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A UNIFIED APPROACH TO COPSON AND BEESACK TYPE INEQUALITIES ON TIME SCALES

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Abstract. Using time scale calculus we will prove some new theorems that unify the proofs of the continuous and discrete Copson type inequalities and indeed extend the Copson type inequalities to general time scales. Our results prove that the inequalities are true when the exponent k in Copson's inequality is negative and then prove that the approach that has been given by Bessack is also valid for the time scale cases.

1. Introduction

In 1928 Copson [8] proved that if $k>1$ and $c>1$, then

$$\sum_{n=1}^{\infty}\frac{\lambda_n}{\Lambda_n^c}A_n^k \leqslant \left(\frac{k}{c-1}\right)^k \sum_{n=1}^{\infty}\lambda_n\Lambda_n^{k-c}a_n^k, \tag{1}$$

where λ_i and $a_i \geqslant 0$, $\Lambda_n=\sum_{i=1}^{n}\lambda_i$ and $A_n=\sum_{i=1}^{n}\lambda_i a_i$. He also proved that if $k>1$ and $0\leqslant c<1$, then

$$\sum_{n=1}^{\infty}\frac{\lambda_n}{\Lambda_n^c}\left(A_n^*\right)^k \leqslant \left(\frac{k}{1-c}\right)^k \sum_{n=1}^{\infty}\lambda_n\Lambda_n^{k-c}a_n^k, \tag{2}$$

where $A_n^*=\sum_{i=n}^{\infty}\lambda_i a_i$. Fifty years later Copson [9, Theorems 1 and 3] proved that the continuous counterparts of the inequalities (1) and (2) are also true. In particular he proved that if $k\geqslant 1$ and $c>1$, then

$$\int_0^b \frac{\lambda(t)}{\Lambda^c(t)}\Phi^k(t)dt \leqslant \left(\frac{k}{c-1}\right)^k \int_0^b \lambda(t)\Lambda^{k-c}(t)g^k(t)dt, \tag{3}$$

where

$$\Lambda(t)=\int_0^t \lambda(s)ds, \text{ and } \Phi(t)=\int_0^t \lambda(s)g(s)ds,$$

and if $k>1$ and $0\leqslant c<1$, then

$$\int_a^{\infty} \frac{\lambda(t)}{\Lambda^c(t)}\left(\Phi^*(t)\right)^k dt \leqslant \left(\frac{k}{1-c}\right)^k \int_a^{\infty} \lambda(t)\Lambda^{k-c}(t)g^k(t)dt, \tag{4}$$

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where

$$\Phi^*(t) = \int_t^\infty \lambda(s)g(s)ds.$$

In 1980 Beesack [4] proved that the inequalities (3)–(4) and all other inequalities proved in [9] are also valid for negative values of k. In Beesack proofs he made some rearrangements of the proofs due to Copson by applying the elementary inequalities (see [10, Theorem 41], [7, p.45])

$$(u+v)^k \geqslant u^k + ku^{k-1}v, \quad \text{if} \quad (k<0 \text{ or } k>1), \tag{5}$$

$$(u+v)^k \leqslant u^k + ku^{k-1}v, \quad \text{if} \quad (0<k<1). \tag{6}$$

In recent years the study of dynamic inequalities on time scales has received a lot of attention and has become a major field in pure and applied mathematics. Many of these disciplines are concerned with the properties of these inequalities of various types (for more details we refer the reader to the book [1]). For more details of dynamic inequalities of Hardy's type on time scales. we refer the reader to the book [2] and the paper [11, 12, 13, 14, 15, 16, 17, 19] and the references they are cited.

In [15] the authors employed a new technique, which is different from those of Copson and Beesack, that depends on the time scale version of the Hölder inequality and the time scales chain rules to unify Copson inequalities (1)–(4) on an arbitrary time scale $\mathbb{T}$. In particular, in [15, Theorems 2.1 and 2.5] it was proved that if $1<c\leqslant k$, then

$$\int_a^\infty \frac{\lambda(t)}{(\Lambda^\sigma(t))^c}(\Phi^\sigma(t))^k\Delta t \leqslant \left(\frac{k}{c-1}\right)^k \int_a^\infty \lambda(t)\frac{(\Lambda^\sigma(t))^{(k-1)c}}{(\Lambda(t))^{k(c-1)}}g^k(t)\Delta t, \tag{7}$$

where

$$\Lambda(t) := \int_a^t \lambda(s)\Delta s, \text{ and } \Phi(t) := \int_a^t \lambda(s)g(s)\Delta s,$$

and if $0\leqslant c<1$ and $k>1$, then

$$\int_a^\infty \frac{\lambda(t)}{(\Lambda^\sigma(t))^c}(\Phi^*(t))^k\Delta t \leqslant \left(\frac{k}{1-c}\right)^k \int_a^\infty \lambda(t)(\Lambda^\sigma(t))^{k-c}g^k(t)\Delta t, \tag{8}$$

where

$$\Phi^*(t) := \int_t^\infty \lambda(s)g(s)\Delta s.$$

In [18] the authors proved the converses of (7) and (8). In particular, they proved that if $0<k<1<c$ and $\Lambda(\infty)=\infty$, then

$$\int_a^\infty \frac{\lambda(t)}{\Lambda^c(t)}(\Phi^\sigma(t))^k\Delta t \geqslant \left(\frac{k}{c-1}\right)^k \int_a^\infty \lambda(t)\Lambda^{k-c}(t)g^k(t)\Delta t, \tag{9}$$

and if $c\leqslant 0<k<1$, then

$$\int_a^\infty \frac{\lambda(t)}{(\Lambda^\sigma(t))^c}(\Phi^*(t))^k\Delta t \geqslant \left(\frac{k}{1-c}\right)^k \int_a^\infty \lambda(t)\,(\Lambda^\sigma(t))^{k-c}g^k(t)\Delta t. \tag{10}$$

It is worth mentioning here that, neither in [15] nor in [18] did the authors discussed the case of negative values of the exponent k in their results. The question that arises now is: Is it possible to unify the proofs of the Copson inequalities for all values of the exponent k to an arbitrary time scale $\mathbb{T}$? Our aim in this paper is to give the affirmative answer for this question and and prove that our results as special cases contain the Copson-Beesack inequalities. The results also complement the dynamic inequalities of Copson-type proved on time scales in the literature and cover the case with negative exponents. The outline of this paper is the following: In Section 2, we give some basics of calculus on time scales which will be used throughout the paper. In Sections 3,4 and 5 we will consider the three cases when $k > 1$, $k < 0$ and $0 < k < 1$, respectively. The main results will be proved by using the time scales Hölder inequality and the time scales chain rules.

