Mathematical
Inequalities
& Applications
Volume 21, Number 4 (2018), 1079–1090 doi:10.7153/mia-2018-21-73

LIMITING CASE HARDY INEQUALITIES ON THE SPHERE

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(*Communicated by M. Praljak*)

Abstract. We give sharp limiting case Hardy inequalities on the sphere $\mathbb{S}^2$ and show that their optimal constants are unattainable by any $f \in H^1(\mathbb{S}^2) \setminus \{0\}$. The singularity of the problem is related to the geodesic distance from a point on the sphere.

1. Introduction

The classical Hardy inequality

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx \geqslant \frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx \tag{1}$$

is valid in dimensions $n \geqslant 3$ for all functions $u \in H^1(\mathbb{R}^n)$ ([1]). It obviously fails on $\mathbb{R}^2$ as the right hand side of (1) no longer makes sense. In order to obtain a version of (1) in the critical case $n = 2$ on bounded domains, a logarithmic weight can be introduced to tame the singularity. In [2, 4–8, 10], for instance, inequalities of the type

$$\int_B |\nabla u|^n dx \geqslant C_n(B) \int_B \frac{|u|^n}{|x|^n \left(\log \frac{1}{|x|}\right)^n} dx$$

were analysed for $u \in W_0^{1,n}(B)$ where B is the unit ball in $\mathbb{R}^n$.

Let $n \geqslant 3$ and $\mathbb{S}^n$ be the unit sphere equipped with its Lebesgue surface measure σ_n in $\mathbb{R}^{n+1}$. Denote by $d(.,p) : \mathbb{S}^n \to [0,\pi]$ the geodesic distance from $p \in \mathbb{S}^n$, and by $\nabla_{\mathbb{S}^n}$ the gradient on $\mathbb{S}^n$. Recently, Xiao [11] proved that if $f \in C^\infty(\mathbb{S}^n)$ then

$$\bar{c}_n \int_{\mathbb{S}^n} f^2 d\sigma_n + \int_{\mathbb{S}^n} |\nabla_{\mathbb{S}^n} f|^2 d\sigma_n \geqslant c_n^2 \int_{\mathbb{S}^n} \left(\frac{f^2}{d(x,p)^2} + \frac{f^2}{(\pi - d(x,p))^2} \right) d\sigma_n \tag{2}$$

with $\bar{c}_n = \left(\frac{4}{3} + \frac{1}{\pi^2}\right) c_n^2 + c_n$, $c_n = \frac{n-2}{2}$. It was also shown in [11] that the constant c_n in (2) is sharp in the sense that

$$c_n^2 = \inf_{f \in C^\infty(\mathbb{S}^n)\setminus\{0\}} \frac{D_n(f)}{\int_{\mathbb{S}^n} \frac{f^2}{d(x,p)^2} d\sigma_n} = \inf_{f \in C^\infty(\mathbb{S}^n)\setminus\{0\}} \frac{D_n(f)}{\int_{\mathbb{S}^n} \frac{f^2}{(\pi - d(x,p))^2} d\sigma_n}$$

Mathematics subject classification (2010): 26D10, 35A23, 46E35.
Keywords and phrases: Critical Hardy inequality, sharp constant, 2-sphere, Sobolev spaces on manifolds.

where

$$D_n(f) := \bar{c}_n \int_{\mathbb{S}^n} f^2 d\sigma_n + \int_{\mathbb{S}^n} |\nabla_{\mathbb{S}^n} f|^2 d\sigma_n, \quad f \in C^\infty(\mathbb{S}^n).$$

Recently, Xiao's result was extended to the case $p \neq 2$ in [9].

We prove L^2 Hardy inequalities with optimal constants on the sphere $\mathbb{S}^2$ in $\mathbb{R}^3$. This is a critical exponent case as the integral $\int_{\mathbb{S}^2} \theta^{-1+\lambda} d\sigma_2$, where θ is the polar angle, diverges for $\lambda \leqslant -1$. We also argue the lack of maximizers for our inequalities. Our approach denies the possibility of an equality in Xiao's inequality (2) as well.

2. Preliminaries

A point on the sphere $\mathbb{S}^2$ will have the standard spherical coordinate parametrization $(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ where $\theta \in [0,\pi]$ refers to the polar angle and $\varphi \in [0,2\pi[$ is the azimuthal angle. Then the surface measure induced by the Lebesgue measure on $\mathbb{R}^3$ is $d\sigma_2 = \sin\theta d\theta d\varphi$, the gradient and the Laplace-Beltrami operator, respectively, are given by

$$\nabla_{\mathbb{S}^2} = \hat{\theta}\frac{\partial}{\partial\theta} + \hat{\varphi}\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}, \quad \Delta_{\mathbb{S}^2} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}.$$

Here $\hat{\theta}, \hat{\varphi}$ denote the orthogonal three-dimensional unit vectors in the direction where θ, φ increase, respectively. The Sobolev space $H^1(\mathbb{S}^2)$ is the completion of $C^\infty(\mathbb{S}^2)$ in the norm

$$\| f \|_{H^1(\mathbb{S}^2)} := \left(\| f \|^2_{L^2(\mathbb{S}^2)} + \| \nabla f \|^2_{L^2(\mathbb{S}^2)} \right)^{\frac{1}{2}}.$$

In order to find the geodesic distance $d(x,p)$ from a point $x \in \mathbb{S}^2$ to a given a point $p \in \mathbb{S}^2$, we rotate the axes, if necessary, to put p on the zenith direction then place the great circle passing through p and x in the azimuth reference direction so that we have $d(x,p) = \theta$.

