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NEW CARLSON–BELLMAN AND HARDY–LITTLEWOOD DYNAMIC INEQUALITIES

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Abstract. In this paper, we will prove some new dynamic inequalities of Carlson and Hardy-Littlewood types on an arbitrary time scale $\mathbb{T}$. These inequalities as special cases contain the classical continuous and discrete Carlson-Bellman and Hardy-Littlewood type inequalities. The results will be proved by employing the time scales Hölder inequality, some algebraic inequalities and some basic lemmas designed and proved for this purpose.

1. Introduction

The first type of inequalities that we shall discuss in this paper is Carlson-type inequalities. This type of inequalities had its inception eighty years ago by Carlson [10], who proved that the following discrete inequality

$$\left(\sum_{n=1}^{\infty} a_n\right)^4 \leqslant \pi^2 \left(\sum_{n=1}^{\infty} a_n^2\right)\left(\sum_{n=1}^{\infty} n^2 a_n^2\right), \tag{1}$$

holds for any nonnegative sequence $\{a_n\}$. In the same paper Carlson proved that the integral form of this inequality also holds. In particular, he proved the validity of the following continuous inequality

$$\left(\int_0^{\infty} f(x)dx\right)^4 \leqslant \pi^2 \left(\int_0^{\infty} f^2(x)dx\right)\left(\int_0^{\infty} x^2 f^2(x)dx\right), \tag{2}$$

where $f(x)$ is a Lebesgue measurable nonnegative function on $[0,\infty)$. It should be mentioned that both (1) and (2) are sharp, in the sense that the constant π^2 cannot be replaced by a smaller constant without violating the inequality. Hardy [17] published two elementary proofs of Carlson's inequality (1), one of Hardy's proofs shows that only the Schwarz inequality is needed to prove (1), provided that a clever trick is used.

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Bellman [4] employed one of Hardy's original ideas and a genuinely new idea of proving a multiplicative inequality by going via an additive inequality to extend the inequality (1). He proved two versions of (1). The first one is

$$\left(\sum_{n=1}^{\infty} a_n\right)^{q\lambda+p\mu} < C_1 \left(\sum_{n=1}^{\infty} n^{q+\mu-1} a_n^q\right)^{\lambda} \left(\sum_{n=1}^{\infty} n^{p-\lambda-1} a_n^p\right)^{\mu}, \tag{3}$$

where $p,\ q>1,\ \lambda,\ \mu>0$ and the second one is

$$\left(\sum_{n=1}^{\infty} a_n\right)^{\alpha\beta+\alpha-\beta} < C_2 \sum_{n=1}^{\infty} a_n^{\alpha} \left(\sum_{n=1}^{\infty} n^{\beta} a_n^{\beta}\right)^{\alpha-1}, \tag{4}$$

where $\alpha,\ \beta>1$. Bellman also proved that the corresponding continuous inequalities are also hold as follows

$$\left(\int_0^{\infty} f(x)dx\right)^{q\lambda+p\mu} < C_3 \left(\int_0^{\infty} x^{q+\mu-1} f^q(x)dx\right)^{\lambda} \left(\int_0^{\infty} x^{p-\lambda-1} f^p(x)dx\right)^{\mu}, \tag{5}$$

and

$$\left(\int_0^{\infty} f(x)dx\right)^{\alpha\beta+\alpha-\beta} < C_4 \left(\int_0^{\infty} f^{\alpha}(x)dx\right) \left(\int_0^{\infty} x^{\beta} f^{\beta}(x)dx\right)^{\alpha-1}. \tag{6}$$

The constants C_1, C_2, C_3 and C_4 are positive constants depend only on the exponents. The sharp constant in the general case was found later by Levin [21]. For more details concerning Carlson type inequalities and their variants and extensions, we refer the reader to the papers [11, 14, 18, 22], the book [20] and the references cited therein.

As an application for the general discrete Bellman's inequality (3), using the simple substitution $p=q$ and $\lambda=\mu$, we could obtain the following inequality due to Gabriel [14]

$$\sum_{n=1}^{\infty} a_n < G(p,\lambda) \left(\sum_{n=1}^{\infty} n^{p-1+\lambda} a_n^p\right)^{\frac{1}{2p}} \left(\sum_{n=1}^{\infty} n^{p-1-\lambda} a_n^p\right)^{\frac{1}{2p}}, \tag{7}$$

where $p>1$ and $\lambda>0$. Also, the same substitutions in (5), leads directly to the continuous analogue of (7) as follows

$$\int_0^{\infty} f(x)dx < G(p,\lambda) \left(\int_0^{\infty} x^{p-1+\lambda} f^p(x)dx\right)^{\frac{1}{2p}} \left(\int_0^{\infty} x^{p-1-\lambda} f^p(x)dx\right)^{\frac{1}{2p}}, \tag{8}$$

which can be found in [23].

The second type of inequalities that we will consider in this paper is Hardy-Littlewood type inequalities which have been proved first by Hardy and Littlewood [15]. In particular, they have used the calculus of variations technique to establish some interesting integral inequalities which contain the function, its first and second derivatives. One of their results in that celebrated paper, reads as

$$\left(\int_0^{\infty} \left(y'\right)^2 dx\right)^2 \leqslant 4 \int_0^{\infty} y^2 dx \int_0^{\infty} \left(y''\right)^2 dx. \tag{9}$$

The constant 4 is the best possible; moreover equality holds in (9) only when

$$y = Ae^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}}{2}x - \frac{\pi}{3}\right),$$

see also [16, Theorem 259]. In the same paper, they also proved a simpler and quite different inequality. In particular, they proved that the following inequality

$$\left(\int_{-\infty}^{\infty} \left(y'\right)^2 dx\right)^2 \leqslant \int_{-\infty}^{\infty} y^2 dx \int_{-\infty}^{\infty} \left(y''\right)^2 dx, \tag{10}$$

holds. The constant 1 is the best possible, see also [16, Theorem 261]. More than forty years later, Copson [12] employed a different approach to investigate Hardy-Littlewood inequality (9).