2. Some basics of time scales calculus

In this section, we recall the following concepts related to the notion of time scales. For more details of time scale analysis, we refer the reader to the two books by Bohner and Peterson [5], [6]. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. The forward jump operator is defined by: $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$. A point $t \in \mathbb{T}$, is said to be right–dense if $\sigma(t) = t$. A function $g : \mathbb{T} \to \mathbb{R}$ is said to be right–dense continuous (rd–continuous) provided g is continuous at right–dense points and at left–dense points in $\mathbb{T}$, left hand limits exist and are finite. The set of all such rd–continuous functions is denoted by $C_{rd}(\mathbb{T})$. If $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}$, then we define

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}.$$

Otherwise, we define

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(s) - f(t)}{s - t}.$$

The time scale interval $[a,b]_{\mathbb{T}}$ is defined by $[a,b]_{\mathbb{T}} := [a,b] \cap \mathbb{T}$. In this paper, we will refer to the (delta) integral which we can define as follows: If $G^{\Delta}(t) = g(t)$, then the Cauchy (delta) integral of g is defined by $\int_a^t g(s)\Delta s := G(t) - G(a)$. It can be shown (see [5]) that if $g \in C_{rd}(\mathbb{T})$, then the Cauchy integral $G(t) := \int_{t_0}^t g(s)\Delta s$ exists, $t_0 \in \mathbb{T}$, and satisfies $G^{\Delta}(t) = g(t)$, $t \in \mathbb{T}$. An infinite integral is defined as $\int_a^{\infty} f(t)\Delta t = \lim_{b \to \infty} \int_a^b f(t)\Delta t$. We will make use of the following product rule for the delta derivative of the product fg of two $\Delta-$differentiable function f and g

$$(fg)^{\Delta} = f^{\Delta}g + f^{\sigma}g^{\Delta} = fg^{\Delta} + f^{\Delta}g^{\sigma}. \tag{11}$$

The following simple consequence of Keller's chain rule [5, Theorem 1.90] on time scales is needed in the proof of the main results

$$(x^{\gamma}(t))^{\Delta} = \gamma \int_0^1 [hx^{\sigma} + (1-h)x]^{\gamma-1}\, dh x^{\Delta}(t). \tag{12}$$

The Hölder inequality, see [3, Theorem 6.2], on time scales is given by

$$\int_a^b |f(t)g(t)|\Delta t \leqslant \left[\int_a^b |f(t)|^{\gamma}\Delta t\right]^{\frac{1}{\gamma}} \left[\int_a^b |g(t)|^{\nu}\Delta t\right]^{\frac{1}{\nu}}, \tag{13}$$

where $a,\ b \in \mathbb{T},\ f,\ g \in C_{rd}(\mathbb{T},\mathbb{R})$ and $\frac{1}{\gamma}+\frac{1}{\nu}=1$. This inequality is reversed if $0<\gamma<1$ and if $\gamma<0$ or $\nu<0$.

3. Inequalities for $k>1$

Throughout this section and latter, we will assume that all the functions in the statements of theorems are nonnegative, rd-continuous functions, $\sup\mathbb{T}=\infty$ and the integrals considered are assumed to exist. Now, we are ready to state and prove our main results when $k>1$.

THEOREM 3.1. *Assume that* $0,\ a,\ b\in\mathbb{T}$, *define* $\Lambda(t)=\int_0^t \lambda(s)\Delta s$ *and* $\Phi(t)=\int_0^t \lambda(s)g(s)\Delta s$. *If* $\Lambda(\infty)=\infty,\ 1<c\leqslant k$ *then for* $0<b<\infty$, *we have*

$$\begin{aligned} &\int_0^b \frac{\lambda(t)}{\Lambda^c(\sigma(t))}\Phi^k(\sigma(t))\Delta t + \left|\frac{k}{1-c}\right|\Phi^k(b)\Lambda^{1-c}(b) \\ &\leqslant \left(\frac{k}{c-1}\right)^k \int_0^b \lambda(t)\frac{(\Lambda^{\sigma}(t))^{(k-1)c}}{(\Lambda(t))^{k(c-1)}}g^k(t)\Delta t, \end{aligned} \tag{14}$$

and for $0<a<\infty$, *we have*

$$\begin{aligned} &\int_a^{\infty} \frac{\lambda(t)}{\Lambda^c(\sigma(t))}\Phi^k(\sigma(t))\Delta t + \lim_{b\to\infty}\left|\frac{k}{1-c}\right|\Phi^k(b)\Lambda^{1-c}(b) \\ &\leqslant \left(\frac{k}{c-1}\right)^k \int_a^{\infty} \lambda(t)\frac{(\Lambda^{\sigma}(t))^{(k-1)c}}{(\Lambda(t))^{k(c-1)}}g^k(t)\Delta t + \left|\frac{k}{1-c}\right|\Phi^k(a)\Lambda^{1-c}(a). \end{aligned} \tag{15}$$

Proof. Assume that $0<a\leqslant t\leqslant b<\infty$, and let $w(t)=\Phi^k(t)\Lambda^{1-c}(t)$. Using (11) we see that

$$w^{\Delta}(t)=\left(\Phi^k(t)\right)^{\Delta}\Lambda^{1-c}(t)+\Phi^k(\sigma(t))\left(\Lambda^{1-c}(t)\right)^{\Delta}. \tag{16}$$

From (12), since $\Phi^{\Delta}(t)=\lambda(t)g(t)\geqslant 0$, we have that

$$\begin{aligned} \left(\Phi^k(t)\right)^{\Delta} &= k\Phi^{\Delta}(t)\int_0^1 [h\Phi(\sigma(t))+(1-h)\Phi(t)]^{k-1}\,dh \\ &\leqslant k\lambda(t)g(t)\int_0^1 [h\Phi(\sigma(t))+(1-h)\Phi(\sigma(t))]^{k-1}\,dh \\ &= k\lambda(t)g(t)\Phi^{k-1}(\sigma(t)). \end{aligned} \tag{17}$$

Again using (12), since $\Lambda^{\Delta}(t)=\lambda(t)\geqslant 0$ and $c>1$, we have that

$$\left(\Lambda^{1-c}(t)\right)^{\Delta}\leqslant(1-c)\lambda(t)\Lambda^{-c}(\sigma(t)). \tag{18}$$

Substituting (17) and (18) into (16), we get that

$$w^{\Delta}(t)\leqslant k\lambda(t)g(t)\Phi^{k-1}(\sigma(t))\Lambda^{1-c}(t)+(1-c)\lambda(t)\Phi^{k}(\sigma(t))\Lambda^{-c}(\sigma(t)). \tag{19}$$