For simplicity, we henceforth denote $d\sigma_2$, $\nabla_{\mathbb{S}^2}$ and $\Delta_{\mathbb{S}^2}$ by $d\sigma$, ∇ and Δ, respectively.

3. Main results

Let $\phi :]0,\pi] \to [1,\infty[$ be defined by $\phi(t) := \log(\pi e/t)$, $\psi :]0,\pi[\to [1+\log\pi, \infty[$ be such that $\psi(t) := \phi(\sin t)$, and $\rho_\phi(t) := t\phi(t)$. Let $A > 0$. Denote by S, T_A, and $Q(.;\phi)$ the positive nonlinear functionals on $H^1(\mathbb{S}^2)$ given by

$$S(f) := \int_{\mathbb{S}^2} |\hat{\theta}.\nabla f|^2 d\sigma + \frac{1}{2\pi^2}\int_{\mathbb{S}^2} f^2 d\sigma,$$

$$T_A(f) := \int_{\mathbb{S}^2} |\nabla f|^2 d\sigma_2 + \frac{A}{4}\int_{\mathbb{S}^2} f^2 d\sigma_2,$$

$$Q(f;\phi) := \frac{1}{4}\int_{\mathbb{S}^2} \left(\frac{f^2}{\rho_\phi{}^2(d(x,p))} + \frac{f^2}{\rho_\phi{}^2(\pi - d(x,p))} \right) d\sigma_2.$$

THEOREM 1. *Assume that* $f \in H^1(\mathbb{S}^2)$. *Then there exists constants* $A, B > 0$, *independent of* f, *such that*

$$Q(f;\phi) \leqslant T_A(f), \tag{3}$$

$$Q(f;\psi) \leqslant T_B(f). \tag{4}$$

Both inequalities (3) and (4) are optimal, but an equality is impossible in either one:

THEOREM 2.

$$\sup_{f\in H^1(\mathbb{S}^2)\setminus\{0\}} \frac{Q(f;\phi)}{T_A(f)} = 1, \tag{5}$$

$$\sup_{f\in H^1(\mathbb{S}^2)\setminus\{0\}} \frac{Q(f;\psi)}{T_B(f)} = 1. \tag{6}$$

THEOREM 3. *There does not exist* $f \in H^1(\mathbb{S}^2)\setminus\{0\}$ *such that* $Q(f;\phi) = T_A(f)$, *or* $Q(f;\psi) = T_B(f)$.

A variant of the above-mentioned results follows via a different approach:

THEOREM 4. *Let* $f \in H^1(\mathbb{S}^2)$. *Then*

$$\frac{1}{4}\int_{\mathbb{S}^2} \frac{f^2}{\rho_\phi{}^2(d(x,p))}\,d\sigma \leqslant S(f) + \frac{1}{2\pi}\int_{\mathbb{S}^2} \frac{f^2}{\pi - d(x,p)}\,d\sigma, \tag{7}$$

$$\frac{1}{4}\int_{\mathbb{S}^2} \frac{f^2}{\rho_\phi{}^2(\pi - d(x,p))}\,d\sigma \leqslant S(f) + \frac{1}{2\pi}\int_{\mathbb{S}^2} \frac{f^2}{d(x,p)}\,d\sigma. \tag{8}$$

Moreover

$$\sup_{f\in H^1(\mathbb{S}^2)\setminus\{0\}} \frac{\frac{1}{4}\int_{\mathbb{S}^2} \frac{f^2}{\rho_\phi{}^2(d(x,p))}\,d\sigma}{S(f) + \frac{1}{2\pi}\int_{\mathbb{S}^2} \frac{f^2}{\pi-d(x,p)}\,d\sigma} = \sup_{f\in H^1(\mathbb{S}^2)\setminus\{0\}} \frac{\frac{1}{4}\int_{\mathbb{S}^2} \frac{f^2}{\rho_\phi{}^2(\pi-d(x,p))}\,d\sigma}{S(f) + \frac{1}{2\pi}\int_{\mathbb{S}^2} \frac{f^2}{d(x,p)}\,d\sigma} = 1, \tag{9}$$

and the suprema in (9) are not attained in $H^1(\mathbb{S}^2)\setminus\{0\}$.

4. Proof of Theorem 1

Proof. Let $f \in C^\infty(\mathbb{S}^2)$. Notice that $\psi > 1$ and write $f(\theta,\varphi) = \sqrt{\psi(\theta)}\,g(\theta,\varphi)$. We have

$$\begin{aligned} |\nabla f|^2 &= |\psi^{\frac{1}{2}}\nabla g + g\nabla\psi^{\frac{1}{2}}|^2 \\ &= \psi|\nabla g|^2 + \langle \psi^{\frac{1}{2}}\nabla g, g\psi^{-\frac{1}{2}}\nabla\psi\rangle + \left|\frac{1}{2}\psi^{-\frac{1}{2}}\nabla\psi\right|^2 g^2 \\ &= \psi|\nabla g|^2 + \frac{1}{2}\langle\nabla\psi, \nabla g^2\rangle + \frac{1}{4}\frac{1}{\psi}|\nabla\psi|^2 g^2. \end{aligned} \tag{10}$$