In [13] Copson used the same approach used in [12] and proved that the discrete analogues of (9) and (10) are also hold. Particularly, he established the following discrete inequalities

$$\left(\sum_{n=0}^{\infty} (\Delta a_n)^2\right)^2 \leqslant 4 \sum_{n=0}^{\infty} (a_n)^2 \sum_{n=0}^{\infty} \left(\Delta^2 a_n\right)^2, \tag{11}$$

and

$$\left(\sum_{n=-\infty}^{\infty} (\Delta a_n)^2\right)^2 \leqslant \sum_{n=-\infty}^{\infty} (a_n)^2 \sum_{n=-\infty}^{\infty} \left(\Delta^2 a_n\right)^2, \tag{12}$$

where $\{a_n\}$ is a non-negative sequence. Since the discovery of these inequalities, they received a lot of attentions, we refer the interested reader to the books [16, Chapter 7][23, Chapter 1], the papers [3, 7, 19] and the references cited therein. Also, Weyl [32] employed the classical Cauchy-Schwarz inequality for sums to prove the following Hardy-Littlewood type inequality

$$\left(\int_{-\infty}^{\infty} y^2 dx\right)^2 \leqslant 4 \int_{-\infty}^{\infty} x^2 y^2 dx \int_{-\infty}^{\infty} \left(y'\right)^2 dx, \tag{13}$$

in order to show the inverse relationship between the uncertainty of the mean value

$$\Delta x = \int_{-\infty}^{\infty} x^2 y^2 dx,$$

of a co-ordinate x and the uncertainty of the mean value of its associate momentum

$$\Delta p = \int_{-\infty}^{\infty} \left(y'\right)^2 dx.$$

In recent years the study of dynamic inequalities on time scales has received a lot of attention. The general idea is to prove a result for a dynamic inequality where the domain of the unknown function is a so-called time scale $\mathbb{T}$, which may be an arbitrary closed subset of the real numbers $\mathbb{R}$. The cases when the time scale is equal to the reals or to the integers represent the classical theories of integral and of discrete inequalities. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus, i.e., when $\mathbb{T}=\mathbb{R}$, $\mathbb{T}=\mathbb{N}$ and $\mathbb{T}=q^{\mathbb{N}_0}=\{q^t : t\in\mathbb{N}_0\}$ where $q>1$. For more details of Hardy's type inequalities, we refer to the papers [24, 25, 26, 27, 28, 29, 30, 31] and the recent book [1].

The question now arises: Is it possible to apply the calculus on time scales to prove some new dynamic inequalities on which as special cases contain the inequalities (5), (6), (9), (10) and (13)?

The main aim of this paper is to give an affirmative answer to this question. The setup of the rest of the paper is as follows. In Section 2, we present some preliminaries about the theory of time scales and some basic lemmas that will be used to prove our main results of this paper. In Section 3, we state and prove the time scales version of Carlson-Bellman type inequalities (5) and (6). In Section 4, we state and prove the time scales versions of Hardy-Littlewood type inequalities (9), (10) and (13).

REMARK 1.1. It is worth to mention here that, for the proof of Hardy-Littlewood inequality, we actually followed the technique of Hardy and Littlewood [15]. Furthermore, the generalization of Copson's approach to an arbitrary time scales is still an open problem.

2. Preliminaries on time scales

For completeness, we recall the following concepts related to the notion of time scales. For more details of time scale analysis we refer the reader to the two books by Bohner and Peterson [5], [6] which summarize and organize much of the time scale calculus.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. The forward jump operator and the backward jump operator are defined by: $\sigma(t):=\inf\{s\in\mathbb{T}: s>t\}$ and $\rho(t):=\sup\{s\in\mathbb{T}: s<t\}$, respectively. A point $t\in\mathbb{T}$, is said to be left–dense if $\rho(t)=t$, is right–dense if $\sigma(t)=t$, is left–scattered if $\rho(t)<t$ and right–scattered if $\sigma(t)>t$. A function $g:\mathbb{T}\to\mathbb{R}$ is said to be right–dense continuous (rd–continuous) provided g is continuous at right–dense points and at left–dense points in $\mathbb{T}$, left hand limits exist and are finite. The set of all such rd–continuous functions is denoted by $C_{rd}(\mathbb{T})$.

The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t):=\sigma(t)-t\geqslant 0$, and for any function $f:\mathbb{T}\to\mathbb{R}$ the notation $f^{\sigma}(t)$ denotes $f(\sigma(t))$. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus, i.e., when $\mathbb{T}=\mathbb{R}$, $\mathbb{T}=\mathbb{N}$ and $\mathbb{T}=q^{\mathbb{N}_0}=\{q^t : t\in\mathbb{N}_0\}$ where $q>1$. The derivative of the product fg and the quotient f/g (where $gg^{\sigma}\neq 0$) of two differentiable functions f and g are given by

$$(fg)^{\Delta}=f^{\Delta}g+f^{\sigma}g^{\Delta}=fg^{\Delta}+f^{\Delta}g^{\sigma}, \tag{14}$$

$$\left(\frac{f}{g}\right)^{\Delta}=\frac{f^{\Delta}g-fg^{\Delta}}{gg^{\sigma}}. \tag{15}$$