Rearranging terms in (19) and integrating from a to b, we get that

$$\begin{aligned}&\int_a^b\lambda(t)\Phi^{k}(\sigma(t))\Lambda^{-c}(\sigma(t))\Delta t-\frac{1}{1-c}\,\Phi^{k}(t)\Lambda^{1-c}(t)\Big|_a^b \\ &\leqslant\frac{k}{c-1}\int_a^b\lambda(t)g(t)\Phi^{k-1}(\sigma(t))\Lambda^{1-c}(t)\Delta t.\end{aligned} \tag{20}$$

Applying Hölder's inequality (13) on the term

$$\int_a^b\lambda(t)g(t)\Phi^{k-1}(\sigma(t))\Lambda^{1-c}(t)\Delta t,$$

with indices $\gamma=k/(k-1)$ and $\nu=k>1$, we have that

$$\begin{aligned}&\int_a^b\lambda(t)g(t)\Phi^{k-1}(\sigma(t))\Lambda^{1-c}(t)\Delta t \\ &=\int_a^b\left(\frac{\lambda(t)g(t)\Lambda^{1-c}(t)}{\lambda^{\frac{k-1}{k}}(t)\left(\Lambda^{-c}(\sigma(t))\right)^{\frac{k-1}{k}}}\right)\left(\frac{\lambda(t)\left(\Phi^{k}(\sigma(t))\right)}{\Lambda^{c}(\sigma(t))}\right)^{\frac{k-1}{k}}\Delta t \\ &\leqslant\left[\int_a^b\left(\frac{\lambda(t)g(t)\Lambda^{1-c}(t)}{\lambda^{\frac{k-1}{k}}(t)\left(\Lambda^{-c}(\sigma(t))\right)^{\frac{k-1}{k}}}\right)^{k}\Delta t\right]^{\frac{1}{k}}\times\left[\int_a^b\lambda(t)\Phi^{k}(\sigma(t))\Lambda^{-c}(\sigma(t))\Delta t\right]^{\frac{k-1}{k}}.\end{aligned} \tag{21}$$

Substituting (21) into (20) and raising both sides to k^{th} power, we get that

$$\begin{aligned}0&\leqslant\left(\int_a^b\lambda(t)\Phi^{k}(\sigma(t))\Lambda^{-c}(\sigma(t))\Delta t-\frac{1}{1-c}\,\Phi^{k}(t)\Lambda^{1-c}(t)\Big|_a^b\right)^{k} \\ &\leqslant\left(\frac{k}{c-1}\right)^{k}\int_a^b\lambda(t)g^{k}(t)\frac{\left(\Lambda^{\sigma}(t)\right)^{(k-1)c}}{(\Lambda(t))^{k(c-1)}}\Delta t\times\left(\int_a^b\lambda(t)\Phi^{k}(\sigma(t))\Lambda^{-c}(\sigma(t))\Delta t\right)^{k-1}.\end{aligned} \tag{22}$$

Applying inequality (5) to the term

$$\left(\int_a^b\lambda(t)\Phi^{k}(\sigma(t))\Lambda^{-c}(\sigma(t))\Delta t-\frac{1}{1-c}\,\Phi^{k}(t)\Lambda^{1-c}(t)\Big|_a^b\right)^{k},$$

with

$$u=\int_a^b \lambda(t)\Phi^k(\sigma(t))\Lambda^{-c}(\sigma(t))\Delta t, \quad \text{and} \quad v=-\frac{1}{1-c}\,\Phi^k(t)\Lambda^{1-c}(t)\Big|_a^b,$$

we have that

$$\begin{aligned}
&\left(\int_a^b \lambda(t)\Phi^k(\sigma(t))\Lambda^{-c}(\sigma(t))\Delta t-\frac{1}{1-c}\,\Phi^k(t)\Lambda^{1-c}(t)\Big|_a^b\right)^k \qquad (23)\\
&\geqslant \left(\int_a^b \lambda(t)\Phi^k(\sigma(t))\Lambda^{-c}(\sigma(t))\Delta t\right)^k-\frac{k}{1-c}\left(\int_a^b \lambda(t)\Phi^k(\sigma(t))\Lambda^{-c}(\sigma(t))\Delta t\right)^{k-1}\\
&\quad\times \Phi^k(t)\Lambda^{1-c}(t)\Big|_a^b\\
&=\left(\int_a^b \lambda(t)\Phi^k(\sigma(t))\Lambda^{-c}(\sigma(t))\Delta t\right)^{k-1}\\
&\quad\times\left(\int_a^b \lambda(t)\Phi^k(\sigma(t))\Lambda^{-c}(\sigma(t))\Delta t-\frac{k}{1-c}\,\Phi^k(t)\Lambda^{1-c}(t)\Big|_a^b\right).
\end{aligned}$$

Substituting (23) into (22), we obtain that

$$\begin{aligned}
&\int_a^b \lambda(t)\Phi^k(\sigma(t))\Lambda^{-c}(\sigma(t))\Delta t-\frac{k}{1-c}\,\Phi^k(t)\Lambda^{1-c}(t)\Big|_a^b\\
&\leqslant\left(\frac{k}{c-1}\right)^k\int_a^b \lambda(t)g^k(t)\frac{(\Lambda^\sigma(t))^{(k-1)c}}{(\Lambda(t))^{k(c-1)}}\Delta t.
\end{aligned}$$

This gives that

$$\begin{aligned}
&\int_a^b \lambda(t)\Phi^k(\sigma(t))\Lambda^{-c}(\sigma(t))\Delta t+\frac{k}{1-c}\Phi^k(a)\Lambda^{1-c}(a) \qquad (24)\\
&\leqslant\left(\frac{k}{c-1}\right)^k\int_a^b \lambda(t)g^k(t)\frac{(\Lambda^\sigma(t))^{(k-1)c}}{(\Lambda(t))^{k(c-1)}}\Delta t+\frac{k}{1-c}\Phi^k(b)\Lambda^{1-c}(b).
\end{aligned}$$

Next, we give two important estimates for the boundary terms $\Phi^k(a)\Lambda^{1-c}(a)$ and $\Phi^k(b)\Lambda^{1-c}(b)$. First, suppose that the integral $\int_a^b \lambda(t)\Phi^k(\sigma(t))\Lambda^{-c}(\sigma(t))\Delta t$ is convergent for $a=0$ or $b=\infty$ and since $\Phi(t)$ is an increasing function, then we have for