Integrating both sides of (10) over $\mathbb{S}^2$ we get

$$\begin{aligned}\int_{\mathbb{S}^2}|\nabla f|^2 d\sigma &= \int_{\mathbb{S}^2}\left(\psi|\nabla g|^2+\frac{1}{2}\langle\nabla\psi,\nabla g^2\rangle+\frac{1}{4}\frac{1}{\psi}|\nabla\psi|^2 g^2\right)d\sigma \\ &\geqslant \frac{1}{4}\int_{\mathbb{S}^2}\frac{1}{\psi}|\nabla\psi|^2 g^2 d\sigma+\frac{1}{2}\int_{\mathbb{S}^2}\langle\nabla\psi,\nabla g^2\rangle d\sigma \qquad (11)\\ &= \frac{1}{4}\int_{\mathbb{S}^2}\frac{1}{\psi}|\psi'|^2 g^2 d\sigma-\frac{1}{2}\int_{\mathbb{S}^2} g^2\Delta\psi d\sigma \qquad (12)\end{aligned}$$

by partial integration over the closed manifold $\mathbb{S}^2$. Calculating, we find

$$\Delta\psi=\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\psi\right)=1. \qquad (13)$$

Returning g to $f/\sqrt{\psi}$ and substituting for $\Delta\psi$ from (13) into (12), we obtain

$$\int_{\mathbb{S}^2}|\nabla f|^2 d\sigma \geqslant \frac{1}{4}\int_{\mathbb{S}^2}\frac{f^2}{\psi^2}\frac{\cos^2\theta}{\sin^2\theta}d\sigma-\frac{1}{2}\int_{\mathbb{S}^2}\frac{f^2}{\psi}d\sigma. \qquad (14)$$

Adding the finite integral $\frac{1}{4}\int_{\mathbb{S}^2}\left(\frac{1}{\theta^2\phi^2(\theta)}+\frac{1}{(\pi-\theta)^2\phi^2(\pi-\theta)}\right)f^2 d\sigma$ to both sides of (14) transforms it into the inequality

$$\begin{aligned}&\frac{1}{4}\int_{\mathbb{S}^2}\left(\frac{1}{\theta^2\phi^2(\theta)}+\frac{1}{(\pi-\theta)^2\phi^2(\pi-\theta)}\right)f^2 d\sigma \\ &\qquad\qquad \leqslant \int_{\mathbb{S}^2}|\nabla f|^2 d\sigma+\frac{1}{4}\int_{\mathbb{S}^2}F(\theta)f^2 d\sigma, \qquad (15)\end{aligned}$$

where

$$F(t):=\frac{1}{t^2\phi^2(t)}+\frac{1}{(\pi-t)^2\phi^2(\pi-t)}-\frac{\cos^2 t}{\sin^2 t}\frac{1}{\phi^2(\sin t)}+\frac{2}{\phi(\sin t)}.$$

Obviously, F is continuous on $]0,\pi[$ and, as expected from the facts that $\phi(t)\to+\infty$ when $t\to 0^+$, $\sin t=t+o(t)$ as $t\to 0$, it turns out

$$\lim_{t\to 0^+}F(t)=\lim_{t\to\pi^-}F(t)=\frac{1}{\pi^2}.$$

Hence, F can be extended to a uniformly continuous, consequently a bounded, function on $[0,\pi]$. Noting this in (15) implies (3). Direct computation also shows

$$A=\sup_{[0,\pi]}|F|=F\left(\frac{\pi}{2}\right)=\frac{2}{1+\log\pi}+\frac{8}{(1+\log 2)^2}\frac{1}{\pi^2}.$$

To prove (4), we add to both sides of (14) the well-defined integral

$$\frac{1}{4}\int_{\mathbb{S}^2}\left(\frac{1}{\theta^2}+\frac{1}{(\pi-\theta)^2}\right)\frac{f^2}{\psi^2(\theta)}d\sigma.$$

We then obtain the following analogue of (15):

$$\frac{1}{4}\int_{\mathbb{S}^2}\left(\frac{1}{\theta^2}+\frac{1}{(\pi-\theta)^2}\right)\frac{f^2}{\psi^2(\theta)}d\sigma \leqslant \int_{\mathbb{S}^2}|\nabla f|^2 d\sigma+\frac{1}{4}\int_{\mathbb{S}^2}G(\theta)f^2 d\sigma, \tag{16}$$

where

$$G(t) := \frac{M(t)}{\psi^2(t)}+\frac{2}{\psi(t)},$$
$$M(t) := \frac{1}{t^2}+\frac{1}{(\pi-t)^2}-\frac{\cos^2 t}{\sin^2 t}. \tag{17}$$

Once the boundedness of G is ensured, we see that (16) yields the inequality (4). Evidently, G has the same features as F. Since

$$\lim_{\theta\to 0}M(\theta)=\lim_{\theta\to\pi}M(\theta)=\frac{2}{3}+\frac{1}{\pi^2},\qquad \lim_{\theta\to 0^+}\psi(t)=\lim_{\theta\to\pi^-}\psi(t)=+\infty \tag{18}$$

then $M\in C[0,\pi]$, and $\lim_{t\to 0^+}G(t)=\lim_{t\to\pi^-}G(t)=0$, which makes G bounded on $[0,\pi]$. Moreover

$$B=\sup_{[0,\pi]}|G|=G\left(\frac{\pi}{2}\right)=\frac{2}{1+\log\pi}+\frac{8}{(1+\log\pi)^2}\frac{1}{\pi^2}. \qquad \square$$