In this paper, we will refer to the (delta) integral which is defined as follows. If $G^{\Delta}(t) = g(t)$, then

$$\int_a^t g(s)\Delta s := G(t) - G(a).$$

It can be shown (see [5]) that if $g \in C_{rd}(\mathbb{T})$, then the Cauchy integral $G(t) := \int_{t_0}^t g(s)\Delta s$ exists, $t_0 \in \mathbb{T}$, and satisfies $G^{\Delta}(t) = g(t)$, $t \in \mathbb{T}$. An improper integral is defined by

$$\int_a^{\infty} g(t)\Delta t = \lim_{b\to\infty} \int_a^b g(t)\Delta t,$$

and the integration by parts formula on time scales is given by

$$\int_a^b u(t)v^{\Delta}(t)\Delta t = [u(t)v(t)]_a^b - \int_a^b u^{\Delta}(t)v^{\sigma}(t)\Delta t. \tag{16}$$

The time scales chain rule (see [5, Theorem 1.87]) is given by

$$(g\circ\delta)^{\Delta}(t) = g'(\delta(d))\,\delta^{\Delta}(t), \text{ where } d \in [t,\sigma(t)], \tag{17}$$

where it is assumed that $g : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and $\delta : \mathbb{T} \to \mathbb{R}$ is delta differentiable. A simple consequence of Keller's chain rule [5, Theorem 1.90] is given by

$$(x^{\gamma}(t))^{\Delta} = \gamma \int_0^1 [hx^{\sigma}(t) + (1-h)x(t)]^{\gamma-1}\, dh\, x^{\Delta}(t). \tag{18}$$

The Hölder inequality, see [5, Theorem 6.13], on time scales is given by

$$\int_a^b |f(t)g(t)|\Delta t \leqslant \left[\int_a^b |f(t)|^{\gamma}\Delta t\right]^{\frac{1}{\gamma}} \left[\int_a^b |g(t)|^{\nu}\Delta t\right]^{\frac{1}{\nu}}, \tag{19}$$

where $a,\ b \in \mathbb{T},\ f,\ g \in \mathrm{C}_{rd}(\mathbb{I},\mathbb{R}),\ \gamma > 1$ and $1/\gamma + 1/\nu = 1$. The special case $\gamma = \nu = 2$ in (19) yields the time scales Cauchy-Schwarz inequality.

Throughout this paper, we will assume that the functions in the statements of the theorems are positive and rd-continuous functions and the integrals considered are assumed to exist. We define the time scale interval $[a,b]_{\mathbb{T}}$ by $[a,b]_{\mathbb{T}} := [a,b] \cap \mathbb{T}$ and u^* is the conjugate of u (in the sense that $1/u + 1/u^* = 1$).

3. Dynamic inequalities of Carlson-Bellman's type

In this section, we will state and prove some generalizations of Carlson-Bellman's type inequality (5) on time scales. To prove the main results, we need the following inequality.

LEMMA 3.1. *If* $Bx^u \leqslant C + Ax^{\nu}$, $\nu > u > 0$, $A,\ B,\ C > 0$ *for all positive* x, *then*

$$C^{\nu-u}A^u \geqslant K(u,\nu)\,B^{\nu}. \tag{20}$$

Proof. By assuming that $x=(Bu/Av)^{\frac{1}{v-u}}$, and substituting in $Bx^u \leqslant C+Ax^v$ and follows a direct simplification we get the desired inequality (20). The proof is complete. $\square$

REMARK 3.1. This lemma may be considered as an extension of the well-known fact that $b^2 \leqslant 4ac$ if $bx \leqslant c+ax^2$ for all $x \geqslant 0$ and $a,\ b,\ c \geqslant 0$.

Now, we are ready to state and prove the main results of this section.

THEOREM 3.1. *Let $\mathbb{T}$ be a time scale. If $p,\ q>1,\ \lambda,\ \eta>0$ and $f(x)$ is a real-valued nonnegative function on $(0,\infty)_{\mathbb{T}}$ such that $x^{q+\eta-1}f^q(x),\ x^{p-\lambda-1}f^p(x)$ are Δ-integrable functions on $(0,\infty)_{\mathbb{T}}$, then*

$$\left(\int_0^\infty f(x)\Delta x\right)^{q\lambda+p\eta} < K(p,q)\left(\int_0^\infty x^{q+\eta-1}f^q(x)\Delta x\right)^{\lambda}\left(\int_0^\infty x^{p-\lambda-1}f^p(x)\Delta x\right)^{\eta}, \tag{21}$$

where

$$K(p,q) := \left(\frac{(p\eta+q\lambda)^{p\eta+q\lambda}}{(p\eta)^{p\eta}(q\lambda)^{q\lambda}}\right)^{\frac{1}{pq}} 2^{\frac{(pq-1)(p\eta+q\lambda)}{pq}} L^{p\eta}M^{q\lambda},$$

$$L := \left(\int_0^\infty \frac{x^{-\frac{p-\lambda-1}{p-1}}\Delta x}{(1+x)^{p^*}}\right)^{\frac{1}{p^*}}, \quad \textit{and} \quad M := \left(\int_0^\infty \frac{x^{-\frac{q+\eta-1}{q-1}}\Delta x}{\left(1+\frac{1}{x}\right)^{q^*}}\right)^{\frac{1}{q^*}}.$$

Proof. Let $0<r+1<p$ and $q<s+1<2q$, then we can write that

$$\int_0^\infty f(x)\Delta x = \int_0^\infty \frac{x^{\frac{r}{p}}f(x)}{x^{\frac{r}{p}}(1+x)}\Delta x + \int_0^\infty \frac{x^{\frac{s}{q}}f(x)}{x^{\frac{s}{q}}\left(1+\frac{1}{x}\right)}\Delta x.$$