$0 < \alpha < \beta < \infty$ that

$$\int_{\alpha}^{\beta} \lambda(t)\Phi^k(\sigma(t))\Lambda^{-c}(\sigma(t))\Delta t \geqslant \Phi^k(\sigma(\alpha))\int_{\alpha}^{\beta} \lambda(t)\Lambda^{-c}(\sigma(t))\Delta t$$

$$\geqslant \frac{1}{1-c}\Phi^k(\sigma(\alpha))\int_{\alpha}^{\beta}\left(\Lambda^{1-c}(t)\right)^{\Delta}\Delta t = \frac{1}{1-c}\Phi^k(\sigma(\alpha))\left[\Lambda^{1-c}(\beta)-\Lambda^{1-c}(\alpha)\right],$$

which leads to (note that $c > 1$)

$$\frac{1}{c-1}\Phi^k(\sigma(\alpha))\Lambda^{1-c}(\alpha)$$
$$\leqslant \int_{\alpha}^{\beta} \lambda(t)\Phi^k(\sigma(t))\Lambda^{-c}(\sigma(t))\Delta t + \frac{1}{c-1}\Phi^k(\sigma(\alpha))\Lambda^{1-c}(\beta).$$

Letting $\alpha \to 0$, we get that

$$0 \leqslant \lim_{\alpha\to 0^+}\frac{1}{c-1}\Phi^k(\sigma(\alpha))\Lambda^{1-c}(\alpha) \leqslant \int_{0}^{\beta} \lambda(t)\Phi^k(\sigma(t))\Lambda^{-c}(\sigma(t))\Delta t, \tag{25}$$

and letting $\beta \to \infty$, we have

$$0 \leqslant \frac{1}{c-1}\Phi^k(\sigma(\alpha))\Lambda^{1-c}(\alpha) \leqslant \int_{\alpha}^{\infty} \lambda(t)\Phi^k(\sigma(t))\Lambda^{-c}(\sigma(t))\Delta t, \text{ if } \Lambda(\infty)=\infty. \tag{26}$$

Second, suppose that the integral

$$\int_{a}^{b} \lambda(t)g^k(t)\frac{(\Lambda^{\sigma}(t))^{(k-1)c}}{(\Lambda(t))^{k(c-1)}}\Delta t,$$

is convergent for $a = 0$ or $b = \infty$, then for $0 < \alpha < \beta < \infty$, we have that

$$\Phi(\beta) = \Phi(\alpha) + \int_{\alpha}^{\beta} \lambda(t)g(t)\Delta t \quad \frac{(\Lambda^{\sigma}(t))^{(k-1)c}}{(\Lambda(t))^{k(c-1)}}$$
$$= \Phi(\alpha) + \int_{\alpha}^{\beta}\left(g(t)(\Lambda^{\sigma}(t))^{\frac{(k-1)c}{k}}(\Lambda(t))^{\frac{k(1-c)}{k}}\lambda^{\frac{1}{k}}(t)\right)$$
$$\times\left((\Lambda^{\sigma}(t))^{\frac{(1-k)c}{k}}(\Lambda(t))^{\frac{k(c-1)}{k}}\lambda^{\frac{k-1}{k}}(t)\right)\Delta t$$

$$\leqslant \Phi(\alpha)+\left[\int_{\alpha}^{\beta}\left(g^{k}(t)\frac{(\Lambda^{\sigma}(t))^{(k-1)c}}{(\Lambda(t))^{k(c-1)}}\lambda(t)\right)\Delta t\right]^{\frac{1}{k}}$$

$$\times\left[\int_{\alpha}^{\beta}(\Lambda^{\sigma}(t))^{-c}(\Lambda(t))^{\frac{k(c-1)}{k-1}}\lambda(t)\Delta t\right]^{\frac{k-1}{k}}$$

$$\leqslant \Phi(\alpha)+\left[\int_{\alpha}^{\beta}\left(g^{k}(t)\frac{(\Lambda^{\sigma}(t))^{(k-1)c}}{(\Lambda(t))^{k(c-1)}}\lambda(t)\right)\Delta t\right]^{\frac{1}{k}}$$

$$\times\left[\left(\frac{k-1}{c-1}\right)\left(\Lambda^{\frac{c-1}{k-1}}(\beta)-\Lambda^{\frac{c-1}{k-1}}(\alpha)\right)\right]^{\frac{k-1}{k}}.$$

Hence, we obtain

$$\Phi(\beta)\Lambda^{\frac{1-c}{k}}(\beta)\leqslant \Phi(\alpha)\Lambda^{\frac{1-c}{k}}(\beta)+\left[\int_{\alpha}^{\beta}g^{k}(t)\frac{(\Lambda^{\sigma}(t))^{(k-1)c}}{(\Lambda(t))^{k(c-1)}}\lambda(t)\Delta t\right]^{\frac{1}{k}}$$

$$\times\left[\left(\frac{k-1}{c-1}\right)\left(1-\left(\frac{\Lambda(\alpha)}{\Lambda(\beta)}\right)^{\frac{c-1}{k-1}}\right)\right]^{\frac{k-1}{k}}.$$

Letting $\alpha\to 0$ we get that

$$0\leqslant \Phi^{k}(\beta)\Lambda^{1-c}(\beta)\leqslant\left(\frac{k-1}{c-1}\right)^{k-1}\int_{0}^{\beta}g^{k}(t)\frac{(\Lambda^{\sigma}(t))^{(k-1)c}}{(\Lambda(t))^{k(c-1)}}\lambda(t)\Delta t, \tag{27}$$

and letting $\beta\to\infty$, we get that

$$0\leqslant \lim_{\beta\to\infty}\Phi^{k}(\beta)\Lambda^{1-c}(\beta)\leqslant\left(\frac{k-1}{c-1}\right)^{k-1}\int_{\alpha}^{\infty}g^{k}(t)\frac{(\Lambda^{\sigma}(t))^{(k-1)c}}{(\Lambda(t))^{k(c-1)}}\lambda(t)\Delta t,\quad \text{if } \Lambda(\infty)=\infty. \tag{28}$$

Now, we can write (24) in the following form

$$\int_{a}^{b}\lambda(t)\Phi^{k}(\sigma(t))\Lambda^{-c}(\sigma(t))\Delta t+\left|\frac{k}{1-c}\right|\Phi^{k}(b)\Lambda^{1-c}(b) \tag{29}$$

$$\leqslant\left(\frac{k}{c-1}\right)^{k}\int_{a}^{b}\lambda(t)g^{k}(t)\frac{(\Lambda^{\sigma}(t))^{(k-1)c}}{(\Lambda(t))^{k(c-1)}}\Delta t+\left|\frac{k}{1-c}\right|\Phi^{k}(a)\Lambda^{1-c}(a).$$

