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FACTORIZATION OF LIPSCHITZ OPERATORS ON BANACH FUNCTION SPACES

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Abstract. Let (X,d) be a pointed metric space. Let $T: X \to Y_1(\mu)$ and $S: X \to Y_2(\mu)$ be two Lipschitz operators into two Banach function spaces Y_1 and Y_2 over the same finite measure μ. We show which are the vector norm inequalities that characterize those T and S for which $T = M_g \circ S$, for some multiplication operator $M_g: Y_2 \to Y_1$. Our ideas give rise to Maurey-Rosenthal type factorization results for Lipschitz operators. We provide some applications on the Lipschitz structure of metric subsets of Banach function spaces.

1. Introduction

Let (X,d) be a pointed metric space. Let (Ω,Σ,μ) be a finite measure space and consider two Lipschitz operators $T: X \to Y_1$ and $S: X \to Y_2$ on Banach function spaces Y_1 and Y_2 over μ. In this paper we analyze when T factors through S as $T = M_g \circ S$ by means of vector norm inequalities, where $M_g: Y_2 \to Y_1$ is a multiplication operator defined by a measurable function g.

In the case of linear operators, this kind of factorization is called *strong factorization* for T through S. A recent study of this problem for the case of linear operators can be found in [8]. We show how these ideas can be adapted to the case of Lipschitz operators. Thus, we characterize the strong factorization of a Lipschitz operator T through a given Lipschitz operator S by using inequalities among these maps.

There are many classical results relating inequalities for linear operators and factorizations. Probably, the ones that have found more applications are the nowadays called Maurey-Rosenthal theorems, that give concavity and convexity conditions on a linear operator to get a strong factorization via L^p spaces (see for instance [15, Proposition III.H.10] and [4, 5, 6]). The direct application of these factorization theorems is the characterization of subspaces and sublattices of Banach function spaces that can be identified isomorphically with an L^p-space for a certain $1 \leqslant p < \infty$. The last part of the present paper is devoted to show how to deal with these arguments when moving to the Lipschitz setting, by extending the main factorization results of the Maurey-Rosenthal

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theory to this non-linear case. In particular, as a consequence of Theorem 4 and Corollary 1 we will find results as the following one. Let $1 < q < p < \infty$ and consider a set of measurable functions $A \subseteq L^q(\mu)$. Take the standard metrics $d_{\|\cdot\|_{L^q(\mu)}}$ and $d_{\|\cdot\|_{L^p(\mu)}}$ provided by the norms in $L^q(\mu)$ and $L^p(\mu)$, respectively. We give sufficient conditions for assuring that there is a Lipschitz isomorphism from the metric space $(A, d_{\|\cdot\|_{L^q(\mu)}})$ to a subset of the metric space $(L^p(\mu), d_{\|\cdot\|_{L^p(\mu)}})$. We will show that it is possible to characterize when the identity map in A is a Lipschitz isomorphism between these spaces by means of domination inequalities.

2. Preliminaries

We use standard Banach space and Banach lattice notation. If X is a Banach space, we write X^* for its dual space. Let (Ω, Σ, μ) be a finite measure space. Let $L^0(\mu)$ be the space of classes of measurable real functions on Ω that are equal μ-a.e. A Banach function space over μ is a Banach space $X(\mu)$ of elements of $L^0(\mu)$ with a norm $\|\cdot\|_{X(\mu)}$ satisfying that if $f \in L^0(\mu)$, $g \in X(\mu)$ and $|f| \leqslant |g|$ μ-a.e. then $f \in X(\mu)$ and $\|f\|_{X(\mu)} \leqslant \|g\|_{X(\mu)}$. In this case, $X(\mu)$ is a Banach lattice with the pointwise μ-a.e. order. Sometimes we will write X instead of $X(\mu)$ for the aim of simplicity if the measure is clearly fixed in the context.

The Köthe dual X' of X is the Banach subspace of X^* formed by those elements that can be represented by means of an integral, that is, for each $x' \in X'$ there is a function $h \in L^0(\mu)$ such that $\langle x, x' \rangle = \int_\Omega x h \, d\mu$ for all $x \in X$ (see [10, p. 29 ff] for issues related to the notion of the Köthe dual). The Banach function space X is order continuous if for every $f, f_n \in X$ such that $0 \leqslant f_n \uparrow f$ μ-a.e., $f_n \to f$ in norm. It is well known that X is order continuous if and only if $X^* = X'$. It has the Fatou property if for every sequence $(f_n) \subset X$ such that $0 \leqslant f_n \uparrow f$ μ-a.e. and $\sup_n \|f_n\|_X < \infty$, it follows that $f \in X$ and $\|f_n\|_X \uparrow \|f\|_X$. The reader can find more information about it in [13, Ch.2], [10, p. 28ff] and [16, Ch. 15].