Applying Hölder's inequality (19) on the first integral of the right hand side with index p and on the the second integral with index q, we get that

$$\begin{aligned}\int_0^\infty f(x)\Delta x &\leqslant \left(\int_0^\infty x^r f^p(x)\Delta x\right)^{\frac{1}{p}}\left(\int_0^\infty \frac{\Delta x}{x^{\frac{r}{p-1}}(1+x)^{p^*}}\right)^{\frac{1}{p^*}} \\ &\quad + \left(\int_0^\infty x^s f^q(x)\Delta x\right)^{\frac{1}{q}}\left(\int_0^\infty \frac{\Delta x}{x^{\frac{s}{q-1}}\left(1+\frac{1}{x}\right)^{q^*}}\right)^{\frac{1}{q^*}} \\ &= L\left(\int_0^\infty x^r f^p(x)\Delta x\right)^{\frac{1}{p}} + M\left(\int_0^\infty x^s f^q(x)\Delta x\right)^{\frac{1}{q}}.\end{aligned} \tag{22}$$

By raising the two sides to the power pq, we get that

$$\left(\int_0^\infty f(x)\Delta x\right)^{pq} \leqslant \left[L\left(\int_0^\infty x^r f^p(x)\Delta x\right)^{\frac{1}{p}} + M\left(\int_0^\infty x^s f^q(x)\Delta x\right)^{\frac{1}{q}}\right]^{pq}. \tag{23}$$

Applying the well-known inequality (see [23, page 500])

$$(a+b)^r \leqslant 2^{r-1}(a^r+b^r), \quad \text{for } r \geqslant 1, \tag{24}$$

to the right-hand side of inequality (23) with $r = pq$, we have that

$$\left(\int_0^\infty f(x)\Delta x\right)^{pq} \leqslant 2^{pq-1}\left[L^{pq}\left(\int_0^\infty x^r f^p(x)\Delta x\right)^q + M^{pq}\left(\int_0^\infty x^s f^q(x)\Delta x\right)^p\right].$$

Multiplying both sides by 1 and observing that $sp+p > pq > qr+q$, we obtain that

$$\begin{aligned} &1^{pq-qr-q}\cdot\left(\int_0^\infty f(x)\Delta x\right)^{pq} \\ &\leqslant 2^{pq-1}\left[1\cdot L^{pq}\left(\int_0^\infty x^r f^p(x)\Delta x\right)^q + 1^{sp+p-qr-q}\cdot M^{pq}\left(\int_0^\infty x^s f^q(x)\Delta x\right)^p\right]. \end{aligned} \tag{25}$$

Now, since Lemma 3.1 is valid for all positive x, we will apply it with the following arguments

$$\begin{aligned} &x = 1, \ \ u = pq-qr-q, \ \ v = sp+p-qr-q, \\ &A = 2^{pq-1}M^{pq}\left(\int_0^\infty x^s f^q(x)\Delta x\right)^p, \\ &B = \left(\int_0^\infty f(x)\Delta x\right)^{pq}, \\ &C = 2^{pq-1}L^{pq}\left(\int_0^\infty x^r f^p(x)\Delta x\right)^q, \end{aligned}$$

to get from (25) that

$$\begin{aligned} \left(\int_0^\infty f(x)\Delta x\right)^{pq(sp+p-qr-q)} &\leqslant K_1(p,q,r,s)\left(\int_0^\infty x^s f^q(x)\Delta x\right)^{p(pq-qr-q)} \\ &\quad\times\left(\int_0^\infty x^r f^p(x)\Delta x\right)^{q(sp-qp+p)}. \end{aligned}$$

Taking $(pq)^{th}$ root of each side, we get that

$$\begin{aligned} \left(\int_0^\infty f(x)\Delta x\right)^{sp+p-qr-q} &\leqslant K_2(p,q,r,s)\left(\int_0^\infty x^s f^q(x)\Delta x\right)^{p-r-1} \\ &\quad\times\left(\int_0^\infty x^r f^p(x)\Delta x\right)^{s-q+1}. \end{aligned}$$

By setting $\lambda = p-r-1$ and $\eta = s-q+1$, we get finally that

$$\begin{aligned} \left(\int_0^\infty f(x)\Delta x\right)^{q\lambda+p\eta} &\leqslant K(p,q)\left(\int_0^\infty x^{q+\eta-1} f^q(x)\Delta x\right)^{\lambda} \\ &\quad\times\left(\int_0^\infty x^{p-\lambda-1} f^p(x)\Delta x\right)^{\eta}, \end{aligned}$$

which is the required inequality (21). The proof is complete. □

REMARK 3.2. If $\mathbb{T}=\mathbb{R}$, then (21) reduces to the following continuous inequality

$$\left(\int_0^\infty f(x)dx\right)^{q\lambda+p\eta} < C\left(\int_0^\infty x^{q+\eta-1}f^q(x)dx\right)^{\lambda}\left(\int_0^\infty x^{p-\lambda-1}f^p(x)dx\right)^{\eta}, \quad (26)$$

due to Bellman [4], where

$$C := 2^{\frac{(pq-1)(p\eta+q\lambda)}{pq}}\left(\frac{(p\eta+q\lambda)^{p\eta+q\lambda}}{(p\eta)^{p\eta}(q\lambda)^{q\lambda}}\right)^{\frac{1}{pq}}\times\beta\left(\frac{\lambda}{p-1},\frac{p-\lambda}{p-1}\right)^{\frac{p-1}{\eta}} \times\beta\left(\frac{\eta}{q-1},\frac{q-\eta}{q-1}\right)^{\frac{q-1}{\lambda}},$$

and $\beta(\cdot,\cdot)$ is the normal beta function. It is worth to mention here that the exact value of the constant C was not stated in the original paper of Bellman.