Using (27), we have from the last inequality for $0<b<\infty$ (letting $a\to 0$), that

$$\int_0^b \lambda(t)\Phi^k(\sigma(t))\Lambda^{-c}(\sigma(t))\Delta t+\left|\frac{k}{1-c}\right|\Phi^k(b)\Lambda^{1-c}(b) \tag{30}$$
$$\leqslant\left(\frac{k}{c-1}\right)^k\int_0^b \lambda(t)g^k(t)\frac{(\Lambda^\sigma(t))^{(k-1)c}}{(\Lambda(t))^{k(c-1)}}\Delta t,$$

which is valid whenever the integral on the right-hand side converges. Similarly, from (29) for $0<a<\infty$ if we let $b\to\infty$, we get that

$$\int_a^\infty \lambda(t)\Phi^k(\sigma(t))\Lambda^{-c}(t)\Delta t+\lim_{b\to\infty}\left|\frac{k}{1-c}\right|\Phi^k(b)\Lambda^{1-c}(b) \tag{31}$$
$$\leqslant\left(\frac{k}{c-1}\right)^k\int_a^\infty \lambda(t)g^k(t)\frac{(\Lambda^\sigma(t))^{(k-1)c}}{(\Lambda(t))^{k(c-1)}}\Delta t+\left|\frac{k}{1-c}\right|\Phi^k(a)\Lambda^{1-c}(a).$$

The inequalities (30) and (31) are the required inequalities (14) and (15). For the case $0<k<1$ and $c>1$, we apply the reversed of Hölder inequality (13) and the inequality (6) instead of (5). This completes the proof. $\square$

As in the proof of Theorem 3.1, we can easily prove the following dual theorem.

THEOREM 3.2. *Assume that* $0,\ a,\ b\in\mathbb{T}$, *define*

$$\Lambda(t)=\int_0^t \lambda(s)\Delta s,\ \text{and}\ \ \Phi^*(t)=\int_t^\infty \lambda(s)g(s)\Delta s.$$

If $c<1<k$ *then for* $0<b<\infty$, *we have*

$$\int_0^b \lambda(t)\left(\Phi^*(t)\right)^k\Lambda^{-c}(t)\Delta t \tag{32}$$
$$\leqslant\left|\frac{k}{1-c}\right|^k\int_0^b \lambda(t)\Lambda^{k-c}(t)g^k(t)\Delta t+\left|\frac{k}{1-c}\right|\left(\Phi^*(b)\right)^k\Lambda^{1-c}(b),$$

and for $0<a<\infty$, *we have*

$$\int_a^\infty \lambda(t)\left(\Phi^*(t)\right)^k\Lambda^{-c}(t)\Delta t+\left|\frac{k}{1-c}\right|\left(\Phi^*(a)\right)^k\Lambda^{1-c}(a) \tag{33}$$
$$\leqslant\left|\frac{k}{1-c}\right|^k\int_a^\infty \lambda(t)\Lambda^{k-c}(t)g^k(t)\Delta t.$$

REMARK 3.1. The two dynamic inequalities (15) and (33) give respectively improvement to dynamic Copson-type inequalities (7) and (8) due to Saker et al. [15] and the two dynamic inequalities (14) and (32) are essentially new.

REMARK 3.2. If $\mathbb{T}=\mathbb{R}$, then $\sigma(t)=t$ and the two dynamic inequalities (14) and (15) reduce respectively to the following continuous inequalities due to Beesack [4]. If $1<c\leqslant k$, then for $0<b<\infty$ we have

$$\int_0^b \frac{\lambda(t)}{\Lambda^c(t)}\Phi^k(t)dt+\left|\frac{k}{1-c}\right|\Phi^k(b)\Lambda^{1-c}(b)$$
$$\leqslant\left(\frac{k}{1-c}\right)^k\int_0^b \lambda(t)\Lambda^{k-c}(t)g^k(t)dt,$$

and for $0<a<\infty$, we have

$$\int_a^\infty \frac{\lambda(t)}{\Lambda^c(t)}\Phi^k(t)dt+\lim_{b\to\infty}\left|\frac{k}{1-c}\right|\Phi^k(b)\Lambda^{1-c}(b)$$
$$\leqslant\left(\frac{k}{1-c}\right)^k\int_a^\infty \lambda(t)\Lambda^{k-c}(t)g^k(t)dt+\left|\frac{k}{1-c}\right|\Phi^k(a)\Lambda^{1-c}(a),$$

where $\Lambda(t)=\int_0^t\lambda(s)ds$ and $\Phi(t)=\int_0^t\lambda(s)g(s)ds$.

REMARK 3.3. If $\mathbb{T}=\mathbb{N}$, then the two dynamic inequalities (14) and (15) reduce respectively to the following discrete Copson-type inequalities. If $1<c\leqslant k$, $\Lambda_n=\sum_{i=1}^n\lambda_i$ and $\Phi_n=\sum_{i=1}^n\lambda_i g_i$, then

$$\sum_{n=1}^m \frac{\lambda_n}{\Lambda_{n+1}^c}\Phi_{n+1}^k+\left|\frac{k}{1-c}\right|\Phi_m^k\Lambda_m^{1-c}\leqslant\left(\frac{k}{c-1}\right)^k\sum_{n=1}^m\lambda_n\frac{(\Lambda_{n+1})^{(k-1)c}}{(\Lambda_n)^{k(c-1)}}\Lambda_n^{k-c}g_n^k,$$

which is an improvement to (1), and

$$\sum_{n=1}^\infty \frac{\lambda_n}{\Lambda_{n+1}^c}\Phi_{n+1}^k+\left|\frac{k}{1-c}\right|\Phi_\infty^k\Lambda_\infty^{1-c}\leqslant\left(\frac{k}{c-1}\right)^k\sum_{n=1}^\infty\lambda_n\frac{(\Lambda_{n+1})^{(k-1)c}}{(\Lambda_n)^{k(c-1)}}g_n^k+\left|\frac{k}{1-c}\right|\Phi_1^k\Lambda_1^{1-c}.$$

4. Inequalities for $k<0$

In this section, we will consider the case when $k<0$.