5. Proof of Theorem 2

Proof. First, we would like to define the weak Laplace-Beltrami gradient of a function $f\in L^1(\mathbb{S}^2)$. Suppose $f\in C^\infty(\mathbb{S}^2)$ and $v(\theta,\varphi)=v_\theta(\theta,\varphi)\hat{\theta}+v_\varphi(\theta,\varphi)\hat{\varphi}$ with $v_\theta, v_\varphi\in C^\infty(\mathbb{S}^2)$. Then

$$\int_{\mathbb{S}^2}\frac{\partial f}{\partial\theta}v_\theta d\sigma=\int_{\mathbb{S}^2}\nabla f\cdot\hat{\theta}\,v_\theta d\sigma=-\int_{\mathbb{S}^2}f\nabla\cdot(v_\theta\hat{\theta})d\sigma,$$
$$\int_{\mathbb{S}^2}\frac{1}{\sin\theta}\frac{\partial f}{\partial\varphi}v_\varphi d\sigma=\int_{\mathbb{S}^2}\nabla f\cdot\hat{\varphi}\,v_\varphi d\sigma=-\int_{\mathbb{S}^2}f\nabla\cdot(v_\varphi\hat{\varphi})d\sigma.$$

Adding these identities we get

$$\int_{\mathbb{S}^2}\nabla f\cdot V\,d\sigma=-\int_{\mathbb{S}^2}f\nabla\cdot V\,d\sigma \tag{19}$$

for any vector field $V\in C^\infty\left(\mathbb{S}^2\to T\left(\mathbb{S}^2\right)\right)$ where $T\left(\mathbb{S}^2\right)$ is the tangent bundle of the smooth manifold $\mathbb{S}^2$. Motivated by (19), f is weakly differentiable if there is a vector field $\vartheta_f\in L^1\left(\mathbb{S}^2\to T\left(\mathbb{S}^2\right)\right)$ such that

$$\int_{\mathbb{S}^2}\vartheta_f\cdot V\,d\sigma=-\int_{\mathbb{S}^2}f\nabla\cdot V\,d\sigma,\quad \forall V\in C^\infty\left(\mathbb{S}^2\to T\left(\mathbb{S}^2\right)\right). \tag{20}$$

This, unique up to a set of zero measure, vector field ϑ_f is the weak surface gradient of f. According to ([3], Proposition 3.2., page 15)

$$H^1\left(\mathbb{S}^2\right) = W^{1,2}\left(\mathbb{S}^2\right) := \left\{ f \in L^2(\mathbb{S}^2) : |\vartheta_f| \in L^2\left(\mathbb{S}^2\right) \right\}.$$

We start with (5). By Theorem 1, it suffices to prove the existence of a sequence $\{f_n\}_{n\geqslant 1}$ in $H^1\left(\mathbb{S}^2\right)$ such that

$$\lim_{n\to\infty} \frac{Q(f_n;\phi)}{T_A(f_n)} = 1. \tag{21}$$

Consider the functions

$$f_n(\theta,\varphi) := \phi(\theta)^{\frac{1}{2}-\frac{1}{n}}. \tag{22}$$

The functions f_n are independent of φ, hence

$$\frac{Q(f_n;\phi)}{T_A(f_n)} = \frac{\int_0^\pi \frac{f_n^2 \sin\theta}{\theta^2\phi^2(\theta)}\,d\theta + \int_0^\pi \frac{f_n^2 \sin\theta}{(\pi-\theta)^2\phi^2(\pi-\theta)}\,d\theta}{4\int_0^\pi \left(\frac{\partial f_n}{\partial\theta}\right)^2 \sin\theta d\theta + A\int_0^\pi f_n^2 \sin\theta d\theta} \tag{23}$$

where the derivative $\partial f_n/\partial\theta$ is understood in the week sense discussed above. Since $\phi \in L^1_{\text{loc}}(\mathbb{R})$ and $\phi \geqslant 1$ on $[0,\pi]$, then

$$\int_0^\pi f_n^2 \sin\theta d\theta = \int_0^\pi \phi(\theta)^{1-\frac{2}{n}} \sin\theta d\theta \leqslant \int_0^\pi \phi(\theta)\,d\theta \approx 1. \tag{24}$$

Thus $f_n \in L^2\left(\mathbb{S}^2\right)$ for all $n \geqslant 1$. Notice also that f_n is smooth on $[0,\pi]\setminus\{0\}$ and its weak derivative

$$\frac{\partial f_n}{\partial\theta} = \frac{\frac{1}{n}-\frac{1}{2}}{\theta\,\phi^{\frac{1}{2}+\frac{1}{n}}}. \tag{25}$$

Therefore

$$\int_0^\pi \left(\frac{\partial f_n}{\partial\theta}\right)^2 \sin\theta d\theta = \frac{a_n}{4}\int_0^\pi \frac{1}{\theta\,\phi^{1+\frac{2}{n}}}\frac{\sin\theta}{\theta}d\theta, \quad a_n := \left(1-\frac{2}{n}\right)^2.$$