If X and Y are Banach function spaces over the same measure μ, the space of multiplication operators from X to Y is

$$Mult(X;Y) = \left\{ h \in L^0(\mu) : hf \in Y \text{ for each } f \in X \right\}.$$

The function $\|h\|_{Mult(X;Y)} := \sup_{f \in B_X} \|hf\|_Y$ for all $h \in Mult(X;Y)$ is clearly a semi-norm on $Mult(X;Y)$. It is also a norm only if X is *saturated*, i.e. there is no $A \in \Sigma$ with $\mu(A) > 0$ such that $f\chi_A = 0$ μ-a.e. for all $f \in X$. In this case, $Mult(X;Y)$ is a Banach function space. If $h \in Mult(X;Y)$, we write $M_h \colon X \to Y$ for the corresponding multiplication operator $f \rightsquigarrow fh$. See [2, 12] for more information on these spaces.

If X and Y are Banach function spaces, the product space $X\pi Y$ is the linear set of functions $f \in L^0(\mu)$ such that $|f| \leqslant \sum_{i \geqslant 1} |x_i y_i|$ μ-a.e. for some sequences $(x_i) \subset X$ and $(y_i) \subset Y$ with $\sum_{i \geqslant 1} \|x_i\|_X \|y_i\|_Y < \infty$. The natural norm for this space is

$$\|f\|_{X\pi Y} = \inf \left\{ \sum_{i \geqslant 1} \|x_i\|_X \|y_i\|_Y \right\}, \quad f \in X\pi Y,$$

where the infimum is computed over all sequences $(x_i) \subset X$ and $(y_i) \subset Y$ such that $|f| \leqslant \sum_{i\geqslant 1} |x_i y_i|$ μ-a.e. and $\sum_{i\geqslant 1} \|x_i\|_X \|y_i\|_Y < \infty$. For instance, if $1/r = 1/p + 1/q$, then $L^p(\mu)\pi L^q(\mu) = L^r(\mu)$. If X, Y and $Mult(X;Y')$ are saturated then $X\pi Y$ is a saturated Banach function space with norm $\|\cdot\|_{X\pi Y}$ (see for example [7]).

Recall that a Banach lattice L – in particular, a Banach function space – is p-convex (resp. p-concave) for $1 \leqslant p < \infty$ if there is a positive constant K (resp. Q) such that for every finite set of elements $x_1, \dots, x_n \in L$,

$$\Big\| \Big(\sum_{i=1}^{n} |x_i|^p \Big)^{1/p} \Big\|_L \leqslant K \Big(\sum_{i=1}^{n} \|x_i\|_L^p \Big)^{1/p}$$

(and

$$\Big(\sum_{i=1}^{n} \|x_i\|_L^p \Big)^{1/p} \leqslant Q \Big\| \Big(\sum_{i=1}^{n} |x_i|^p \Big)^{1/p} \Big\|_L$$

for the case of p-concavity.)

Regarding Lipschitz operators, a map from a metric space (X,d) on a Banach space Z is called a *Lipschitz operator* if there is a constant $K > 0$ such that for any pair of points $x_1, x_2 \in X$, we have

$$\|T(x^1) - T(x^2)\| \leqslant K d(x^1, x^2).$$

We will work with pointed metric spaces X with base point 0 that is, 0 is any arbitrary fixed point of X. The notation of such a distinguished point indicates that this is actually the null vector when the metric space X is indeed a normed linear space. We will consider Lipschitz operators that map 0 to 0, and then the infimum of all constants $K > 0$ as above determines a (complete) norm on the space $Lip_0(X,Z)$ of all such maps. The space $\mathscr{M}(X)$ of molecules associated to a pointed metric space (X,d) is given by the linear span of all functions $m_{x^1,x^2} : X \to \mathbb{R}$ that are differences of characteristic functions,

$$m_{x^1,x^2}(w) := \chi_{\{x^1\}}(w) - \chi_{\{x^2\}}(w), \quad w, x^1, x^2 \in X.$$

The completion of $\mathscr{M}(X)$, when endowed with the norm

$$\|m\| := \inf\{ \sum_{j=1}^{n} |a_j| d(x_j^1, x_j^2) \},$$

where the infimum is taken over all finite representations of the molecule

$$m = \sum_{j=1}^{N} a_j m_{x_j^1, x_j^2},$$

is denoted $Æ(X)$ and called the Arens-Eells space associated to X (see [1]).