THEOREM 4.1. *Assume that* $0,\ a,\ b\in\mathbb{T}$, *define*

$$\Lambda(t)=\int_0^t\lambda(s)\Delta s,\text{ and }\Phi(t)=\int_0^t\lambda(s)g(s)\Delta s.$$

If $k<0$ *and* $c<1$, *then for* $0<b<\infty$, *we have*

$$\int_0^b \frac{\lambda(t)}{\Lambda^c(\sigma(t))}\Phi^k(t)\Delta t+\left|\frac{k}{1-c}\right|\Phi^k(b)\Lambda^{1-c}(b) \leqslant \left|\frac{k}{c-1}\right|^k \int_0^b \lambda(t)\Lambda^{k-c}(\sigma(t))g^k(t)\Delta t, \tag{34}$$

and for $0<a<\infty$, *we have*

$$\int_a^\infty \frac{\lambda(t)}{\Lambda^c(\sigma(t))}\Phi^k(t)\Delta t+\lim_{b\to\infty}\left|\frac{k}{1-c}\right|\Phi^k(b)\Lambda^{1-c}(b) \leqslant \left|\frac{k}{c-1}\right|^k \int_a^\infty \lambda(t)\Lambda^{k-c}(\sigma(t))g^k(t)\Delta t+\left|\frac{k}{1-c}\right|\Phi^k(a)\Lambda^{1-c}(a). \tag{35}$$

Proof. As in the proof of Theorem 3.1, we consider the function $w(t)=\Phi^k(t)\Lambda^{1-c}(t)$ and get that

$$w^\Delta(t)=\Phi^k(t)\left(\Lambda^{1-c}(t)\right)^\Delta+\left(\Phi^k(t)\right)^\Delta\Lambda^{1-c}(\sigma(t)). \tag{36}$$

Applying the time scales chain rule twice, we get that

$$w^\Delta(t)\geqslant(1-c)\lambda(t)\Phi^k(t)\Lambda^{-c}(\sigma(t))+k\lambda(t)g(t)\Phi^{k-1}(t)\Lambda^{1-c}(\sigma(t)). \tag{37}$$

Rearranging terms in (37) and integrating from a to b, we have that

$$\int_a^b \lambda(t)\Phi^k(t)\Lambda^{-c}(\sigma(t))\Delta t-\frac{1}{1-c}\,\Phi^k(t)\Lambda^{1-c}(t)\Big|_a^b \geqslant \left|\frac{k}{c-1}\right|\int_a^b \lambda(t)g(t)\Phi^{k-1}(t)\Lambda^{1-c}(\sigma(t))\Delta t. \tag{38}$$

Applying the reverse of Hölder inequality (13) on the right hand side of (38) with indices $\gamma=k/(k-1)$ and $\nu=k<0$, we have that

$$\begin{aligned}\int_a^b \lambda(t)\Phi^k(t)\Lambda^{-c}(\sigma(t))\Delta t-\frac{1}{1-c}\,\Phi^k(t)\Lambda^{1-c}(t)\Big|_a^b &\geqslant \left|\frac{k}{c-1}\right|\left(\int_a^b \lambda(t)g^k(t)\Lambda^{k-c}(\sigma(t))\Delta t\right)^{\frac{1}{k}}\\ &\quad\times\left(\int_a^b \lambda(t)\Phi^k(t)\Lambda^{-c}(\sigma(t))\Delta t\right)^{\frac{k-1}{k}}.\end{aligned} \tag{39}$$

By raising both sides of (39) to the power $k<0$ and then applying inequality (5) on the left hand-side of the resulting inequality with

$$u=\int_a^b \lambda(t)\Phi^k(t)\Lambda^{-c}(\sigma(t))\Delta t, \text{ and } v=-\frac{1}{1-c}\,\Phi^k(t)\Lambda^{1-c}(t)\Big|_a^b,$$

we get that

$$\begin{aligned}&\left(\int_a^b \lambda(t)\Phi^k(t)\Lambda^{-c}(\sigma(t))\Delta t-\frac{1}{1-c}\,\Phi^k(t)\Lambda^{1-c}(t)\Big|_a^b\right)^k\\ &\geqslant\left(\int_a^b \lambda(t)\Phi^k(t)\Lambda^{-c}(\sigma(t))\Delta t\right)^{k-1}\\ &\quad\times\left[\int_a^b \lambda(t)\Phi^k(t)\Lambda^{-c}(\sigma(t))\Delta t-\frac{k}{1-c}\,\Phi^k(t)\Lambda^{1-c}(t)\Big|_a^b\right].\end{aligned}$$

The rest of the proof is similar to the proof of Theorem 3.1 and hence it is omitted. This completes the proof. □

As in the proof of Theorem 4.1, we can easily prove the following dual theorem.

THEOREM 4.2. *Suppose that $\mathbb{T}$ be a time scale with $0,\ a,\ b\in\mathbb{T}$, define*

$$\Lambda(t)=\int_0^t \lambda(s)\Delta s, \text{ and } \ \Phi^*(t)=\int_t^\infty \lambda(s)g(s)\Delta s.$$

If $k<0$ and $c>1$, then for $0<b<\infty$, we have

$$\begin{aligned}&\int_0^b \frac{\lambda(t)}{\Lambda^c(t)}\left(\Phi^*(\sigma(t))\right)^k\Delta t \\ &\leqslant\left|\frac{k}{1-c}\right|^k\int_0^b \lambda(t)\Lambda^{k-c}(t)g^k(t)\Delta t+\left|\frac{k}{1-c}\right|\left(\Phi^*(b)\right)^k\Lambda^{1-c}(b),\end{aligned} \tag{40}$$

and for $0<a<\infty$, we have

$$\begin{aligned}&\int_a^\infty \frac{\lambda(t)}{\Lambda^c(t)}\left(\Phi^*(\sigma(t))\right)^k\Delta t+\left|\frac{k}{1-c}\right|\left(\Phi^*(a)\right)^k\Lambda^{1-c}(a) \\ &\leqslant\left|\frac{k}{1-c}\right|^k\int_a^\infty \lambda(t)\Lambda^{k-c}(t)g^k(t)\Delta t.\end{aligned} \tag{41}$$

REMARK 4.1. If $\mathbb{T}=\mathbb{R}$, then $\sigma(t)=t$ and the two dynamic inequalities (34) and (35) reduce respectively to the following continuous inequalities due to Beesack [4]. If $k<0$ and $c<1$, then for $0<b<\infty$ we have

$$\int_0^b \frac{\lambda(t)}{\Lambda^c(t)}\Phi^k(t)dt+\left|\frac{k}{1-c}\right|\Phi^k(b)\Lambda^{1-c}(b)\leqslant\left|\frac{k}{c-1}\right|^k\int_0^b\lambda(t)\Lambda^{k-c}(t)g^k(t)dt,$$

where

$$\Lambda(t)=\int_0^t\lambda(s)ds \text{ and } \Phi(t)=\int_0^t\lambda(s)g(s)ds,$$

and for $0<a<\infty$, we have

$$\begin{aligned}&\int_a^\infty \frac{\lambda(t)}{\Lambda^c(t)}\Phi^k(t)dt+\lim_{b\to\infty}\left|\frac{k}{1-c}\right|\Phi^k(b)\Lambda^{1-c}(b)\\ \leqslant&\left|\frac{k}{c-1}\right|^k\int_a^\infty\lambda(t)\Lambda^{k-c}(t)g^k(t)dt+\left|\frac{k}{1-c}\right|\Phi^k(a)\Lambda^{1-c}(a).\end{aligned}$$