REMARK 3.3. As a special case of (26), for $p=q=2$ and $\lambda=\eta=1$, we get the following consequence of the original Carlson's inequality

$$\left(\int_0^\infty f(x)dx\right)^4 \leqslant 16\left(\int_0^\infty f^2(x)dx\right)\left(\int_0^\infty x^2f^2(x)dx\right), \quad (27)$$

with the constant π^2 replaced by 16, which means that the constant in Theorem 3.1 can be improved to give the best one due to Carlson.

REMARK 3.4. If $\mathbb{T}=\mathbb{N}$, then (21) reduces to the discrete inequality (3) due to Bellman.

In the following, we will state and prove the time scales version of (6).

THEOREM 3.2. *Let $\mathbb{T}$ be a time scale. If $\alpha,\ \beta>1$ and $f(x)$ is a real-valued nonnegative function on $(0,\infty)_{\mathbb{T}}$ such that $f^{\alpha}(x)$, $x^{\beta}f^{\beta}(x)$ are Δ-integrable functions on $(0,\infty)_{\mathbb{T}}$, then*

$$\left(\int_0^\infty f(x)\Delta x\right)^{\alpha\beta+\alpha-\beta} < K(\alpha,\beta)\int_0^\infty f^{\alpha}(x)\Delta x\left(\int_0^\infty x^{\beta}f^{\beta}(x)\Delta x\right)^{\alpha-1}, \quad (28)$$

where

$$K(\alpha,\beta) := \left(\frac{(\alpha+\beta)^{\alpha+\beta}}{(\alpha)^{\alpha}(\beta)^{\beta}}\right)^{\frac{1}{\alpha\beta}} 2^{\frac{(\alpha\beta-1)(\alpha+\beta)}{\alpha\beta}} L_1^{\alpha}M_1^{\beta},$$

$$L_1 := \left(\int_0^\infty \frac{\Delta x}{(1+x^2)^{\frac{1}{\alpha^*}}}\right)^{\frac{1}{\alpha^*}}, \quad \text{and} \quad M_1 := \left(\int_0^\infty\left(\frac{x}{1+x^2}\right)^{\beta^*}\Delta x\right)^{\frac{1}{\beta^*}}.$$

Proof. Let $f(x) \geqslant 0$ and $\alpha,\ \beta > 1$, then we can write that

$$\int_0^\infty f(x)\Delta x = \int_0^\infty \frac{f(x)}{1+x^2}\Delta x + \int_0^\infty \frac{x^2 f(x)}{1+x^2}\Delta x.$$

Applying the time scales Hölder's inequality (19) to the two integrals on the right-hand side, the first integral with index α and the second integral with index β, we get that

$$\begin{aligned}
\int_0^\infty f(x)\Delta x &\leqslant \left(\int_0^\infty f^\alpha(x)\Delta x\right)^{\frac{1}{\alpha}} \left(\int_0^\infty \frac{\Delta x}{(1+x^2)^{\frac{1}{\alpha^*}}}\right)^{\frac{1}{\alpha^*}} \\
&\quad + \left(\int_0^\infty x^\beta f^\beta(x)\Delta x\right)^{\frac{1}{\beta}} \left(\int_0^\infty \left(\frac{x}{1+x^2}\right)^{\beta^*} \Delta x\right)^{\frac{1}{\beta^*}} \\
&= L_1 \left(\int_0^\infty f^\alpha(x)\Delta x\right)^{\frac{1}{\alpha}} + M_1 \left(\int_0^\infty x^\beta f^\beta(x)\Delta x\right)^{\frac{1}{\beta}}.
\end{aligned}$$

The rest of the proof is similar to the proof of Theorem 3.1 and will be omitted. The proof is complete. □

REMARK 3.5. If $\mathbb{T} = \mathbb{R}$, then (28) reduces to the continuous inequality (6) due to Bellman.

REMARK 3.6. If $\mathbb{T} = \mathbb{N}$, then (28) reduces to the discrete inequality (4) due to Bellman.

4. Dynamic inequalities of Hardy-Littlewood's type

In this section, we will state and prove some new dynamic inequalities of Hardy-Littlewood's type which as special cases contain the inequalities (9), (10) and (13). Before we begin, we present the definition of Δ-measurable functions on time scales and we set out the method used in [6, Chapter 5] by Bohner and Guseinov to define the Lebesgue Δ-measure on $\mathbb{T}$. First, by defining the measure m_1 which assigns to each interval $[a,b) \cap \mathbb{T}$ its length, that is $m_1([a,b)) = b-a$. Using m_1, they generate the outer measure m_1^*, defined for each subset E of $\mathbb{T}$ as

$$m_1^*(E) = \begin{cases} \inf_{\mathscr{R}} \left\{ \sum_{i \in I_{\mathscr{R}}} (b_i - a_i) \right\} \in \mathbb{R}^+, & \text{if } b \notin E, \\ +\infty, & \text{if } b \in E, \end{cases}$$

with

$$\mathscr{R} = \left\{ \{[a_i,b_i) \cap \mathbb{T}\}_{i \in I_{\mathscr{R}}} : I_{\mathscr{R}} \subset \mathbb{N},\ E \subset \bigcup_{i \in I_{\mathscr{R}}} ([a_i,b_i) \cap \mathbb{T}) \right\}.$$

A set $A \subset \mathbb{T}$ is said to be Δ-measurable if the following equality

$$m_1^*(E) = m_1^*(E \cap A) + m_1^*(E \cap (\mathbb{T} \backslash A)),$$

holds for all subset $E \subset \mathbb{T}$.

DEFINITION 4.1. We say that $f:\mathbb{T}\to\mathbb{R}$ is Δ-measurable if for every $\alpha\in\mathbb{R}$, the set

$$f^{-1}([-\infty,\alpha))=\{t\in\mathbb{T}:f(t)<\alpha\},$$

is Δ-measurable.