And since $\displaystyle\int_0^\pi \frac{d\theta}{\theta\,\phi^{1+\frac{2}{n}}} = \frac{n}{2}$, $\sin\theta \leqslant \theta$, then $\partial f_n/\partial\theta \in L^2\left(\mathbb{S}^2\right)$ for all $n \geqslant 1$. Substituting for f_n from (22) and for $\partial f_n/\partial\theta$ from (25) into (23) implies

$$\frac{Q(f_n;\phi)}{T_A(f_n)} = \frac{\alpha_n+\beta_n}{a_n\alpha_n+\gamma_n} = \frac{1}{a_n}\left(1+\frac{\beta_n-\gamma_n/a_n}{\alpha_n+\gamma_n/a_n}\right) \tag{26}$$

where

$$\alpha_n := \int_0^\pi \frac{1}{\theta\,\phi^{1+\frac{2}{n}}}\frac{\sin\theta}{\theta}d\theta,$$

$$\beta_n := \int_0^\pi \frac{\phi^{1-\frac{2}{n}}(\theta)\sin\theta}{(\pi-\theta)^2\,\phi^2(\pi-\theta)}d\theta,$$

$$\gamma_n := A\int_0^\pi \phi^{1-\frac{2}{n}} \sin\theta d\theta.$$

Observe that $\lim_{n\to+\infty} a_n = 1$. We shall show that, while $\lim_{n\to+\infty} \alpha_n = +\infty$, the sequences $\{\beta_n\}_{n\geqslant 1}$ and $\{\gamma_n\}_{n\geqslant 1}$ are both convergent. Using this in (26) proves (21).

Exploiting the continuity and positivity of $\sin\theta/\left(\theta^2\phi^{1+\frac{2}{n}}\right)$ on $[\pi/2,\pi]$, then applying the inequality $\sin\theta/\theta \geqslant 2/\pi$ when $0\leqslant\theta\leqslant\pi/2$, we obtain

$$\alpha_n = \int_0^{\pi/2} \frac{1}{\theta\,\phi^{1+\frac{2}{n}}}\frac{\sin\theta}{\theta}d\theta + \int_{\pi/2}^{\pi}\frac{\sin\theta}{\theta^2\,\phi^{1+\frac{2}{n}}}d\theta$$
$$\geqslant \frac{2}{\pi}\int_0^{\pi/2}\frac{1}{\theta\,\phi^{1+\frac{2}{n}}}d\theta = \frac{n}{\pi(1+\log(2))^{\frac{2}{n}}}. \tag{27}$$

This proves the divergence of $\{\alpha_n\}$. Next, by the dominated convergence theorem and (24) we readily find

$$\lim_{n\to+\infty}\gamma_n = A\lim_{n\to+\infty}\int_0^{\pi}\phi^{1-\frac{2}{n}}(\theta)\sin\theta d\theta = \int_0^{\pi}\phi(\theta)\sin\theta d\theta \lesssim 1.$$

Finally, since $\theta\mapsto\sin\theta/\left((\pi-\theta)^2\,\phi^2(\pi-\theta)\right)\in C([0,\pi/2])$, then using the local integrability of ϕ and the dominated convergence theorem again implies

$$\lim_{n\to\infty}\int_0^{\pi/2}\frac{\phi^{1-\frac{1}{n}}(\theta)\sin\theta}{(\pi-\theta)^2\,\phi^2(\pi-\theta)}d\theta = \int_0^{\pi/2}\frac{\phi(\theta)\sin\theta}{(\pi-\theta)^2\,\phi^2(\pi-\theta)}d\theta \lesssim 1. \tag{28}$$

Furthermore, since $\phi\in C([\pi/2,\pi])$, and $\dfrac{\sin\theta}{\pi-\theta} = \dfrac{\sin(\pi-\theta)}{\pi-\theta}\leqslant 1$, on $[\pi/2,\pi]$, then

$$\int_{\pi/2}^{\pi}\frac{\phi^{1-\frac{1}{n}}(\theta)\sin\theta}{(\pi-\theta)^2\,\phi^2(\pi-\theta)}d\theta \lesssim \int_{\pi/2}^{\pi}\frac{d\theta}{(\pi-\theta)\,\phi^2(\pi-\theta)} \approx 1. \tag{29}$$

The convergence of $\{\beta_n\}$ follows from (28) together with (29).

The proof of (6) shares the main idea of (5). The functions $g_n(\theta,\varphi) := \psi(\theta)^{\frac{1}{2}-\frac{1}{n}}\in L^2\left(\mathbb{S}^2\right)$, $n\geqslant 1$, and satisfy $\lim\limits_{n\to\infty}\dfrac{Q(g_n;\psi)}{T_B(g_n)} = 1$. Indeed, we have

$$\frac{Q(g_n;\psi)}{T_B(g_n)} = \frac{\int_0^{\pi}\frac{g_n^2\sin\theta}{\theta^2\,\psi^2(\theta)}d\theta + \int_0^{\pi}\frac{g_n^2\sin\theta}{(\pi-\theta)^2\,\psi^2(\pi-\theta)}d\theta}{4\int_0^{\pi}\left(\frac{\partial g_n}{\partial\theta}\right)^2\sin\theta d\theta + B\int_0^{\pi}g_n^2\sin\theta d\theta}$$
$$= \frac{\tilde{\alpha}_n}{a_n\tilde{\alpha}_n+\tilde{\beta}_n} = \frac{1}{a_n}\left(1-\frac{\tilde{\beta}_n/a_n}{\tilde{\alpha}_n+\tilde{\beta}_n/a_n}\right)$$

