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SHARP REDHEFFER–TYPE AND BECKER–STARK–TYPE INEQUALITIES WITH AN APPLICATION

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Abstract. In this paper, we give sharp Redheffer-type and Becker-Stark-type inequalities for trigonometric functions. As an application of Redheffer-type inequality, we improve the well-known Yang Le inequality.

1. Introduction

Redheffer [21] proposed, then Williams [25] proved that, for $x \in \mathbb{R}$,

$$\frac{1-x^2}{1+x^2} \leqslant \frac{\sin \pi x}{\pi x}, \tag{1}$$

or alternatively

$$\frac{\pi^2-x^2}{\pi^2+x^2} \leqslant \frac{\sin x}{x}. \tag{2}$$

In 2012, He and Huang [16] pointed out (without proof) that, for $0<x<\pi$,

$$\frac{\pi^2-x^2}{\pi^2+\alpha x^2} < \frac{\sin x}{x} < \frac{\pi^2-x^2}{\pi^2+\beta x^2}, \tag{3}$$

with the best possible constants $\alpha = 1$ and $\beta = \frac{\pi^2}{6}-1$. In 2013, Aharonov and Elias [2] rediscovered and proved this inequality. In 2016, Bhayo and Sándor [8, Theorem 7] proved the right-hand side of (3). We notice that the proof from [8] is more elegant.

Some Redheffer-type inequalities for trigonometric and hyperbolic functions were established in [8, 11, 17, 22, 37, 41]. For example, Chen et al. [11] proved that, for $|x| \leqslant \pi/2$,

$$\frac{\pi^2-4x^2}{\pi^2+4x^2} \leqslant \cos x. \tag{4}$$

Also in [16], He and Huang pointed out (without proof) that, for $0<x<\pi/2$,

$$\frac{\pi^2-4x^2}{\pi^2+\left(\frac{16}{\pi}-4\right)x^2} < \cos x < \frac{\pi^2-4x^2}{\pi^2+\left(\frac{\pi^2}{2}-4\right)x^2}, \tag{5}$$

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where the constants $\frac{16}{\pi}-4$ and $\frac{\pi^2}{2}-4$ are the best possible. Aharonov and Elias [2], Bhayo and Sándor [8, Theorem 9] rediscovered and proved this inequality. The inequality (5) can be written for $0<x<\pi/2$ as

$$\frac{\pi^2-(2x)^2}{\pi^2+(\frac{4}{\pi}-1)(2x)^2}<\cos x<\frac{\pi^2-(2x)^2}{\pi^2+(\frac{\pi^2}{8}-1)(2x)^2}. \tag{6}$$

It is known in the literature that

$$\frac{4/\pi}{\pi-2x}<\frac{\tan x}{x}<\frac{\pi}{\pi-2x} \tag{7}$$

for $0<x<\pi/2$. The left-hand side inequality (7) was presented by Stečkin [23], while the right-hand side inequality (7) was proved by Ge [14].

Becker and Stark [7] showed that for $0<x<\pi/2$,

$$\frac{8}{\pi^2-4x^2}<\frac{\tan x}{x}<\frac{\pi^2}{\pi^2-4x^2}. \tag{8}$$

The Becker-Stark inequality (8) has attracted much interest of many mathematicians and has motivated a large number of research papers (cf. [6, 9, 10, 12, 19, 24, 38, 39, 40] and the references cited therein).

All results of the present paper are motivated by the papers [22] and [8]. In view of the inequalities above, in this paper we give sharp Redheffer-type and Becker-Stark-type inequalities. As an application of Redheffer-type inequality for $\sin x/x$, we improve the well-known Yang Le inequality.

The numerical values given have been calculated using the computer program MAPLE 13.

2. Redheffer-type inequality

Theorem 1 gives a alternative proof of (3).

THEOREM 1. *The inequalities* (3) *hold for* $0<x<\pi$, *where the constants* $\alpha=1$ *and* $\beta=\frac{\pi^2}{6}-1=0.644934\ldots$ *are the best possible, in the sense that* $\alpha=1$ *can not be replaced by a smaller number, and* $\beta=\frac{\pi^2}{6}-1$ *can not be replaced by a larger number.*

Proof. Clearly, the left-hand side of (3) holds for $\alpha=1$. We now prove the right-hand side of (3) with $\beta=\frac{\pi^2}{6}-1$, i.e.,

$$\frac{\sin\pi x}{\pi x}<\frac{1-x^2}{1+\beta x^2}, \qquad 0<x<1. \tag{9}$$

We consider two cases.

Case 1: $0<x\leqslant 0.6$.

The following inequality is obtained by truncation of an alternating series,

$$\frac{\sin \pi x}{\pi x} < 1 - \frac{(\pi x)^2}{6} + \frac{(\pi x)^4}{120} \tag{10}$$

for $x \in \mathbb{R}$ and $x \neq 0$. We have, using (10),

$$\frac{1-x^2}{1+\beta x^2} =: 1 - \frac{(\pi x)^2}{6} + \frac{(\pi x)^4}{120} + f(x) > \frac{\sin \pi x}{\pi x} + f(x)$$

and it is sufficient to prove that $f(x) > 0$. This can be written in the form

$$\begin{aligned} f(x) &= \frac{1}{1+\beta x^2}\left[1 - x^2 - \left(1 - \frac{(\pi x)^2}{6} + \frac{(\pi x)^4}{120}\right)(1+\beta x^2)\right] \\ &= \frac{\pi^2 x^4}{6(1+\beta x^2)}\left(\frac{7\pi^2}{60} - 1 + \frac{\beta \pi^2}{20} x^2\right). \end{aligned}$$

Thus, $f(x) > 0$ for $x \in (0, 0.6]$.

Case 2: $0.6 < x < 1$.