3. Factoring Lipschitz operators

Let X be a pointed metric space and Y be a Banach function space (or just a Banach space). If we consider the canonical Lipschitz isometry $\delta_X : X \to Æ(X)$ given by $\delta_X(x) := m_{x,0}$, $x \in X$, then a Lipschitz map $T \in Lip_0(X,Y)$ always factors through the corresponding Arens-Eells space as

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, "\delta_X"'] & & Y \\ & Æ(X) \arrow[ur, "T_L"'] & \end{tikzcd}$$

where T_L is the unique continuous linear map such that $T_L \circ \delta_X = T$. The operator T_L is referred to as the linearization of T (see [14, Theorem 2.2.4 (b)].

As δ_X takes values in $\mathscr{M}(X)$ we can take the restriction of T_L to $\mathscr{M}(X)$, that we still denote T_L, and then we can consider the following alternative commutative diagram

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, "\delta_X"'] & & Y. \\ & \mathscr{M}(X) \arrow[ur, "T_L"'] & \end{tikzcd}$$

For instance, in [9] it is proved that $Æ(\ell_1)$ and $Æ(X)$, over any finite-dimensional Banach space X, are isomorphic to $L_1(\mathbb{R})$.

3.1. Strong factorizations between couples of Lipschitz operators

Let (X,d) be a metric space and let Y_1 and Y_2 be two Banach function spaces over a finite measure μ, such that Y_1 and $Y_2\pi Y_1'$ are order continuous and that Y_1 has the Fatou property. Under these conditions, the product space $Y_2\pi Y_1'$ is a saturated Banach function space, as a consequence of [7, Proposition 2.2]. For the sake of clarity, let us explain some topological aspects concerning these Banach function spaces. Since $Y_2\pi Y_1'$ is order continuous, by [10, p. 29] the topological dual $(Y_2\pi Y_1')^*$ and the Köthe dual $(Y_2\pi Y_1')'$ coincide. The Fatou property of Y_1 provides the isometry $Y_1 = Y_1''$ ([10, p. 30]), and by [7, Proposition 2.2] we obtain the isometric equalities $(Y_2\pi Y_1')' = Mult(Y_2;Y_1'') = Mult(Y_2;Y_1)$. Then for each $\xi \in Y_2\pi Y_1'$ there exists $\xi' \in B_{Mult(Y_2;Y_1)}$ such that $\|\xi\|_{Y_2\pi Y_1'} = \langle \xi', \xi \rangle = \int \xi \xi' d\mu$. Conditions under which the product space of two Banach function spaces is order continuous – which is a requirement in the results above – can be found in Section 5 of [8].

Consider two Lipschitz operators $T \in Lip_0(X,Y_1)$ and $S \in Lip_0(X,Y_2)$. In this section we characterize when T factors strongly through S, that is, when there is a function $g \in Mult(Y_2;Y_1)$ so that the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[dr, "S"'] & Y_1 \\ & Y_2 \arrow[u, dotted, "M_g"'] \end{tikzcd} \quad . \tag{1}$$

This factorization is inspired in the following result regarding linear operators between Banach function spaces:

THEOREM 1. ([8, Theorem 4.1]) *Let X_1, X_2, Y_1, Y_2 be Banach function spaces and let $T : X_1 \to Y_1$ and $S : X_2 \to Y_2$ be continuous linear operators. Assume that Y_1 and $Y_2 \pi Y_1'$ are order continuous and that Y_1 has the Fatou property. The following statements are equivalent.*

(i) There exists a function $h \in Mult(X_1;X_2)$ such that

$$\sum_{i=1}^{n} \int T(x_i) y_i' \, d\mu \leqslant \Big\| \sum_{i=1}^{n} S(hx_i) y_i' \Big\|_{Y_2 \pi Y_1'}$$

for each $n \in \mathbb{N}$, every $x_1, \ldots, x_n \in X_1$ and $y_1', \ldots, y_n' \in Y_1'$.

(ii) There exist functions $f \in Mult(X_1;X_2)$ and $g \in Mult(Y_2;Y_1)$ such that $T(x) = g(S(fx))$ for all $x \in X_1$.

It must be mentioned that if we take $X_0 := X_1 = X_2$ in the above theorem, the Banach lattice structure on X_0 does not play any relevant role in its proof. The reader can easily check that the theorem remains valid for any normed linear space X as domain of both linear operators T and S. Now, we are going to see that we can use such a variant of Theorem 1 for metric domains in order to prove a strong factorization theorem for Lipschitz operators. Recall that in the next result X is a pointed metric space.