where

$$\tilde{\alpha}_n := \int_0^{\pi}\frac{\sin\theta\,d\theta}{\theta^2\,\psi^{1+\frac{2}{n}}} + \int_0^{\pi}\frac{\sin\theta d\theta}{(\pi-\theta)^2\,\psi^{1+\frac{2}{n}}} = 2\int_0^{\pi}\frac{\sin\theta\,d\theta}{\theta^2\,\psi^{1+\frac{2}{n}}},$$
$$\tilde{\beta}_n := B\int_0^{\pi}\psi^{1-\frac{2}{n}}\sin\theta d\theta - a_n\int_0^{\pi}M(\theta)\frac{\sin\theta}{\psi^{1+\frac{2}{n}}}d\theta.$$

Similarly to (27), we have

$$\begin{aligned}
\tilde{\alpha}_n &= 2\int_0^1 \frac{\sin\theta}{\theta^2}\frac{1}{\psi^{1+\frac{2}{n}}}\,d\theta + 2\int_1^\pi \frac{\sin\theta}{\theta^2}\frac{1}{\psi^{1+\frac{2}{n}}}\,d\theta \\
&\geqslant 2\int_0^1 \frac{\sin\theta}{\theta^2}\frac{1}{\psi^{1+\frac{2}{n}}}\,d\theta = 2\int_0^1 \frac{\sin^2\theta}{\theta^2\cos\theta}\frac{1}{\psi^{1+\frac{2}{n}}}\frac{\cos\theta}{\sin\theta}\,d\theta \\
&\geqslant \frac{8}{\pi^2}\int_0^1 \frac{1}{\psi^{1+\frac{2}{n}}}\frac{\cos\theta}{\sin\theta}\,d\theta = \frac{4n}{\pi^2}\frac{1}{(1+\log\pi)^{\frac{2}{n}}}.
\end{aligned}$$

Hence $\lim_{n\to\infty}\tilde{\alpha}_n = \infty$. Recall from (17) and (18) that $M \in C([0,\pi])$. Also, since $\psi \in L^1_{\text{loc}}(\mathbb{R})$, $\psi > 1$ uniformly, then $\lim_{n\to\infty}\tilde{\beta}_n$ exists by the dominated convergence theorem. $\square$

6. Proof of Theorem 3

Proof. The transition to the inequalities (3) and (4) from their respective stronger versions, (15) and (16), comes from the bounds

$$\int_{\mathbb{S}^2} F(\theta) f^2 d\sigma \leqslant A\int_{\mathbb{S}^2} f^2 d\sigma, \quad \int_{\mathbb{S}^2} G(\theta) f^2 d\sigma \leqslant B\int_{\mathbb{S}^2} f^2 d\sigma$$

where the bounded functions F and G are both positive and independent of f. Interestingly, as seen in Section 5, the size of $0 < A, B < \infty$ played no role in optimising (3) and (4).

Up to the inequality (15) or (16) an equality relation persists except for the only inequality (11). So a sufficient and necessary condition for an equality in (15) or (16) (and a necessary condition for an equality in (3) and (4)) is an equality in (11). But an equality in (11) occurs if and only if

$$\int_{\mathbb{S}^2} \psi|\nabla g|^2 d\sigma = 0. \tag{30}$$

Recalling that $g = f/\sqrt{\psi}$, we compute

$$\begin{aligned}
\psi|\nabla g|^2 &= \psi\left|\frac{\nabla f}{\sqrt{\psi}} - \frac{1}{2}\frac{f}{\psi^{\frac{3}{2}}}\frac{\partial\psi}{\partial\theta}\hat{\theta}\right|^2 \\
&= |\nabla f|^2 - \frac{f}{\psi}\frac{\partial\psi}{\partial\theta}\nabla f\cdot\hat{\theta} + \frac{1}{4}\frac{f^2}{\psi^2}\left(\frac{\partial\psi}{\partial\theta}\right)^2 \\
&= |\nabla f|^2 - \left(\frac{\partial f}{\partial\theta}\right)^2 + \left(\frac{\partial f}{\partial\theta}\right)^2 - \frac{f}{\psi}\frac{\partial\psi}{\partial\theta}\frac{\partial f}{\partial\theta} + \frac{1}{4}\frac{f^2}{\psi^2}\left(\frac{\partial\psi}{\partial\theta}\right)^2 \\
&= |\nabla f|^2 - \left(\frac{\partial f}{\partial\theta}\right)^2 + \left(\frac{\partial f}{\partial\theta} - \frac{1}{2}\frac{f}{\psi}\frac{\partial\psi}{\partial\theta}\right)^2.
\end{aligned} \tag{31}$$

Since $|\nabla f|^2 - \left(\frac{\partial f}{\partial \theta}\right)^2 = \frac{1}{\sin^2\theta}\left(\frac{\partial f}{\partial \varphi}\right)^2 \geqslant 0$, then, by (31), the equality (30) is equivalent to

$$\int_{\mathbb{S}^2} |\nabla f|^2 - \left(\frac{\partial f}{\partial \theta}\right)^2 d\sigma = \int_{\mathbb{S}^2} \left(\frac{\partial f}{\partial \theta} - \frac{1}{2}\frac{f}{\psi}\frac{\partial \psi}{\partial \theta}\right)^2 d\sigma = 0. \tag{32}$$