REMARK 4.2. If $\mathbb{T}=\mathbb{N}$, then the two dynamic inequalities (34) and (35) reduce respectively to the following discrete Copson-type inequalities. If $k<0$, $c<1$, $\Lambda_n=\sum_{i=1}^n\lambda_i$ and $\Phi_n=\sum_{i=1}^n\lambda_i g_i$, then

$$\sum_{n=1}^m\frac{\lambda_n}{\Lambda_n^c}\Phi_{n+1}^k+\left|\frac{k}{1-c}\right|\Phi_m^k\Lambda_m^{1-c}\leqslant\left|\frac{k}{c-1}\right|^k\sum_{n=1}^m\lambda_n\Lambda_n^{k-c}g_n^k,$$

and

$$\sum_{n=1}^\infty\frac{\lambda_n}{\Lambda_n^c}\Phi_{n+1}^k+\left|\frac{k}{1-c}\right|\Phi_\infty^k\Lambda_\infty^{1-c}\leqslant\left|\frac{k}{c-1}\right|^k\sum_{n=1}^\infty\lambda_n\Lambda_n^{k-c}g_n^k+\left|\frac{k}{1-c}\right|\Phi_n^k\Lambda_n^{1-c}.$$

5. Inequalities for $0<k<1$

In this section, we will consider the case when $0<k<1$.

THEOREM 5.1. *Assume that* $0,\ a,\ b\in\mathbb{T}$*, define*

$$\Lambda(t)=\int_0^t\lambda(s)\Delta s, \text{ and } \Phi(t)=\int_0^t\lambda(s)g(s)\Delta s.$$

If $\Lambda(\infty)=\infty$ *and* $0<k<1<c$*, then for* $0<b<\infty$*, we have*

$$\begin{aligned}&\int_0^b\frac{\lambda(t)}{\Lambda^c(t)}\Phi^k(\sigma(t))\Delta t+\left|\frac{k}{1-c}\right|\Phi^k(b)\Lambda^{1-c}(b) \qquad (42)\\ \geqslant&\left(\frac{k}{c-1}\right)^k\int_0^b\lambda(t)\Lambda^{k-c}(t)g^k(t)\Delta t,\end{aligned}$$

and for $0 < a < \infty$, *we have*

$$\int_a^\infty \frac{\lambda(t)}{\Lambda^c(t)} \Phi^k(\sigma(t))\Delta t + \lim_{b\to\infty} \left|\frac{k}{1-c}\right| \Phi^k(b)\Lambda^{1-c}(b) \tag{43}$$
$$\geqslant \left(\frac{k}{c-1}\right)^k \int_a^\infty \lambda(t)\Lambda^{k-c}(t) g^k(t)\Delta t + \left|\frac{k}{1-c}\right| \Phi^k(a)\Lambda^{1-c}(a).$$

Proof. Assume that $0 < a \leqslant t \leqslant b < \infty$, and let $w(t) = \Phi^k(t)\Lambda^{1-c}(t)$. Using (11) we see that

$$w^\Delta(t) = \left(\Phi^k(t)\right)^\Delta \Lambda^{1-c}(t) + \Phi^k(\sigma(t)) \left(\Lambda^{1-c}(t)\right)^\Delta. \tag{44}$$

From (12), since $\Phi^\Delta(t) = \lambda(t) g(t) \geqslant 0$, we have that

$$\left(\Phi^k(t)\right)^\Delta = k\Phi^\Delta(t) \int_0^1 [h\Phi(\sigma(t)) + (1-h)\Phi(t)]^{k-1}\, dh \tag{45}$$
$$\geqslant k\lambda(t) g(t) \int_0^1 [h\Phi(\sigma(t)) + (1-h)\Phi(\sigma(t))]^{k-1}\, dh$$
$$= k\lambda(t) g(t) \Phi^{k-1}(\sigma(t)).$$

Again using (12), since $\Lambda^\Delta(t) = \lambda(t) \geqslant 0$ and $c > 1$, we have that

$$\left(\Lambda^{1-c}(t)\right)^\Delta \geqslant (1-c)\lambda(t)\Lambda^{-c}(t). \tag{46}$$

Substituting (45) and (46) into (44), we get that

$$w^\Delta(t) \geqslant k\lambda(t) g(t)\Phi^{k-1}(\sigma(t))\Lambda^{1-c}(t) + (1-c)\lambda(t)\Phi^k(\sigma(t))\Lambda^{-c}(t). \tag{47}$$

Rearranging terms in (47) and integrating from a to b, we get that

$$\int_a^b \lambda(t)\Phi^k(\sigma(t))\Lambda^{-c}(t)\Delta t - \frac{1}{1-c}\, \Phi^k(t)\Lambda^{1-c}(t)\Big|_a^b \tag{48}$$
$$\geqslant \frac{k}{c-1} \int_a^b \lambda(t) g(t)\Phi^{k-1}(\sigma(t))\Lambda^{1-c}(t)\Delta t.$$

Applying the reverse of Hölder inequality (13) on the term

$$\int_a^b \lambda(t) g(t)\Phi^{k-1}(\sigma(t))\Lambda^{1-c}(t)\Delta t,$$

with indices $\gamma = k/(k-1) < 0$ and $\nu = k < 1$, we have that

$$\int_a^b \lambda(t) g(t) \Phi^{k-1}(\sigma(t)) \Lambda^{1-c}(t) \Delta t \tag{49}$$

$$\geqslant \left[\int_a^b \lambda(t) \Lambda^{k-c}(t) g^k(t) \Delta t\right]^{\frac{1}{k}} \times \left[\int_a^b \lambda(t) \Phi^k(\sigma(t)) \Lambda^{-c}(t) \Delta t\right]^{\frac{k-1}{k}}.$$

Substituting (49) into (48) and raising both sides to k^{th} power, we get that

$$\left(\int_a^b \lambda(t) \Phi^k(\sigma(t)) \Lambda^{-c}(t) \Delta t - \frac{1}{1-c} \Phi^k(t) \Lambda^{1-c}(t)\Big|_a^b\right)^k \tag{50}$$

$$\geqslant \left(\frac{k}{c-1}\right)^k \int_a^b \lambda(t) \Lambda^{k-c}(t) g^k(t) \Delta t \times \left(\int_a^b \lambda(t) \Phi^k(\sigma(t)) \Lambda^{-c}(t) \Delta t\right)^{k-1}.$$