Replacing x by $1-x$ leads to equivalent inequality:

$$\frac{\sin \pi x}{\pi(1-x)} < \frac{2x - x^2}{1+\beta(1-x)^2}, \qquad 0 < x < 0.4,$$

so it is sufficient to prove that

$$g(x) := \frac{2x - x^2}{1+\beta(1-x)^2} \cdot \frac{1-x}{x} - 1 + \frac{(\pi x)^2}{6} - \frac{(\pi x)^4}{120} > 0$$

for $0 < x < 0.4$. We can write

$$g(x) = \frac{x}{1+\beta(1-x)^2} g_1(x),$$

where

$$\begin{aligned} g_1(x) = 2 &- \frac{\pi^2}{6} - 5x + \frac{\pi^2 x}{3} + 2x^2 - \frac{\pi^2 x^2}{6} + \frac{\pi^4 x^2}{36} \\ &+ \frac{\pi^2 x^3}{3} - \frac{\pi^4 x^3}{18} - \frac{\pi^2 x^4}{6} + \frac{\pi^4 x^4}{36} - \frac{\pi^6 x^4}{720} - \frac{\pi^4 x^5}{60} + \frac{\pi^6 x^5}{360} + \frac{\pi^4 x^6}{120} - \frac{\pi^6 x^6}{720}. \end{aligned}$$

This can be written as

$$\begin{aligned} g_1(x) = &\left(2 - \frac{\pi^2}{6} - 5x + \frac{\pi^2 x}{3} + x^2 - \frac{\pi^2 x^2}{6} + \frac{\pi^4 x^2}{36}\right) \\ &+ \left(36 + 12\pi^2 x - 2\pi^4 x - 6\pi^2 x^2 + \pi^4 x^2 - \frac{\pi^6 x^2}{20}\right)\frac{x^2}{36} \\ &+ \left(-12 + 2\pi^2 + 6x - \pi^2 x\right)\frac{\pi^4 x^5}{720}. \end{aligned}$$

Each function in parenthesis is positive on $(0,0.4)$, so g_1 and g are also positive on this interval.

Hence, the right-hand side of (9) holds with $\beta = \frac{\pi^2}{6} - 1$.

If we write (3) as

$$\beta < \frac{1}{x^2}\left(\frac{\pi x(1-x^2)}{\sin(\pi x)} - 1\right) < \alpha, \qquad 0 < x < 1,$$

we find that

$$\lim_{x\to 0^+} \frac{1}{x^2}\left(\frac{\pi x(1-x^2)}{\sin(\pi x)} - 1\right) = \frac{\pi^2}{6} - 1 \quad \text{and} \quad \lim_{x\to 1^-} \frac{1}{x^2}\left(\frac{\pi x(1-x^2)}{\sin(\pi x)} - 1\right) = 1.$$

Hence, the inequalities (3) hold, and the constants $\alpha = 1$ and $\beta = \frac{\pi^2}{6} - 1$ are the best possible. The proof is complete. □

REMARK 1. Following the same method as was used in the proof of Theorem 4 below, we can prove that the function

$$I(x) = \frac{1}{x^2}\left(\frac{\pi x(1-x^2)}{\sin(\pi x)} - 1\right)$$

is strictly increasing for $0 < x < 1$, and

$$\lim_{x\to 0^+} I(x) = \frac{\pi^2}{6} - 1 \quad \text{and} \quad \lim_{x\to 1^-} I(x) = 1$$

(we omit the proof). Thus, Theorem 1 is proved.

Theorem 2 gives another upper bound in (2).

THEOREM 2. *The inequality*

$$\frac{\sin x}{x} < \frac{\pi^3 - x^3}{\pi^3 + \theta x^3}, \qquad 0 < x < \pi \tag{11}$$

holds, where the constant $\theta = 2$ *is the best possible.*

Proof. We first prove (11) with $\theta = 2$, i.e.,

$$\frac{\sin(\pi x)}{\pi x} < \frac{1 - x^3}{1 + \theta x^3}, \qquad 0 < x < 1 \tag{12}$$

We consider two cases.

Case 1: $0 < x \leqslant 0.6$.

Denote

$$\frac{1 - x^3}{1 + 2x^3} =: 1 - \frac{\pi^2 x^2}{6} + \frac{\pi^4 x^4}{120} + G(x).$$

Because of (10), it is sufficient to prove that G is positive on $(0,0.6]$.

$$\begin{aligned} G(x) &= \frac{1}{1+2x^3}\left[1-x^3-\left(1-\frac{\pi^2x^2}{6}+\frac{\pi^4x^4}{120}\right)(1+2x^3)\right] \\ &= \frac{\pi^2x^2}{6(1+2x^3)}\left[1-\frac{18x}{\pi^2}-\frac{\pi^2x^2}{20}+2x^3-\frac{\pi^2x^5}{10}\right] =: \frac{\pi^2x^2}{6(1+2x^3)}G_1(x). \end{aligned}$$

We shall prove that G_1 is decreasing on $[0,0.6]$.

$$G_1'(x) = -\frac{18}{\pi^2}-\frac{\pi^2x}{10}+6x^2-\frac{\pi^2x^4}{2} < -\frac{18}{\pi^2}+\left(6-\frac{\pi^2}{10}\right)x^2-\frac{\pi^2x^4}{2} < 0, \quad \forall\, x\in\mathbb{R}.$$

Hence, G_1 is decreasing on $[0,0.6]$. Since $G_1(0.6)\approx 0.08333>0$, we conclude that G_1 and therefore G are positive on $(0,0.6]$.

Case 2: $0.6<x<1$.

Replacing x by $1-x$ leads to equivalent inequality:

$$\frac{\sin \pi x}{\pi(1-x)} < \frac{1-(1-x)^3}{1+2(1-x)^3}, \qquad 0<x<0.4.$$

We continue as in the first case. Because of (10), it is sufficient to prove

$$h(x) := (1-x)\frac{1-(1-x)^3}{1+2(1-x)^3} - x\left[1-\frac{\pi^2x^2}{6}+\frac{\pi^4x^4}{120}\right] > 0$$

for $0<x<0.4$. Let us denote

$$h(x) = \frac{x^3}{1+2(1-x)^3}h_1(x),$$

where

$$h_1(x) = -2+\frac{\pi^2}{2}+x-\pi^2x+\pi^2x^2-\frac{\pi^4x^2}{40}-\frac{\pi^2x^3}{3}+\frac{\pi^4x^3}{20}-\frac{\pi^4x^4}{20}+\frac{\pi^4x^5}{60}.$$