DEFINITION 4.2. Let $E\subset\mathbb{T}$ be a Δ-measurable set and let $p\geqslant 1$ and let $f:E\to\mathbb{R}$ be a Δ-measurable function. We say that f belongs to $L_{\Delta}^{p}(E)$ provided that either

$$\textstyle\int_E |f|^p\Delta s<\infty,\quad \text{if}\quad 1\leqslant p<\infty,$$

or there exists a constant $C\in\mathbb{R}$ such that

$$|f|\leqslant C,\quad \text{if}\quad p=+\infty,$$

where f is Δ-almost every where on E.

Now, we use the above definitions to prove some lemmas that will be needed in the proofs of the main results for this section.

LEMMA 4.1. *Let $\mathbb{T}$ be a time scale and $f:[t_0,\infty)_{\mathbb{T}}\to[0,\infty)$ be Δ-differentiable such that*

$$f\in L_{\Delta}^{2}[t_0,\infty)_{\mathbb{T}},\quad \textit{and}\quad f^{\Delta}\in L_{\Delta}^{2}[t_0,\infty)_{\mathbb{T}}.$$

Then

$$\lim_{t\to\infty}f(t)=0.$$

Proof. Since $f^{\Delta}\in L_{\Delta}^{2}[t_0,\infty)$, for every $\varepsilon>0$, there exist $t_2>t_1\geqslant t_0$ such that

$$\int_{t_1}^{\infty}\left|f^{\triangle}(t)\right|^2\Delta t<\varepsilon/2,\qquad \int_{t_2}^{\infty}\left|f^{\triangle}(t)\right|^2\Delta t<\varepsilon/2.$$

From this, we see that

$$\begin{aligned}|f(t_2)-f(t_1)|^2&=\left|\int_{t_1}^{t_2}f^{\Delta}(t)\Delta t\right|^2=\left|\int_{t_1}^{\infty}f^{\Delta}(t)\Delta t-\int_{t_2}^{\infty}f^{\Delta}(t)\Delta t\right|^2\\&\leqslant\int_{t_1}^{\infty}\left(\left|f^{\Delta}(t)\right|\right)^2\Delta t+\int_{t_1}^{\infty}\left(\left|f^{\Delta}(t)\right|\right)^2\Delta t<\varepsilon.\end{aligned}$$

It follows that $\lim_{t\to\infty}f(t)$ exists. We only prove that $\lim_{t\to\infty}f^2(t)=0$. To show this, argue by contradiction: Suppose that $f^2(t)\nrightarrow 0$ as $t\to\infty$. This means that, we may find a positive constant ε and a positive value of $t>t_0$ for which

$$|f(t)|^2>\varepsilon, \tag{29}$$

and which is so large that

$$\int_{t}^{\infty}\left|f^{\triangle}(x)\right|^2\Delta x\leqslant\frac{1}{4}. \tag{30}$$

The time scales Hölder inequality (19) gives us for some $0 \leqslant t < y < t+\varepsilon$ that

$$\left| \int_t^y f^{\triangle}(x) \Delta x \right|^2 \leqslant (y-t) \int_t^y \left| f^{\triangle}(x) \right|^2 \Delta x,$$

which leads us directly after using (30) to the following

$$|f(y) - f(t)|^2 \leqslant \frac{1}{4}\varepsilon, \quad \text{whence } f(y) \geqslant f(t) - \frac{1}{2}\sqrt{\varepsilon} > \frac{1}{2}\sqrt{\varepsilon}. \tag{31}$$

Thus

$$\int_t^y |f(x)|^2 \Delta x > \frac{\varepsilon}{4}\varepsilon = \frac{\varepsilon^2}{4}.$$

This implies that $\int_{t_0}^{\infty} |f(t)|^2 \Delta t$ diverges which asserts that f is not L_{Δ}^2-integrable which is a contradiction. Then $\lim_{t\to\infty} f(t) = 0$. The proof is complete. □

The proof of the following lemma is similar to the proof of Lemma 4.1 and hence it is omitted.

LEMMA 4.2. *Let $\mathbb{T}$ be a time scale and $f : [t_0,\infty)_{\mathbb{T}} \to [0,\infty)$ be $\triangle^2-$differentiable such that*

$$f^{\Delta} \in L_{\Delta}^2[t_0,\infty)_{\mathbb{T}}, \quad \textit{and} \quad f^{\Delta\Delta} \in L_{\Delta}^2[t_0,\infty)_{\mathbb{T}}.$$

Then

$$\lim_{t\to\infty} f^{\Delta}(t) = 0.$$

By combining the above two lemmas, we get the following result.

LEMMA 4.3. *Let $\mathbb{T}$ be a time scale $f : [t_0,\infty)_{\mathbb{T}} \to [0,\infty)$. If $f,\ f^{\triangle},\ f^{\triangle\triangle} \in L_{\triangle}^2[t_0,\infty)_{\mathbb{T}}$, then*

$$\lim_{x\to\infty} \left[f(x) + f^{\triangle}(x) \right]^2 = 0. \tag{32}$$

The following lemma plays an important role in proving our main results and its proof depends on the application of the result given in Lemma 4.3.