The equalities (32) are, in their turn, equivalent to

$$\frac{1}{\sin\theta}\left|\frac{\partial f}{\partial \varphi}\right| = \left|\frac{\partial f}{\partial \theta} - \frac{1}{2}\frac{f}{\psi}\frac{\partial \psi}{\partial \theta}\right| = 0. \tag{33}$$

Suppose that f is not the zero function. Then (33) are possible if and only if

$$f = f(\theta), \quad \frac{df}{f} = \frac{1}{2}\frac{d\psi}{\psi}.$$

That is $f = c\sqrt{\psi}$, c is a constant. But such $f \notin H^1\left(\mathbb{S}^2\right)$ because

$$\begin{aligned}\int_{\mathbb{S}^2} |\nabla f|^2 d\sigma = 2\pi \int_0^\pi \left(\frac{\partial f}{\partial \theta}\right)^2 \sin\theta\, d\theta &\gtrsim \int_0^1 \frac{\cos^2\theta}{\sin\theta}\frac{1}{\psi} d\theta \\ &\gtrsim \int_0^1 \frac{d\theta}{\sin\theta\, \phi(\sin\theta)} \approx \int_0^1 \frac{d\theta}{\theta\, \phi(\theta)} = +\infty. \quad \square\end{aligned}$$

7. Proof of Theorem 4

Proof. Write

$$\frac{1}{\theta}\frac{1}{\phi^2(\theta)} = \nabla\left(\frac{1}{\phi(\theta)}\right)\cdot\hat{\theta}.$$

Assume that f is smooth. Then integrating by parts w.r.t. the surface measure σ we get

$$\begin{aligned}\int_{\mathbb{S}^2} \frac{f^2}{\theta^2\phi^2(\theta)} d\sigma &= \int_{\mathbb{S}^2} \nabla\left(\frac{1}{\phi(\theta)}\right)\cdot\frac{f^2}{\theta}\hat{\theta} d\sigma \\ &= -\int_{\mathbb{S}^2} \frac{1}{\phi(\theta)}\nabla\cdot\left(\frac{f^2}{\theta}\hat{\theta}\right) d\sigma \\ &= -2\int_{\mathbb{S}^2} \frac{f\nabla f.\hat{\theta}}{\theta\,\phi(\theta)} d\sigma + \int_{\mathbb{S}^2} \frac{f^2}{\theta^2\,\phi(\theta)} d\sigma - \int_{\mathbb{S}^2} \frac{f^2}{\theta\,\phi(\theta)}\frac{\cos\theta}{\sin\theta} d\sigma.\end{aligned} \tag{34}$$

Observe here that each of the last two integrals on the right hand side of (34) can diverge. They suffer nonintegrable singularities at $\theta = 0$. But, when put together, their sum

$$I := \int_{\mathbb{S}^2} \frac{f^2}{\theta^2\,\phi(\theta)} d\sigma - \int_{\mathbb{S}^2} \frac{f^2}{\theta\,\phi(\theta)}\frac{\cos\theta}{\sin\theta} d\sigma = \int_{\mathbb{S}^2} \frac{1}{\theta\,\phi(\theta)}\left(\frac{1}{\theta} - \frac{\cos\theta}{\sin\theta}\right) f^2 d\sigma \tag{35}$$

is convergent. In fact

$$\lim_{\theta\to 0^+}\frac{1}{\theta\,\phi(\theta)}\left(\frac{1}{\theta}-\frac{\cos\theta}{\sin\theta}\right)=0.$$

Also, $\theta\mapsto 1/\left(\theta^2\phi(\theta)\right)$ is continuous on a neighborhood of $\theta=\pi$. Furthermore, if we fix $\delta>0$ and let $D:=\left\{x(\theta,\varphi)\in\mathbb{S}^2:0\leqslant\theta<\delta\right\}$, then the integral

$$\int_{\mathbb{S}^2\setminus D}\frac{f^2}{\theta\,\phi(\theta)}\frac{\cos\theta}{\sin\theta}d\sigma$$

does exist. Unfortunately, we can not control the integral I by $\int_{\mathbb{S}^2}f^2d\sigma$, up to a constant factor. The reason is

$$\lim_{\theta\to\pi^-}\frac{1}{\theta\,\phi(\theta)}\frac{\cos\theta}{\sin\theta}=-\infty.$$

But since

$$\lim_{\theta\to\pi^-}\left(\frac{1}{\theta\,\phi(\theta)}\frac{\cos\theta}{\sin\theta}+\frac{1}{\pi}\frac{1}{(\pi-\theta)}\right)=0$$

then, we may introduce the convergent integral $J:=\dfrac{1}{\pi}\displaystyle\int_{\mathbb{S}^2}\frac{f^2}{\pi-\theta}d\sigma$ to the integral I to get

$$I=I-J+J=\int_{\mathbb{S}^2}K(\theta)f^2\,d\sigma+J \tag{36}$$

where

$$K(\theta):=\frac{1}{\theta\,\phi(\theta)}\left(\frac{1}{\theta}-\frac{\cos\theta}{\sin\theta}\right)-\frac{1}{\pi}\frac{1}{(\pi-\theta)}.$$