Now we have

$$\begin{aligned} h_1'(x) &= 1-\pi^2+2\pi^2x-\frac{\pi^4x}{20}-\pi^2x^2+\frac{3\pi^4x^2}{20}-\frac{\pi^4x^3}{5}+\frac{\pi^4x^4}{12} \\ &< 1-\pi^2+2\pi^2x-\pi^2x^2\left[1-\frac{\pi^2}{10}+\frac{\pi^2x}{5}-\frac{\pi^2x^2}{12}\right]. \end{aligned}$$

Noting that

$$1-\pi^2+2\pi^2x<0 \quad \text{and} \quad 1-\frac{\pi^2}{10}+\frac{\pi^2x}{5}-\frac{\pi^2x^2}{12}>0$$

hold on $[0,0.4]$, we obtain that $h_1'(x)<0$ on $[0,0.4]$. Hence, $h_1(x)$ is decreasing on $[0,0.4]$. Since $h_1(0.4)\approx 0.56956>0$ it follows that h_1 and therefore h are positive on $(0,0.4)$.

Hence, the inequality (12) holds with $\theta = 2$.

If we write (11) as

$$\frac{1}{x^3}\left(\frac{\pi x(1-x^3)}{\sin(\pi x)} - 1\right) > \theta,$$

we find that

$$\lim_{x\to 1^-} \frac{1}{x^3}\left(\frac{\pi x(1-x^3)}{\sin(\pi x)} - 1\right) = 2.$$

Hence, the inequality (11) holds, and the constant $\theta = 2$ is the best possible. The proof is complete. □

It follows from the left-hand side of (3) and inequality (11) that

$$\frac{\pi^2 - t^2}{\pi^2 + \alpha t^2} < \frac{\sin t}{t} < \frac{\pi^3 - t^3}{\pi^3 + \theta t^3}, \qquad 0 < t < \pi, \tag{13}$$

or alternatively

$$\frac{1 - x^2}{1 + \alpha x^2} < \frac{\sin \pi x}{\pi x} < \frac{1 - x^3}{1 + \theta x^3}, \qquad 0 < x < 1, \tag{14}$$

where the constants $\alpha = 1$ and $\theta = 2$ are the best possible. In particular, we have

$$\frac{\pi^2 - x^2}{\pi^2 + x^2} < \frac{\sin x}{x} < \frac{\pi^3 - x^3}{\pi^3 + 2x^3}, \qquad 0 < x < \pi. \tag{15}$$

The choice $\alpha = 1$ and $\beta = \frac{\pi^2}{6} - 1$ in (3) yields

$$\frac{\pi^2 - x^2}{\pi^2 + x^2} < \frac{\sin x}{x} < \frac{\pi^2 - x^2}{\pi^2 + \left(\frac{\pi^2}{6} - 1\right)x^2}, \qquad 0 < x < \pi. \tag{16}$$

Sándor and Bhayo [22] established the following inequalities:

$$\frac{\sin x}{x} < c_1 \frac{\pi^2 - x^2}{\pi^2 + x^2}, \qquad 0 < x < \pi, \tag{17}$$

where $c_1 = 1.07514$, and

$$\frac{\sin x}{x} < \frac{\pi^a - x^a}{\pi^a + x^a}, \qquad 0 < x < \pi, \tag{18}$$

where $a = 2.175$.

Let $x_0 = \frac{-18\pi + \pi^3 + \sqrt{324\pi^2 + 36\pi^4 - 3\pi^6}}{2(18 - \pi^2)} = 2.2302\ldots$. By MAPLE 13, we find that, for $x < x_0$, the upper bound in (16) is better than the one in (15). For $x_0 < x < \pi$, the upper bound in (15) is better than the one in (16).

Let $x_1 = 0.742276\ldots$ and $x_2 = 2.668968\ldots$. By MAPLE 13, we find that, for $x_1 < x < x_2$, the upper bound in (17) is better than the one in (15). For $x \in (0, x_1) \cup (x_2, \pi)$, the upper bound in (15) is better than the one in (17).

Let $x^* = 2.602792\ldots$. By MAPLE 13, we find that, for $0 < x < x^*$, the upper bound in (18) is better than the one in (15). For $x^* < x < \pi$, the upper bound in (15) is better than the one in (18).

REMARK 2. The gamma function has the following reflection formula (see [1, p. 256]):

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \tag{19}$$

and recurrence formula

$$\Gamma(z+1) = z\Gamma(z).$$

For the representation of trigonometric functions in terms of gamma function, Bhayo and Sándor [8] pointed out that, (19) can be written as

$$\frac{x}{\sin x} = \Gamma\left(1+\frac{x}{\pi}\right)\Gamma\left(1-\frac{x}{\pi}\right) = B\left(1+\frac{x}{\pi}\right)B\left(1-\frac{x}{\pi}\right), \qquad 0<x<\pi, \tag{20}$$

where

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

denotes the beta function. The logarithmic differentiation to both sides of (19) gives the following reflection formula:

$$\psi(1-t) - \psi(t) = \frac{\pi}{\tan(\pi t)}, \tag{21}$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function. Replacing z by $t+1/2$ in (19) yieds

$$\frac{x}{\cos x} = \frac{x}{\pi}\Gamma\left(\frac{1}{2}+\frac{x}{\pi}\right)\Gamma\left(\frac{1}{2}-\frac{x}{\pi}\right), \qquad 0<x<\frac{\pi}{2}. \tag{22}$$

Also in [8], Bhayo and Sándor established some inequalities for the gamma, digamma and beta functions. For example, Bhayo and Sándor [8, Theorem 6] proved that, for $y \in (0,1)$, the following inequality holds:

$$B(x,y) < \frac{1}{xy}\frac{x+y}{1+xy}, \qquad 0<x<1. \tag{23}$$

The inequality (23) is reversed for $x>1$.

We obtain from (14) that

$$\frac{1-x^2}{1+x^2} < \frac{1}{\Gamma(1+x)\Gamma(1-x)} = \frac{\sin \pi x}{\pi x} < \frac{1-x^3}{1+2x^3}, \qquad 0<x<1. \tag{24}$$

Theorem 3 presents a more general result that includes (15) as its special case.