LEMMA 4.4. *Let $\mathbb{T}$ be a time scale. If $f^{\triangle} \in C_{rd}[0,\infty)_{\mathbb{T}}$ and $f,\ f^{\triangle},\ f^{\triangle\triangle} \in L_{\triangle}^2[0,\infty)_{\mathbb{T}}$, then*

$$\int_0^{\infty} \left(f^{\triangle}(x) \right)^2 \Delta x < \int_0^{\infty} f^2(x) \Delta x + \int_0^{\infty} \left(f^{\triangle\triangle}(x) \right)^2 \Delta x. \tag{33}$$

Proof. Note that for all $t>0,$ we have

$$\int_0^\infty \left[(f)^2-\left(f^{\triangle}\right)^2+\left(f^{\triangle\triangle}\right)^2-(f+f^{\triangle}+f^{\triangle\triangle})^2\right](x)\Delta x \tag{34}$$
$$=\lim_{t\to\infty}\int_0^t \left[(f)^2-\left(f^{\triangle}\right)^2+\left(f^{\triangle\triangle}\right)^2-(f+f^{\triangle}+f^{\triangle\triangle})^2\right](x)\Delta x$$
$$=-2\lim_{t\to\infty}\int_0^t \left[\left(f^{\triangle}\right)^2+ff^{\triangle}+ff^{\triangle\triangle}+f^{\triangle}f^{\triangle\triangle}\right](x)\Delta x$$
$$=-2\lim_{t\to\infty}\int_0^t \left[\left(f+f^{\triangle}\right)\left(f^{\triangle}+f^{\triangle\triangle}\right)\right](x)\Delta x.$$

Using the time scales chain rule (18) with $\gamma=2$ and $g=f+f^{\triangle}$, we get that

$$\left(g^2\right)^{\Delta}=2\int_0^1 [hg^{\sigma}+(1-h)g]\,dhg^{\Delta} \tag{35}$$
$$\geqslant 2\int_0^1 [hg+(1-h)g]\,dhg^{\Delta}=2gg^{\Delta}.$$

Substituting (35) into (34), we have that

$$\int_0^\infty \left[(f)^2-\left(f^{\triangle}\right)^2+\left(f^{\triangle\triangle}\right)^2-(f+f^{\triangle}+f^{\triangle\triangle})^2\right](x)\Delta x$$
$$>-\lim_{X\to\infty}\int_0^X \left[\left(f(x)+f^{\triangle}(x)\right)^2\right]^{\triangle}\Delta x$$
$$=\lim_{X\to\infty}\left(\left[f(0)+f^{\triangle}(0)\right]^2-\left[f(X)+f^{\triangle}(X)\right]^2\right)$$
$$=\left[f(0)+f^{\triangle}(0)\right]^2-\lim_{X\to\infty}\left[f^2(X)+2f(X)f^{\triangle}(X)+\left(f^{\triangle}(X)\right)^2\right]. \tag{36}$$

Applying Lemma 4.3, we find that all limits on the right-hand side of (36) should tend to zero and hence (36) reduces to

$$\int_0^\infty \left[(f)^2-\left(f^{\triangle}\right)^2+\left(f^{\triangle\triangle}\right)^2\right](x)\Delta x \tag{37}$$
$$>\left[f(0)+f^{\triangle}(0)\right]^2+\int_0^\infty (f+f^{\triangle}+f^{\triangle\triangle})^2(x)\Delta x,$$

which can be written as

$$\left[f(0)+f^{\triangle}(0)\right]^2+\int_0^\infty (f+f^{\triangle}+f^{\triangle\triangle})^2(x)\Delta x+\int_0^\infty \left(f^{\triangle}\right)^2(x)\Delta x \tag{38}$$
$$<\int_0^\infty \left[(f)^2+\left(f^{\triangle\triangle}\right)^2\right](x)\Delta x.$$

The inequality (33) follows from (38), with strict inequality unless f satisfies

$$f(0)+f^{\triangle}(0)=0 \quad \text{and} \quad f+f^{\triangle}+f^{\triangle\triangle}=0.$$

This completes the proof. □

Now, we are ready to state and prove our main results for this section. We begin with the time scales version of Hardy-Littlewood's inequality (9).

THEOREM 4.1. *Let $\mathbb{T}$ be a time scale. If $f,\ f^{\triangle},\ f^{\triangle\triangle}\in L^2_{\triangle}(0,\infty)_{\mathbb{T}}$, then*

$$\left(\int_0^{\infty}\left(f^{\triangle}(x)\right)^2\Delta x\right)^2<4\int_0^{\infty}(f(x))^2\,\Delta x\int_0^{\infty}\left(f^{\triangle\triangle}(x)\right)^2\Delta x. \tag{39}$$

Proof. From Lemma 4.4 we see that (33) holds, and then setting $y=\beta x$ for $\beta>0$, we have

$$\beta^2\int_0^{\infty}\left(f^{\triangle}(y)\right)^2\Delta y\leqslant\int_0^{\infty}(f(y))^2\,\Delta y+\beta^4\int_0^{\infty}\left(f^{\triangle\triangle}(y)\right)^2\Delta y. \tag{40}$$

Dividing (40) by β^2, we get

$$\int_0^{\infty}\left(f^{\triangle}(y)\right)^2\Delta y\leqslant\beta^{-2}\int_0^{\infty}(f(y))^2\,\Delta y+\beta^2\int_0^{\infty}\left(f^{\triangle\triangle}(y)\right)^2\Delta y. \tag{41}$$

Using the substitution

$$\beta^2=\left(\frac{\int_0^{\infty}(f(y))^2\,\Delta y}{\int_0^{\infty}\left(f^{\triangle\triangle}(y)\right)^2\Delta y}\right)^{\frac{1}{2}},$$

in inequality (41), we can find easily that this reduces to (39). This completes the proof. □

REMARK 4.1. If we choose $\mathbb{T}=\mathbb{N}$, then (39) reduces to the discrete inequality (11) due to Copson.

REMARK 4.2. If we choose $\mathbb{T}=\mathbb{R}$, then (39) reduces to the continuous inequality (9) due to Hardy and Littlewood.