Applying inequality (5) to the term

$$\left(\int_a^b \lambda(t) \Phi^k(\sigma(t)) \Lambda^{-c}(t) \Delta t - \frac{1}{1-c} \Phi^k(t) \Lambda^{1-c}(t)\Big|_a^b\right)^k,$$

with

$$u = \int_a^b \lambda(t) \Phi^k(\sigma(t)) \Lambda^{-c}(t) \Delta t, \quad \text{and} \quad v = -\frac{1}{1-c} \Phi^k(t) \Lambda^{1-c}(t)\Big|_a^b,$$

we have that

$$\left(\int_a^b \lambda(t) \Phi^k(\sigma(t)) \Lambda^{-c}(t) \Delta t - \frac{1}{1-c} \Phi^k(t) \Lambda^{1-c}(t)\Big|_a^b\right)^k \tag{51}$$

$$\leqslant \left(\int_a^b \lambda(t) \Phi^k(\sigma(t)) \Lambda^{-c}(t) \Delta t\right)^{k-1}$$

$$\times \left(\int_a^b \lambda(t) \Phi^k(\sigma(t)) \Lambda^{-c}(t) \Delta t - \frac{k}{1-c} \Phi^k(t) \Lambda^{1-c}(t)\Big|_a^b\right).$$

Substituting (51) into (50), we obtain that

$$\int_a^b \lambda(t) \Phi^k(\sigma(t)) \Lambda^{-c}(\sigma(t)) \Delta t - \frac{k}{1-c} \Phi^k(t) \Lambda^{1-c}(t)\Big|_a^b$$

$$\geqslant \left(\frac{k}{c-1}\right)^k \int_a^b \lambda(t) g^k(t) \frac{(\Lambda^\sigma(t))^{(k-1)c}}{(\Lambda(t))^{k(c-1)}} \Delta t.$$

This gives that

$$\int_a^b \lambda(t)\Phi^k(\sigma(t))\Lambda^{-c}(\sigma(t))\Delta t + \frac{k}{1-c}\Phi^k(a)\Lambda^{1-c}(a) \tag{52}$$
$$\geqslant \left(\frac{k}{c-1}\right)^k \int_a^b \lambda(t)\Lambda^{k-c}(t)g^k(t)\Delta t + \frac{k}{1-c}\Phi^k(b)\Lambda^{1-c}(b).$$

The rest of the proof is similar to the proof of Theorem 3.1 and hence it is omitted. This completes the proof. □

As in the proof of Theorem 3.1, we can easily prove the following dual theorem.

THEOREM 5.2. *Assume that* $0,\ a,\ b \in \mathbb{T}$, *define*

$$\Lambda(t) = \int_0^t \lambda(s)\Delta s \text{ and } \Phi^*(t) = \int_t^\infty \lambda(s)g(s)\Delta s.$$

If $0 < k < 1$ *and* $c < 1$, *then for* $0 < b < \infty$, *we have*

$$\int_0^b \lambda(t)\left(\Phi^*(t)\right)^k \Lambda^{-c}(\sigma(t))\Delta t \tag{53}$$
$$\geqslant \left(\frac{k}{1-c}\right)^k \int_0^b \lambda(t)\Lambda^{k-c}(\sigma(t))g^k(t)\Delta t + \left|\frac{k}{1-c}\right| \left(\Phi^*(b)\right)^k \Lambda^{1-c}(b),$$

and for $0 < a < \infty$, *we have*

$$\int_a^\infty \lambda(t)\left(\Phi^*(t)\right)^k \Lambda^{-c}(\sigma(t))\Delta t + \left|\frac{k}{1-c}\right| \left(\Phi^*(a)\right)^k \Lambda^{1-c}(a) \tag{54}$$
$$\geqslant \left(\frac{k}{1-c}\right)^k \int_a^\infty \lambda(t)\Lambda^{k-c}(\sigma(t))g^k(t)\Delta t.$$

REMARK 5.1. The two dynamic inequalities (43) and (54) give respectively improvement to dynamic Copson-type inequalities (9) and (10) due to Saker et al. [15] and the two dynamic inequalities (42) and (53) are essentially new.

REMARK 5.2. If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$ and the two dynamic inequalities (42) and (43) reduce respectively to the following continuous inequalities due to Beesack [4]. If $0 < k < 1 < c$, then for $0 < b < \infty$, we have

$$\int_0^b \frac{\lambda(t)}{\Lambda^c(t)}\Phi^k(t)dt + \left|\frac{k}{1-c}\right| \Phi^k(b)\Lambda^{1-c}(b)$$
$$\geqslant \left(\frac{k}{c-1}\right)^k \int_0^b \lambda(t)\Lambda^{k-c}(t)g^k(t)dt,$$

and for $0<a<\infty$, we have

$$\int_a^\infty \frac{\lambda(t)}{\Lambda^c(t)}\Phi^k(t)dt+\lim_{b\to\infty}\left|\frac{k}{1-c}\right|\Phi^k(b)\Lambda^{1-c}(b)$$
$$\geqslant\left(\frac{k}{c-1}\right)^k\int_a^\infty \lambda(t)\Lambda^{k-c}(t)g^k(t)dt+\left|\frac{k}{1-c}\right|\Phi^k(a)\Lambda^{1-c}(a),$$

where $\Lambda(t)=\int_0^t\lambda(s)ds$ and $\Phi(t)=\int_0^t\lambda(s)g(s)ds$.

REMARK 5.3. If $\mathbb{T}=\mathbb{N}$, then the two dynamic inequalities (42) and (43) reduce respectively to the following discrete Copson-type inequalities. If $0<k<1<c$, $\Lambda_n=\sum_{i=1}^n\lambda_i$ and $\Phi_n=\sum_{i=1}^n\lambda_i g_i$, then

$$\sum_{n=1}^m\frac{\lambda_n}{\Lambda_n^c}\Phi_{n+1}^k+\left|\frac{k}{1-c}\right|\Phi_m^k\Lambda_m^{1-c}\geqslant\left(\frac{k}{c-1}\right)^k\sum_{n=1}^m\lambda_n\Lambda_n^{k-c}g_n^k,$$

and

$$\sum_{n=1}^\infty\frac{\lambda_n}{\Lambda_n^c}\Phi_{n+1}^k+\left|\frac{k}{1-c}\right|\Phi_\infty^k\Lambda_\infty^{1-c}\geqslant\left(\frac{k}{c-1}\right)^k\sum_{n=1}^\infty\lambda_n\Lambda_n^{k-c}g_n^k+\left|\frac{k}{1-c}\right|\Phi_n^k\Lambda_n^{1-c}.$$

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