THEOREM 3. *Let $p \geqslant 3$ and $q \leqslant 2$ be real numbers. Then, we have*

$$\frac{\pi^q - x^q}{\pi^q + (q-1)x^q} < \frac{\sin x}{x} < \frac{\pi^p - x^p}{\pi^p + (p-1)x^p}, \qquad 0<x<\pi, \tag{25}$$

where the constants $q-1$ and $p-1$ are he best possible.

Proof. Replacing x by πx in (25) leads to equivalent inequality:

$$\frac{1-x^q}{1+(q-1)x^q} < \frac{\sin \pi x}{\pi x} < \frac{1-x^p}{1+(p-1)x^p}, \qquad 0 < x < 1. \tag{26}$$

Denote for real p, real positive a and $0 < x < 1$,

$$w(p) = \begin{cases} \dfrac{1-x^p}{1+(ap-1)x^p}, & p \neq 0, \\ \dfrac{\ln x}{\ln x - a}, & p = 0. \end{cases} \tag{27}$$

Differentiation yields

$$w'(p) = \begin{cases} \dfrac{ax^p(x^p - p\ln x - 1)}{(1+(ap-1)x^p)^2}, & p \neq 0, \\ \dfrac{a}{2}\left(\dfrac{\ln x}{a-\ln x}\right)^2, & p = 0. \end{cases}$$

We have $w'(p) > 0$ for $p \in \mathbb{R}$, because of the well known inequality $t-1 > \ln t$, for all $t > 0$. Hence, the function w is strictly increasing for $p \in \mathbb{R}$.

From (24) and the monotonicity of function w, we obtain

$$\frac{1-x^q}{1+(q-1)x^q} \leqslant \frac{1-x^2}{1+x^2} < \frac{\sin \pi x}{\pi x} < \frac{1-x^3}{1+2x^3} \leqslant \frac{1-x^p}{1+(p-1)x^p},$$

where $0 < x < 1$, $p \geqslant 3$ and $q \leqslant 2$.

For $p \geqslant 3$, the right-hand side of (26) can be written as

$$\frac{1}{x^p}\left(\frac{\pi x(1-x^p)}{\sin(\pi x)} - 1\right) > p - 1.$$

We find that

$$\lim_{x \to 1^-} \frac{1}{x^p}\left(\frac{\pi x(1-x^p)}{\sin(\pi x)} - 1\right) = p - 1.$$

Hence, the constant $p-1$ in the upper bound is the best possible.

We note that

$$\lim_{q \to 0} \frac{1-x^q}{1+(q-1)x^q} = \frac{\ln x}{\ln x - 1} < \frac{\sin \pi x}{\pi x}, \qquad 0 < x < 1.$$

Hence, the left-hand side of (26) holds for $q = 0$.

For $0 < q \leqslant 2$ and $q < 0$, the left-hand side of (26) can be written respectively as

$$\frac{1}{x^q}\left(\frac{\pi x(1-x^q)}{\sin(\pi x)} - 1\right) < q - 1 \quad \text{and} \quad \frac{1}{x^q}\left(\frac{\pi x(1-x^q)}{\sin(\pi x)} - 1\right) > q - 1.$$

We find that

$$\lim_{x \to 1^-} \frac{1}{x^q}\left(\frac{\pi x(1-x^q)}{\sin(\pi x)} - 1\right) = q - 1.$$

Hence, the constant $q-1$ in the lower bound is the best possible. The proof is complete. $\square$

REMARK 3. We thank a referee for suggesting the short argument given below. For fixed $0<x<1$, let

$$f(u)=\begin{cases}\dfrac{1-x^u}{1+(u-1)x^u}, & u\neq 0,\\[2mm] \dfrac{\ln x}{\ln x-1}, & u=0.\end{cases}$$

Differentiation yields

$$f'(u)=\begin{cases}\dfrac{x^u(x^u-u\ln x-1)}{(1+ux^u-x^u)^2}, & u\neq 0,\\[2mm] \dfrac{1}{2}\left(\dfrac{\ln x}{1-\ln x}\right)^2, & u=0.\end{cases}$$

We find by the well known inequality $t-1>\ln t$ for $t>0$ that

$$f'(u)>0, \qquad u\in(-\infty,\infty).$$

Hence, $f(u)$ is strictly increasing for $u\in\mathbb{R}$. This shows that $f(q)\leqslant f(2)$ for $q\leqslant 2$, so the left-hand side of (26) follows by Redheffer's inequality; and $f(3)\leqslant f(p)$ for $p\geqslant 3$, so the right-hand side of (26) follows by Theorem 2. This shows also that, the best results are obtained for $q=2$ (the left-hand side of (26)) and $p=3$ (the right-hand side of (26)).

REMARK 4. For $2<p<3$, the function

$$x\longmapsto \frac{\sin\pi x}{\pi x}-\frac{1-x^p}{1+(p-1)x^p}$$

change its sign on $(0,1)$, hence Redheffer type inequality is not valid for such exponents.

Theorem 4 presents a sharp Redheffer-type inequality for $\cos x$.

THEOREM 4. *Let $p\geqslant 3$ be a real number. Then, we have*

$$\cos x<\frac{\pi^p-(2x)^p}{\pi^p+\lambda(2x)^p}, \qquad 0<x<\frac{\pi}{2}, \tag{28}$$

or alternatively

$$\cos\left(\frac{\pi t}{2}\right)<\frac{1-t^p}{1+\lambda t^p}, \qquad 0<t<1, \tag{29}$$

with the best possible constant $\lambda=\frac{2}{\pi}p-1$.