Next, we get the time scales version of Hardy-Littlewood's inequality (10).

THEOREM 4.2. *Let $\mathbb{T}$ be a time scale. If $f,\ f^{\triangle},\ f^{\triangle\triangle}\in L^2_{\triangle}(-\infty,\infty)_{\mathbb{T}}$, then*

$$\left(\int_{-\infty}^{\infty}\left(f^{\triangle}(x)\right)^2\Delta x\right)^2<\int_{-\infty}^{\infty}(f^{\sigma}(x))^2\,\Delta x\int_{-\infty}^{\infty}\left(f^{\triangle\triangle}(x)\right)^2\Delta x. \tag{42}$$

Proof. From Lemma 4.3, we have that

$$\lim_{t\to\pm\infty}\left[f(t)+f^{\triangle}(t)\right]^2=0.$$

Integrating the left hand side of (42) by parts and then applying the time scales Hölder's inequality (19), we have that

$$\begin{aligned}\left(\int_{-\infty}^{\infty}\left(f^{\triangle}(x)\right)^2\Delta x\right)^2&=\left(-\int_{-\infty}^{\infty}f^{\sigma}(x)f^{\triangle\triangle}(x)\Delta x\right)^2\\&\leqslant\int_{-\infty}^{\infty}(f^{\sigma}(x))^2\Delta x\int_{-\infty}^{\infty}\left(f^{\triangle\triangle}(x)\right)^2\Delta x,\end{aligned}$$

which is the desired inequality (42). The proof is complete. □

REMARK 4.3. If we choose $\mathbb{T}=\mathbb{N}$, then (42) reduces to the discrete inequality (12) due to Copson.

REMARK 4.4. If we choose $\mathbb{T}=\mathbb{R}$, then (42) reduces to the continuous inequality (10) due to Hardy and Littlewood.

Finally, we are ready to state and prove the time scales version of (13) as follows.

THEOREM 4.3. *Let $\mathbb{T}$ be a time scale. If $f,\ f^{\triangle},\ f^{\triangle\triangle}\in L^2_{\triangle}(-\infty,\infty)_{\mathbb{T}}$, then*

$$\left(\int_{-\infty}^{\infty}(f^{\sigma})^2(x)\Delta x\right)^2\leqslant 2\int_{-\infty}^{\infty}x^2(f+f^{\sigma})^2(x)\Delta x\int_{-\infty}^{\infty}\left(f^{\triangle}\right)^2(x)\Delta x. \tag{43}$$

Proof. From (14), we may write that

$$x(f^2)^{\Delta}=xff^{\triangle}+xf^{\sigma}f^{\triangle}. \tag{44}$$

Applying the classical Hölder's inequality for sums with $n=2,\ a_1=xf,\ b_1=f^{\triangle}$, $a_2=xf^{\sigma}$ and $b_2=f^{\triangle}$, we get that

$$\begin{aligned}x(f^2)^{\Delta}&=xff^{\triangle}+xf^{\sigma}f^{\triangle}\\&\leqslant\sqrt{2}\sqrt{(xf)^2+(xf^{\sigma})^2}f^{\triangle}.\end{aligned}$$

Hence

$$\begin{aligned}\left(\int_{-\infty}^{\infty}x(f^2)^{\Delta}(x)\Delta x\right)^2&\leqslant 2\left(\int_{-\infty}^{\infty}\sqrt{(xf)^2+(xf^{\sigma})^2}f^{\triangle}\Delta x\right)^2\\&\leqslant 2\left(\int_{-\infty}^{\infty}x^2(f^2+(f^{\sigma})^2)(x)\Delta x\right)\left(\int_{-\infty}^{\infty}\left(f^{\triangle}\right)^2(x)\Delta x\right),\end{aligned} \tag{45}$$

where we have used the time scales Hölder's inequality (19). Using integration by parts (16) on the left hand-side of (45), we obtain that

$$\int_{-\infty}^{\infty} x(f^2)^{\Delta}(x)\Delta x = xf^2(x)\big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (f^{\sigma})^2(x)\Delta x.$$

Applying the same ideas of Lemma 4.3, we can get that

$$\lim_{t\to\pm\infty} [tf(t)]^2 = 0,$$

which leads to

$$\int_{-\infty}^{\infty} x(f^2)^{\Delta}(x)\Delta x = -\int_{-\infty}^{\infty} (f^{\sigma})^2(x)\Delta x.$$

Squaring both sides gives us that

$$\left(\int_{-\infty}^{\infty} x(f^2)^{\Delta}(x)\Delta x\right)^2 = \left(\int_{-\infty}^{\infty} (f^{\sigma})^2(x)\Delta x\right)^2. \tag{46}$$

Combining (45) and (46), we obtain that

$$\left(\int_{-\infty}^{\infty} (f^{\sigma})^2(x)\Delta x\right)^2 \leqslant 2\int_{-\infty}^{\infty} x^2(f+f^{\sigma})^2(x)\Delta x \int_{-\infty}^{\infty} \left(f^{\triangle}\right)^2(x)\Delta x,$$

which is the desired inequality (43). The proof is complete. □

REMARK 4.5. If we choose $\mathbb{T}=\mathbb{N}$, then (43) reduces to the following discrete inequality

$$\left(\sum_{-\infty}^{\infty} a_{n+1}^2\right)^2 \leqslant \sum_{-\infty}^{\infty} n^2(a_n+a_{n+1})^2 \sum_{-\infty}^{\infty} (\Delta a_n)^2,$$

which is essentially new.

REMARK 4.6. If $\mathbb{T}=\mathbb{R}$, then $f=f^{\sigma}$ and (43) reduces to the continuous inequality (13) due to Weyl.

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