\bullet$ denotes the scalar product in the Hilbert space structure of $\mathbb{C}$. Clearly $\varphi(t)$ is a real-valued function on $[a,b]$ which is two times differentiable in (a,b). Using the Taylor's theorem with Lagrangian remainder

(a) 1 degree Taylor polynomial

$$\varphi(b)=\varphi(a)+(b-a)\varphi'(c)\ \longrightarrow\ \varphi(b)-\varphi(a)=(b-a)z\bullet f'(c)$$

for some $c\in(a,b)$. On the other hand

$$\varphi(b)-\varphi(a)=z\bullet f(b)-z\bullet f(a)=z\bullet z=|z|^2$$

The Cauchy Schwarz inequality gives

$$|z|^2=(b-a)|z\bullet f'(c)|\leqslant(b-a)|z||f'(c)|$$

Henze $|z|\leqslant(b-a)|f'(c)|$, the (10) inequality desired.

(b) 2 degree Taylor polynomial

$$\varphi(b)=\varphi(a)+(b-a)\varphi'(a)+\frac{(b-a)^2}{2}\varphi''(c)\ \longrightarrow$$

$$\varphi(b)-\varphi(a)=(b-a)z\bullet f'(a)+\frac{(b-a)^2}{2}z\bullet f''(c)$$

for some $c\in(a,b)$. The Cauchy Schwarz inequality now gives

$$\begin{aligned}|z|^2&\leqslant(b-a)|z\bullet f'(a)|+\frac{(b-a)^2}{2}|z\bullet f''(c)|\\&\leqslant(b-a)|z||f'(a)|+\frac{(b-a)^2}{2}|z||f''(c)|\end{aligned}$$

Henze $|z|\leqslant(b-a)|f'(a)|+\dfrac{(b-a)^2}{2}|f''(c)|$, the (11) inequality desired. $\square$

We are now ready to state the main result of this section

THEOREM 1. (Main) *Let* $x>0,\ \alpha\geqslant 0,\ y\in\mathbb{R}$. *Then:*

$$\frac{1}{\Gamma(x+\alpha+iy)}=\frac{e^{x}x^{-\alpha-x-iy+1/2}}{\sqrt{2\pi}}\left(1+\tilde{R}(\alpha,x,y)\right) \tag{12}$$

where the remainder $\tilde{R}(\alpha,x,y)$ *verifies:*

$$|\tilde{R}(\alpha,x,y)|\leqslant\frac{H(\alpha,y)}{x}\ \text{ with } H(\alpha,y):=e^{\frac{|y|\pi}{2}+\alpha}\left(\frac{|\alpha+iy|}{2}+|\alpha+iy|^{2}+\frac{1}{12}\right). \tag{13}$$

Proof. Our starting point is the Binet representation [9, eq. 3.22] of the reciprocal Gamma function:

$$\frac{1}{\Gamma(z)}=\frac{z^{-z+1/2}e^{z}}{\sqrt{2\pi}}e^{-\mu(z)},$$

where $\mu(z)$ can be expressed:

$$\mu(z):=\int_{0}^{\infty}\frac{1}{t^{2}}\left(\frac{t}{e^{t}-1}-1+\frac{t}{2}\right)e^{-zt}dt. \tag{14}$$

A first substitution $z=\alpha+x+iy$ gives (12):

$$\frac{1}{\Gamma(\alpha+x+iy)}=\frac{e^{x}x^{-\alpha-x-iy+1/2}}{\sqrt{2\pi}}\left[1+\tilde{R}(\alpha,x,y)\right], \tag{15}$$

with

$$\tilde{R}(\alpha,x,y)=\left(1+\frac{\alpha+iy}{x}\right)^{-\alpha-x-iy+1/2}e^{\alpha+iy}e^{-\mu(\alpha+x+iy)}-1.$$

Defining $X:=1/x$ we have

$$\begin{aligned}R(\alpha,X,y)&\equiv\tilde{R}(\alpha,1/X,y)\\&=\left(1+X(\alpha+iy)\right)^{-\alpha-\frac{1}{X}-iy+1/2}e^{\alpha+iy}e^{-\mu\left(\frac{1}{X}+\alpha+iy\right)}-1.\end{aligned} \tag{16}$$

In order to derive (13) we apply the property (10) to the function $R(\alpha,X,y)$ in the range $[0,X]$, considering α and y fixed and $R(\alpha,0,y)=\lim_{X\to 0^{+}}R(\alpha,X,y)=e^{-\alpha-iy}e^{\alpha+iy}e^{0}-1=0$, to find:

$$|R(\alpha,X,y)|\leqslant X\cdot\left|\frac{\partial R}{\partial X}(\alpha,X,y)\right|_{X=C}\quad\text{with } C\in(0,X) \tag{17}$$

Now, we write $\dfrac{\partial R}{\partial X}(\alpha,X,y)$ in the form

$$\begin{aligned}\frac{\partial R}{\partial X}(\alpha,X,y)&=\left(1+X(\alpha+iy)\right)^{-\alpha-\frac{1}{X}-iy-1/2}e^{\alpha+iy}e^{-\mu\left(\frac{1}{X}+\alpha+iy\right)}\\&\quad\times\left[\frac{\alpha+iy}{2}+(1+X(\alpha+iy))\frac{\log(1+X(\alpha+iy))-X(\alpha+iy)}{X^{2}}\right.\\&\quad\left.+(1+X(\alpha+iy))\mu'\left(\alpha+\frac{1}{X}+iy\right)\right]\end{aligned}$$

and finally, we are going to find a bound for the modulus of each factor in the above expression for $X > 0$:

(a) $\left|(1+X(\alpha+iy))^{-\alpha-\frac{1}{X}-iy-1/2}\right| = \left((1+X\alpha)^2+(Xy)^2\right)^{-\frac{1}{2X}-\frac{2\alpha+1}{4}} e^{-y\theta}$ with $\theta = Arg(1+X(\alpha+iy)) \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

We have $(1+X\alpha)^2+(Xy)^2 \geqslant 1$ and $-\dfrac{1}{2X}-\dfrac{2\alpha+1}{4} \leqslant 0$ $(\alpha \geqslant 0)$. Then

$$\left|(1+X(\alpha+iy))^{-\alpha-\frac{1}{X}-iy-1/2}\right| \leqslant e^{\pi|y|/2}.$$

(b) $\left|e^{\alpha+iy}\right| = e^{\alpha}$.

(c) We have the inequality $\left|e^{-\mu\left(\alpha+\frac{1}{X}+iy\right)}\right| \leqslant 1$ from [7, pp. 8].

(d) We denote $G(X) = (1+X(\alpha+iy))(\log(1+X(\alpha+iy))-X(\alpha+iy))$ and applying (11) in the range $[0,X]$

$$\begin{aligned}|G(X)| = |G(X)-G(0)| &\leqslant \frac{X^2}{2}\cdot|G''(C)|\\ &= \frac{X^2}{2}|\alpha+iy|^2\left|1+\frac{(\alpha+iy)C}{1+(\alpha+iy)C}\right| \quad \text{with } C\in(0,X)\end{aligned}$$

we obtain

$$\left|(1+X(\alpha+iy))\frac{\log(1+X(\alpha+iy))-X(\alpha+iy)}{X^2}\right| \leqslant |\alpha+iy|^2.$$

(e) Finally

$$\left|(1+X(\alpha+iy))\mu'\left(\frac{1}{X}+\alpha+iy\right)\right|$$

$$= \frac{1}{X}\left|\int_0^{\infty}\frac{1}{t}\left(\frac{t}{e^t-1}-1+\frac{t}{2}\right)\left(\frac{1}{X}+\alpha+iy\right)e^{-\left(\alpha+\frac{1}{X}+iy\right)t}dt\right|$$

Integrating by parts we find that the above integral reads

$$\frac{1}{X}\left|\int_0^{\infty}\left(\frac{1}{t^2}-\frac{e^t}{(e^t-1)^2}\right)e^{-\left(\alpha+\frac{1}{X}+iy\right)t}dt\right| \leqslant \frac{1}{X}\int_0^{\infty}\left|\frac{1}{t^2}-\frac{e^t}{(e^t-1)^2}\right|e^{-\left(\alpha+\frac{1}{X}\right)t}dt.$$

Using the inequality [5, eq. (8)]:

$$0 \leqslant \frac{1}{t^2}-\frac{e^t}{(e^t-1)^2} \leqslant \frac{1}{12}\ , \ \forall t\in(0,\infty),$$

we find

$$\frac{1}{X}\int_0^\infty \left|\frac{1}{t^2}-\frac{e^t}{(e^t-1)^2}\right| e^{-\left(\alpha+\frac{1}{X}\right)t}dt \leqslant \frac{1}{12X}\int_0^\infty e^{-\left(\alpha+\frac{1}{X}\right)t}dt$$
$$= \frac{1}{12(1+X\alpha)}\left(-e^{-\left(\alpha+\frac{1}{X}\right)t}\right)\Big|_0^\infty$$
$$= \frac{1}{12(1+X\alpha)} \leqslant \frac{1}{12}.$$

Using the bounds (a)–(e) and replacing X by x^{-1} in (17) we obtain the desired result (13). □

As a direct consequence of Theorem 1, taking the particular case $\alpha=0$, we can obtain the Stirling formula for reciprocal Gamma function $1/\Gamma(z)$ when real part of $z=x+iy$ is large and establish a new explicit error bound

COROLLARY 1. *For $x>0$,*

$$\frac{1}{\Gamma(x+iy)} = \frac{e^x x^{-x-iy+1/2}}{\sqrt{2\pi}}\left(1+\tilde{R}(x,y)\right)\ ,\ \forall y\in\mathbb{R}, \tag{18}$$

with

$$|\tilde{R}(x,y)| \leqslant \frac{e^{\frac{|y|\pi}{2}}\left(\frac{|y|}{2}+y^2+\frac{1}{12}\right)}{x}. \tag{19}$$

If we set $y=0$ in Corollary 1, that is, the argument of reciprocal Gamma function is real and positive, we obtain the well known formula

$$\frac{1}{\Gamma(x)} = \frac{e^x x^{-x+1/2}}{\sqrt{2\pi}}\left(1+\tilde{R}(x,0)\right) \text{ with } |\tilde{R}(x,0)| \leqslant \frac{1}{12x}.$$

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