Proof. Because of the monotonicity of function w in (27) it is sufficient to prove that

$$\cos\left(\frac{\pi t}{2}\right) < \frac{1-t^3}{1+(\frac{6}{\pi}-1)t^3}, \qquad 0<t<1. \tag{30}$$

To this end, we define the function $u(t)$ by

$$u(t) = \frac{1}{t^3}\left(\frac{(1-t^3)}{\cos(\pi t/2)} - 1\right), \qquad 0<t<1.$$

Differentiation yields

$$-2t^4\cos^2(\pi t/2)u'(t) = \left(-\pi t + \pi t^4\right)\sin\left(\frac{\pi t}{2}\right) + 6\cos\left(\frac{\pi t}{2}\right)\left(1-\cos\left(\frac{\pi t}{2}\right)\right) := v(t).$$

We are in a position to prove $v(t)>0$ for $0<t<1$. Let

$$V(t) = \begin{cases} \mu, & t=0, \\ \dfrac{v(t)}{t^2(1-t)^2}, & 0<t<1, \\ \nu, & t=1, \end{cases}$$

where μ and ν are constants determined with limits:

$$\mu = \lim_{t\to 0^+}\frac{v(t)}{t^2(1-t)^2} = \frac{\pi^2}{4} = 2.467401101\ldots,$$
$$\nu = \lim_{t\to 1^-}\frac{v(t)}{t^2(1-t)^2} = 6\pi - \frac{3}{2}\pi^2 = 4.045149316\ldots.$$

Using Maple we determine Taylor approximation for the function $V(t)$ by the polynomial of the fourth order:

$$Q(t) = \frac{1}{4}\pi^2 + \frac{1}{2}\pi^2 t + \left(\frac{3}{4}\pi^2 - \frac{17}{192}\pi^4\right)t^2 + \left(\frac{3}{2}\pi^2 - \frac{17}{96}\pi^4\right)t^3$$
$$+ \left(\frac{9}{4}\pi^2 - \frac{17}{64}\pi^4 + \frac{29}{7680}\pi^6\right)t^4,$$

which has a bound of absolute error

$$\varepsilon = 6\pi - \frac{27}{4}\pi^2 + \frac{17}{32}\pi^4 - \frac{29}{7680}\pi^6 = 0.348060174\ldots$$

for values $0<t<1$. It is true that

$$V(t) - (Q(t) - \varepsilon) \geqslant 0$$

and

$$Q(t) - \varepsilon = -6\pi + 7\pi^2 - \frac{17}{32}\pi^4 + \frac{29}{7680}\pi^6 + \frac{1}{2}\pi^2 t + \left(\frac{3}{4}\pi^2 - \frac{17}{192}\pi^4\right)t^2$$
$$+ \left(\frac{3}{2}\pi^2 - \frac{17}{96}\pi^4\right)t^3 + \left(\frac{9}{4}\pi^2 - \frac{17}{64}\pi^4 + \frac{29}{7680}\pi^6\right)t^4 > 0$$

for $0<t<1$. Hence, for $t\in[0,1]$ it is true that $V(t)>0$ and therefore $v(t)>0$ and $u'(t)<0$ for $t\in(0,1)$. Therefore, $u(t)$ is strictly decreasing for $0<t<1$, and we have

$$u(t)=\frac{1}{t^3}\left(\frac{(1-t^3)}{\cos(\pi t/2)}-1\right)>\lim_{x\to 1^-}u(x)=\frac{6}{\pi}-1$$

for $0<t<1$. This proves (30).

If we write (29) as

$$\frac{1}{t^p}\left(\frac{(1-t^p)}{\cos(\pi t/2)}-1\right)>\lambda,$$

we find that

$$\lim_{t\to 1^-}\frac{1}{t^p}\left(\frac{(1-t^p)}{\cos(\pi t/2)}-1\right)=\frac{2}{\pi}p-1.$$

Hence, the inequality (29) holds for $0<t<1$ and $p\geqslant 3$, and the constant $\lambda=\frac{2}{\pi}p-1$ is the best possible. The proof is complete. □

REMARK 5. In view of (28) and the left-hand side of (6), we find that the inequality

$$\cos x<\frac{\pi^p-(2x)^p}{\pi^p+(\frac{2}{\pi}p-1)(2x)^p},\qquad 0<x<\frac{\pi}{2}, \tag{31}$$

is valid for $p\geqslant 3$, the inequality (31) is reversed for $p=2$. In particular, we have

$$\frac{\pi^2-(2x)^2}{\pi^2+(\frac{4}{\pi}-1)(2x)^2}<\cos x<\frac{\pi^3-(2x)^3}{\pi^3+(\frac{6}{\pi}-1)(2x)^3},\qquad 0<x<\frac{\pi}{2}, \tag{32}$$

or alternatively

$$\frac{1-x^2}{1+(\frac{4}{\pi}-1)x^2}<\cos\left(\frac{\pi x}{2}\right)<\frac{1-x^3}{1+(\frac{6}{\pi}-1)x^3},\qquad 0<x<1. \tag{33}$$

The left-hand side of (32) is exactly the left-hand side of Theorem 9 of [8].

From (33) and the monotonicity of function w in (27), we obtain that

$$\frac{1-x^q}{1+(\frac{2}{\pi}q-1)x^q}<\cos\left(\frac{\pi x}{2}\right)<\frac{1-x^p}{1+(\frac{2}{\pi}p-1)x^p} \tag{34}$$

for $0<x<1$, $q\leqslant 2$ and $p\geqslant 3$. For $q=0$, the first inequality in (34) is understood as

$$\frac{\ln x}{\ln x-\frac{2}{\pi}}<\cos\left(\frac{\pi x}{2}\right),\qquad 0<x<1. \tag{35}$$

For $0<q\leqslant 2$ and $q<0$, the left-hand side of (34) can be written respectively as

$$\frac{1}{x^q}\left(\frac{1-x^q}{\cos\left(\frac{\pi x}{2}\right)}-1\right)<\frac{2}{\pi}q-1\quad\text{and}\quad\frac{1}{x^q}\left(\frac{\pi x(1-x^q)}{\sin(\pi x)}-1\right)>\frac{2}{\pi}q-1.$$

We find that

$$\lim_{x\to 1^-}\frac{1}{x^q}\left(\frac{1-x^q}{\cos\left(\frac{\pi x}{2}\right)}-1\right)=\frac{2}{\pi}q-1.$$

Hence, the constant $\frac{2}{\pi}q-1$ in the lower bound of (34) is the best possible.