By the continuity of K on $]0,\pi[$ and since

$$\lim_{\theta\to 0^+}K(\theta)=-\lim_{\theta\to\pi^-}K(\theta)=-\frac{1}{\pi^2}$$

then K is bounded on $[0,\pi]$. Actually, K is monotonically increasing. Thus

$$\sup_{[0,\pi]}|K|=\frac{1}{\pi^2}. \tag{37}$$

Using (37) in (36) we deduce that

$$I\leqslant\frac{1}{\pi^2}\int_{\mathbb{S}^2}f^2\,d\sigma+J. \tag{38}$$

Returning with (38) to the inequality (34) in the light of (35) we obtain

$$\int_{\mathbb{S}^2}\frac{f^2}{\theta^2\phi^2(\theta)}d\sigma\leqslant-2\int_{\mathbb{S}^2}\frac{f\nabla f.\hat{\theta}}{\theta\,\phi(\theta)}d\sigma+\frac{1}{\pi^2}\int_{\mathbb{S}^2}f^2\,d\sigma+\frac{1}{\pi}\int_{\mathbb{S}^2}\frac{f^2}{\pi-\theta}d\sigma. \tag{39}$$

Applying Cauchy's inequality with an ε we find

$$-2\int_{\mathbb{S}^2} \frac{f\nabla f.\hat{\theta}}{\theta\phi(\theta)}d\sigma \leqslant 2\varepsilon \int_{\mathbb{S}^2} \frac{f^2}{\theta^2\phi^2(\theta)}d\sigma + \frac{1}{2\varepsilon}\int_{\mathbb{S}^2} |\hat{\theta}.\nabla f|^2 d\sigma. \tag{40}$$

Therefore, it follows from (39) and (40) that

$$2\varepsilon(1-2\varepsilon)\int_{\mathbb{S}^2} \frac{f^2}{\theta^2\phi^2(\theta)}d\sigma \leqslant \int_{\mathbb{S}^2} |\hat{\theta}.\nabla f|^2 d\sigma + \frac{2\varepsilon}{\pi^2}\int_{\mathbb{S}^2} f^2\, d\sigma + \frac{2\varepsilon}{\pi}\int_{\mathbb{S}^2} \frac{f^2}{\pi-\theta}d\sigma, \quad 0<\varepsilon<\frac{1}{2}. \tag{41}$$

The choice $\varepsilon = 1/4$ maximizes the factor $2\varepsilon(1-2\varepsilon)$ and, consequently, the left hand side of (41). This proves (7). The inequality (8) can be obtained analogously.

In the fashion of the proof of Theorem 2, the sequence $f_n = \phi^{\frac{1}{2}-\frac{1}{n}}$ clearly satisfies

$$\lim_{n\to\infty} \frac{\frac{1}{4}\int_0^\pi \frac{f_n^2}{\rho_\phi{}^2(\theta)}\sin\theta\, d\theta}{U(f_n)+\frac{1}{2\pi}\int_0^\pi \frac{f_n^2}{\pi-\theta}\sin\theta\, d\theta} = \lim_{n\to\infty} \frac{\frac{1}{4}\int_0^\pi \frac{f_n^2}{\rho_\phi{}^2(\pi-\theta)}\sin\theta\, d\theta}{U(f_n)+\frac{1}{2\pi}\int_0^\pi \frac{f_n^2}{\theta}\sin\theta\, d\theta} = 1$$

where

$$U(f) = \int_0^\pi \left(\frac{\partial f}{\partial\theta}\right)^2 \sin\theta\, d\theta + \frac{1}{2\pi^2}\int_0^\pi f^2 \sin\theta\, d\theta.$$

One only needs to inspect the convergence of $\int_0^\pi \left(\phi^{1-\frac{2}{n}}\sin\theta/\theta\right)d\theta$, $\int_0^\pi \left(\phi^{1-\frac{2}{n}}\sin\theta/(\pi-\theta)\right)d\theta$ as $n\to\infty$. This is obvious from the bound $\sin\theta \leqslant \min\{\theta,\pi-\theta\}$ on $[0,\pi]$ and the fact $\phi\in L^1([0,\pi])$.

Finally, careful review of the proof of (7) above reveals that a necessary condition for a function $f\in H^1(\mathbb{S}^2)\setminus\{0\}$ to achieve an equality in (7) is that it yields an equality in (40). This is equivalent to

$$\nabla f.\hat{\theta} = -\frac{1}{2}\frac{f}{\theta\phi(\theta)}. \tag{42}$$

Suppose (42) was true. Then by (34) and (35) we must have

$$\int_{\mathbb{S}^2} \frac{h(\theta)f^2}{\theta\phi(\theta)}d\sigma = 0 \tag{43}$$

where

$$h(\theta) := \frac{1}{\theta} - \frac{\cos\theta}{\sin\theta}.$$

On the other hand

$$\lim_{\theta\to 0^+} h(\theta) = 0, \quad h'(\theta) = \frac{\theta^2-\sin^2\theta}{\theta^2\sin^2\theta} > 0,\ \ 0<\theta<\pi.$$

This shows h is strictly positive on $]0,\pi]$ and since $\theta\phi(\theta)\geqslant 0$ then (43) is a contradiction. $\square$

Acknowledgement. The author would like to thank the referee for his/her careful reading of this manuscript and for the valuable suggestions.

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(Received November 2, 2017)

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