REMARK 6. For $2<p<3$, the function

$$x\longmapsto \cos\left(\frac{\pi x}{2}\right)-\frac{1-x^p}{1+(\frac{2}{\pi}p-1)x^p}$$

change its sign on $(0,1)$, hence the inequality (34) is not valid for such exponents.

3. Becker-Stark-type inequality

In view of (7) and (8), we establish a sharp Becker-Stark-type inequality (Theorem 5). The inequality (26) can be rewritten as

$$\frac{\pi^q-(2x)^q}{\pi^q+(q-1)(2x)^q}<\frac{\sin(2x)}{2x}<\frac{\pi^p-(2x)^p}{\pi^p+(p-1)(2x)^p} \tag{36}$$

for $0<x<\pi/2$, $q\leqslant 2$ and $p\geqslant 3$.

The proof of Theorem 5 makes use of the inequality (36).

THEOREM 5. *Let $p>0$ be a given real number. Then, for $p\geqslant 3$,*

$$\frac{a}{\pi^p-(2x)^p}<\frac{\tan x}{x}<\frac{b}{\pi^p-(2x)^p},\qquad 0<x<\frac{\pi}{2}, \tag{37}$$

with the best possible constants

$$a=\pi^p\quad \textit{and}\quad b=4p\pi^{p-2}. \tag{38}$$

If $0<p\leqslant 2$, then the inequalities (37) are reversed.

Proof. For $0<x<\pi/2$ and $p>0$, let

$$F(x)=\frac{(\pi^p-(2x)^p)\tan x}{x}.$$

Differentiation yields

$$x\cos^2 xF'(x)=\Big(\pi^p+(p-1)(2x)^p\Big)\left(\frac{\pi^p-(2x)^p}{\pi^p+(p-1)(2x)^p}-\frac{\sin(2x)}{2x}\right).$$

For $p\geqslant 3$, we find by (36) that

$$F'(x)>0,\qquad 0<x<\frac{\pi}{2}.$$

Hence, for every $p \geqslant 3$, $F(x)$ is strictly increasing for $0 < x < \pi/2$, and we have

$$\pi^p = \lim_{x\to 0^+} F(x) < F(x) = \frac{(\pi^p - (2x)^p)\tan x}{x} < \lim_{x\to\pi/2^-} F(x) = 4p\pi^{p-2}.$$

Therefore, the inequalities (37) hold, and the constants $a = \pi^p$ and $b = 4p\pi^{p-2}$ are the best possible.

For $0 < p \leqslant 2$, we find by (36) that

$$F'(x) < 0, \qquad 0 < x < \frac{\pi}{2}.$$

Hence, for every $p \in (0,2]$, $F(x)$ is strictly decreasing for $0 < x < \pi/2$, and we have

$$\pi^p = \lim_{x\to 0^+} F(x) > F(x) = \frac{(\pi^p - (2x)^p)\tan x}{x} > \lim_{x\to\pi/2^-} F(x) = 4p\pi^{p-2}.$$

Therefore, the inequalities (37) are reversed. The proof is complete. □

REMARK 7. The inequality (37) can be written as

$$\frac{1}{1-t^p} < \frac{\tan(\pi t/2)}{\pi t/2} < \frac{(4/\pi^2)p}{1-t^p} \tag{39}$$

for $0 < t < 1$ and $p \geqslant 3$.

We note that, for every $t \in (0,1)$, the function $p \mapsto \frac{p}{1-t^p}$ is strictly increasing for $p \in \mathbb{R}$, and the function $p \mapsto \frac{1}{1-t^p}$ is strictly decreasing on $(0,\infty)$. In particular, the choice $p = 3$ in (39) then yields

$$\frac{1}{1-t^3} < \frac{\tan(\pi t/2)}{\pi t/2} < \frac{12/\pi^2}{1-t^3}, \qquad 0 < t < 1, \tag{40}$$

or alternatively

$$\frac{\pi^3}{\pi^3 - (2x)^3} < \frac{\tan x}{x} < \frac{12\pi}{\pi^3 - (2x)^3}, \qquad 0 < x < \frac{\pi}{2}. \tag{41}$$

REMARK 8. In order to ensure that the lower bound of (37) is positive, we restrict $p > 0$. In Theorem 5, we do not think about the case $p = 0$, since

$$\lim_{p\to 0^+} \frac{a}{\pi^p - (2x)^p} = \infty.$$

Computing limit of the upper bound in (37) yields

$$\lim_{p\to 0} \frac{4p\pi^{p-2}}{\pi^p - (2x)^p} = \frac{4/\pi^2}{\ln\left(\frac{\pi}{2x}\right)}. \tag{42}$$

For $p = 0$, the second inequality in (37) is reversed, which is understood as

$$\frac{\tan x}{x} > \frac{4/\pi^2}{\ln\left(\frac{\pi}{2x}\right)}, \qquad 0 < x < \frac{\pi}{2}. \tag{43}$$

We omit the proof.

Bhayo and Sándor [8, Corollary 3] proved that, for $0 < t < 1$,

$$\frac{4}{\pi}\frac{t}{1-t^2} < \tan\left(\frac{\pi t}{2}\right) < \frac{\pi}{2}\frac{t}{1-t^2}. \tag{44}$$

Theorems 6 and 7 below are motivated by (44). Theorem 6 shows that, in fact, the left-hand side of (39) is valid for $p \geqslant \pi^2/4 = 2.4674\ldots$. The proof of Theorem 6 makes use of the following lemma.

LEMMA 1. (see [3,4,5]) *Let $-\infty < a < b < \infty$, and let f, $g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$, differentiable on (a,b). Let $g'(x) \neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), then so are*

$$[f(x) - f(a)]/[g(x) - g(a)] \text{ and } [f(x) - f(b)]/[g(x) - g(b)].$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

THEOREM 6. *Let $p > 0$ be a real number. The inequality*

$$\frac{1}{1-t^p} < \frac{\tan(\pi t/2)}{\pi t/2} \tag{45}$$

holds for $0 < t < 1$ if and only if $p \geqslant \pi^2/4$, while the reversed inequality holds if and only if $0 < p \leqslant 2$.

Proof. The inequality (45) can be written for $p > 0$ as

$$\frac{\ln\left(1 - \frac{\pi t/2}{\tan(\pi t/2)}\right)}{\ln t} < p, \qquad 0 < t < 1.$$

For $0 < t < 1$, let

$$f_1(t) = \ln\left(1 - \frac{\pi t/2}{\tan(\pi t/2)}\right) \quad \text{and} \quad f_2(t) = \ln t,$$

and let

$$J(t) = \frac{f_1(t)}{f_2(t)} = \frac{\ln\left(1 - \frac{\pi t/2}{\tan(\pi t/2)}\right)}{\ln t}.$$

Then,

$$\frac{f_1'(t)}{f_2'(t)} = \frac{\pi t\left(2\tan(\frac{\pi t}{2}) - \pi t \sec^2(\frac{\pi t}{2})\right)}{2\tan(\frac{\pi t}{2})\left(\pi t - 2\tan(\frac{\pi}{2})\right)} =: J_1(t).$$

Differentiating $J_1(t)$, after some elementary computations we obtain

$$\begin{aligned}
&\frac{4}{\pi}\sin^2\left(\frac{\pi t}{2}\right)\left(2\sin\left(\frac{\pi t}{2}\right)-\pi t\cos\left(\frac{\pi t}{2}\right)\right)^2 J_1'(t)\\
&\quad=-(3\pi^2t^2+2)\sin(\pi t)+(\pi^3t^3-6\pi t)\cos(\pi t)+\sin(2\pi t)+6\pi t\\
&\quad=\sum_{n=4}^{\infty}(-1)^n u_n(t)=\frac{1}{2160}(\pi t)^9-\frac{1}{30240}(\pi t)^{11}+\ldots, \qquad (46)
\end{aligned}$$

where

$$u_n(t)=\frac{2(4^n-4n^3+6n^2-2n-4)}{(2n+1)!}(\pi t)^{2n+1}.$$

We find that, for $0<t<1$,

$$\begin{aligned}
\frac{u_{n+1}(t)}{u_n(t)}&=\frac{t^2(4^{n+1}-4n^3-6n^2-2n-4)}{2(n+1)(2n+3)(4^n-4n^3+6n^2-2n-4)}\\
&<\frac{4^{n+1}}{2(n+1)(2n+3)(4^n-4n^3+6n^2-2n-4)}\\
&=\frac{2}{(n+1)(2n+3)\left(1-\frac{4n^3-6n^2+2n+4}{4^4}\right)}
\end{aligned}$$

Noting that the sequence $\left\{\frac{4n^3-6n^2+2n+4}{4^4}\right\}$ is strictly decreasing for $n\geqslant 4$, we have, $n\geqslant 4$,

$$\frac{4n^3-6n^2+2n+4}{4^4}\leqslant\left[\frac{4n^3-6n^2+2n+4}{4^4}\right]_{n=4}=\frac{43}{64}.$$

We then obtain that for $0<t<1$ and $n\geqslant 4$,

$$\frac{u_{n+1}(t)}{u_n(t)}<\frac{2}{(n+1)(2n+3)\left(1-\frac{43}{64}\right)}=\frac{128}{21(n+1)(2n+3)}<1.$$

Therefore, for fixed $t\in(0,1)$, the sequence $n\mapsto u_n(t)$ is strictly decreasing for $n\geqslant 4$. We then obtain from (46) that, for $0<t<1$,

$$\frac{4}{\pi}\sin^2\left(\frac{\pi t}{2}\right)\left(2\sin\left(\frac{\pi t}{2}\right)-\pi t\cos\left(\frac{\pi t}{2}\right)\right)^2 J_1'(t)>(\pi t)^9\left(\frac{1}{2160}-\frac{(\pi t)^2}{30240}\right)>0.$$

Therefore, the functions $J_1(t)$ and $f_1'(t)/f_2'(t)$ are strictly increasing on $(0,1)$. By Lemma 1, the function

$$J(t)=\frac{f_1(t)}{f_2(t)}=\frac{f_1(t)-f_1(1)}{f_2(t)-f_2(1)}$$

is strictly increasing on $(0,1)$. And hence,

$$2=\lim_{u\to0^+}J(u)<J(t)=\frac{\ln\left(1-\frac{\pi t/2}{\tan(\pi t/2)}\right)}{\ln t}<\lim_{u\to1^-}J(u)=\pi^2/4.$$

Hence, the inequality (45) holds for $0<t<1$ if and only if $p\geqslant\pi^2/4$, while the reversed inequality holds if and only if $0<p\leqslant 2$. The proof is complete. □

In particular, we have, for $0<t<1$,

$$\frac{1}{1-t^{\pi^2/4}}<\frac{\tan(\pi t/2)}{\pi t/2}<\frac{1}{1-t^2}, \tag{47}$$

or alternatively

$$\frac{\pi}{2}\frac{t}{1-t^{\pi^2/4}}<\tan\left(\frac{\pi t}{2}\right)<\frac{\pi}{2}\frac{t}{1-t^2}. \tag{48}$$

We here point out that the lower bound in (48) is better than the one in (44).

Theorem 7 below considers the right-hand side of (39) and its reversed inequality. The proof of Theorem 7 makes use of the left-hand side of (47).

THEOREM 7. *The inequality*

$$\frac{\tan(\pi t/2)}{\pi t/2}<\frac{(4/\pi^2)p}{1-t^p} \tag{49}$$

holds for $0<t<1$ *if and only if* $p\geqslant 3$, *while the reversed inequality holds if and only if* $p\leqslant\pi^2/4$.

Proof. For $p\geqslant 3$, (49) has been shown.

As t approaches 1, with $t<1$, we find that

$$\frac{\tan(\pi t/2)}{\pi t/2}-\frac{(4/\pi^2)p}{1-t^p}=\frac{2(3-p)}{\pi^2}+\frac{-\pi^2+13-p^2}{3\pi^2}(1-t)+O\big((1-t)^2\big).$$

It then follows that it is necessary to have $p\geqslant 3$ for $\frac{\tan(\pi t/2)}{\pi t/2}-\frac{(4/\pi^2)p}{1-t^p}$ to be negative on $(0,1)$.

We now consider the reversed inequality of (49). From (47) and the monotonically increasing property of function $p\mapsto\frac{p}{1-t^p}$ (for $p\in\mathbb{R}$), we obtain that, for $p\leqslant\pi^2/4$,

$$\frac{(4/\pi^2)p}{1-t^p}\leqslant\frac{1}{1-t^{\pi^2/4}}<\frac{\tan(\pi t/2)}{\pi t/2}. \tag{50}$$

This shows that, for $p\leqslant\pi^2/4$, the reversed inequality of (49) holds.

As t approaches 0, with $t>0$, we find that

$$\frac{\tan(\pi t/2)}{\pi t/2}-\frac{(4/\pi^2)p}{1-t^p}=\frac{\pi^2-4p}{\pi^2}+\frac{\pi^2}{12}t^2-\frac{4p}{\pi^2}t^{2p}+\dots.$$

It then follows that it is necessary to have $p \leqslant \pi^2/4$ for $\frac{\tan(\pi t/2)}{\pi t/2} - \frac{(4/\pi^2)p}{1-t^p}$ to be positive on $(0,1)$.

Hence, the inequality (49) holds for $0 < t < 1$ if and only if $p \geqslant 3$, while the reversed inequality holds if and only if $p \leqslant \pi^2/4$. The proof is complete. □

4. Application to the improvement of the Yang Le inequality

It is well-known that the Yang Le inequality plays an important role in the theory of distribution of values of functions (see [31] for details). This inequality is stated below:

If $A_1 > 0$, $A_2 > 0$, $A_1 + A_2 \leqslant \pi$ and $0 \leqslant \mu \leqslant 1$, then,

$$\cos^2 \mu A_1 + \cos^2 \mu A_2 - 2\cos\mu\pi \cos\mu A_1 \cos\mu A_2 \geqslant \sin^2 \mu\pi. \tag{51}$$

The Yang Le inequality has been improved (see [13, 15, 20, 26, 27, 28, 29, 30, 32, 34, 35, 36]), by using generalized and sharp versions of Jordan's inequality. For example, Debnath and Zhao [13, Theorem 1] obtained an improvement of the Yang Le inequality and proved:

Let $A_i > 0$ $(i = 1, 2, \ldots, n)$ with $\sum_{i=1}^{n} A_i \leqslant \pi$, $0 \leqslant \lambda \leqslant 1$, and let $n \geqslant 2$ be a natural number. Then

$$N_1(\lambda) \leqslant (n-1)\sum_{k=1}^{n} \cos^2 \lambda A_k - 2\cos\lambda\pi \sum_{1\leqslant i<j\leqslant n} \cos\lambda A_i \cos\lambda A_j \leqslant M_1(\lambda), \tag{52}$$

where

$$N_1(\lambda) = \binom{n}{2}\left(3-\lambda^2\right)^2 \left(\lambda \cos\frac{\lambda\pi}{2}\right)^2 \quad \text{and} \quad M_1(\lambda) = \binom{n}{2}\lambda^2\pi^2.$$

By using Redheffer-type inequality (13), we here present an improvement of the Yang Le inequality.

THEOREM 8. *Let* $A_i > 0$ $(i = 1, 2, \ldots, n)$ *with* $\sum_{i=1}^{n} A_i \leqslant \pi$, $0 \leqslant \lambda \leqslant 1$, *and let* $n \geqslant 2$ *be a natural number. Then*

$$N(\lambda) \leqslant (n-1)\sum_{k=1}^{n} \cos^2 \lambda A_k - 2\cos\lambda\pi \sum_{1\leqslant i<j\leqslant n} \cos\lambda A_i \cos\lambda A_j \leqslant M(\lambda), \tag{53}$$

where

$$N(\lambda) = \binom{n}{2}\left(\frac{1-(\frac{\lambda}{2})^2}{1+(\frac{\lambda}{2})^2}\right)^2 \pi^2 \left(\lambda \cos\frac{\lambda\pi}{2}\right)^2$$

and

$$M(\lambda)=\binom{n}{2}\left(\frac{1-(\frac{\lambda}{2})^3}{1+2(\frac{\lambda}{2})^3}\right)^2\pi^2\lambda^2.$$

Proof. Let

$$H_{ij}=\cos^2\lambda A_i+\cos^2\lambda A_j-2\cos\lambda\pi\cos\lambda A_i\cos\lambda A_j.$$

It follows from [33] that

$$\sin^2\lambda\pi\leqslant H_{ij}\leqslant 4\sin^2\frac{\lambda}{2}\pi,\quad 1\leqslant i<j\leqslant n. \tag{54}$$

By summing all of the inequalities in (54), we obtain

$$\sum_{1\leqslant i<j\leqslant n}\sin^2\lambda\pi\leqslant\sum_{1\leqslant i<j\leqslant n}H_{ij}\leqslant\sum_{1\leqslant i<j\leqslant n}4\sin^2\frac{\lambda}{2}\pi,$$

that is,

$$4\binom{n}{2}\sin^2\frac{\lambda}{2}\pi\cos^2\frac{\lambda}{2}\pi\leqslant(n-1)\sum_{k=1}^{n}\cos^2\lambda A_k-2\cos\lambda\pi\sum_{1\leqslant i<j\leqslant n}\cos\lambda A_i\cos\lambda A_j$$
$$\leqslant 4\binom{n}{2}\sin^2\frac{\lambda}{2}\pi. \tag{55}$$

On the other hand, it follows from the inequality (13), by a direct calculation, that

$$\frac{1-(\frac{\lambda}{2})^2}{1+(\frac{\lambda}{2})^2}\left(\frac{\pi}{2}\right)\lambda<\sin\frac{\pi\lambda}{2}<\frac{1-(\frac{\lambda}{2})^3}{1+2(\frac{\lambda}{2})^3}\left(\frac{\pi}{2}\right)\lambda,\qquad 0<\lambda<2. \tag{56}$$

Applying the inequality (56) to (55) leads to the desired inequality (53). The proof is complete. □

REMARK 9. The upper bound in inequality (53) is sharper than the one in inequality (52). There is no strict comparison between the two lower bounds in inequalities (52) and (53).

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