THEOREM 2. *Let $T \in Lip_0(X, Y_1(\mu))$ and $S \in Lip_0(X, Y_2(\mu))$ be Lipschitz operators. Suppose that $Y_1(\mu)$ and $Y_2(\mu) \pi Y_1(\mu)'$ are order continuous and that Y_1 has the Fatou property. The following statements are equivalent.*

(i) The inequality

$$\int \sum_{i=1}^{n} \big(T(x_i^1) - T(x_i^2) \big) y_i' \, d\mu \leqslant \Big\| \sum_{i=1}^{n} \big(S(x_i^1) - S(x_i^2) \big) y_i' \Big\|_{Y_2(\mu) \pi Y_1(\mu)'}$$

holds for each $n \in \mathbb{N}$ and each finite sets $x_1^1, \ldots, x_n^1, x_1^2, \ldots, x_n^2 \in X$ and $y_1', \ldots, y_n' \in Y_1(\mu)'$.

(ii) There exists a function $g \in Mult(Y_2(\mu);Y_1(\mu))$ such that

$$T(x^1) - T(x^2) = g(S(x^1) - S(x^2))$$

for all $x^1, x^2 \in X$.

(iii) There is a function $g \in Mult(Y_2(\mu);Y_1(\mu))$ such that $T(x) = gS(x)$ for all $x \in X$, i.e. T factors through S as

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[dr, "S"'] & Y_1(\mu). \\ & Y_2(\mu) \arrow[u, dotted, "M_g"'] \end{tikzcd}$$

(iv) T_L factors through S_L as

$$\begin{tikzcd} Æ(X) \arrow[r, "T_L"] \arrow[dr, "S_L"'] & Y_1(\mu) \\ & Y_2(\mu) \arrow[u, dotted, "M_g"'] \end{tikzcd}$$

for a certain function $g \in Mult(Y_2(\mu);Y_1(\mu))$.

In that case, the g's in (ii), (iii) *and* (iv) *coincide.*

Proof. We will write Y_1 and Y_2 instead of $Y_1(\mu)$ and $Y_2(\mu)$ for the aim of clarity in the proof. If we assume (i), we can rewrite the inequality as

$$\sum_{i=1}^{n} \int T_L(m_{x_i^1,x_i^2})\, y_i'\, d\mu \leqslant \Big\| \sum_{i=1}^{n} \big(S_L(m_{x_i^1,x_i^2})\big) y_i' \Big\|_{Y_2 \pi Y_1'}$$

for every finite sets $x_1^1,\dots,x_n^1,x_1^2,\dots,x_n^2 \in X$ and $y_1',\dots,y_n' \in Y_1'$. It is clear then that the inequality

$$\sum_{i=1}^{n} \int T_L(m_i)\, y_i'\, d\mu \leqslant \Big\| \sum_{i=1}^{n} \big(S_L(m_i)\big) y_i' \Big\|_{Y_2 \pi Y_1'}$$

holds for every $m_1,\dots,m_n \in \mathscr{M}(X)$. Note that, when considering molecules as linear combinations of $m_{x_i^1,x_i^2}$, the scalar coefficients can be absorbed by the y_i'. By Theorem 1, there exists a function $g \in Mult(Y_2;Y_1)$ such that $T_L(m) = gS_L(m)$ for all $m \in \mathscr{M}(X)$. In particular, $T_L(m_{x^1,x^2}) = gS_L(m_{x^1,x^2})$ for all $x^1,x^2 \in X$. Therefore, $T(x^1) - T(x^2) = g(S(x^1) - S(x^2))$ for all $x^1,x^2 \in X$, and (ii) is proved. The implication (ii) $\Rightarrow$ (iii) is obvious. If we assume (iii), the linearization of S fulfills $T(x) = gS_L(m_{x,0})$ for all $x \in X$. Therefore, $T_L(m) = gS_L(m)$ for all $m \in \mathscr{M}(X)$ and (iv) follows. That (iv) implies (i) follows easily from Theorem 1. $\square$

3.2. Maurey-Rosenthal type Theorems on factorization of Lipschitz maps through L^p-spaces

Chávez-Domínguez [3] has found several results concerning the factorization of Lipschitz operators through L^p-spaces, including some variants of the Maurey-Rosenthal Theorem. In this section we will show a different way of proving similar results on factorization. However, it must be noted that unlike what happens in the results of [3], the measure μ over which $Y(\mu)$ is defined as a Banach function space is the same that appears in the factorization space $L^p(\mu)$. Also the fact that the closing operator is a multiplication map $L^p(\mu) \to Y(\mu)$ provides a different meaning to our result when comparing with the ones of Chávez-Domínguez, that are given for abstract Banach lattices.

Let X be a pointed metric space and let $Y(\mu)$ be a Banach function space. Although the definition given in [3] for Lipschitz p-convexity allows to prove factorization results, for the aim of coherence we prefer to define Lipschitz p-convexity in a slightly different way. This will allow to obtain Maurey-Rosenthal type factorizations in the classical sense. We will say that $T \in \mathrm{Lip}_0(X, Y(\mu))$ is Lipschitz strongly p-convex ($1 \leqslant p < \infty$), if there exists a constant $K \geqslant 0$ such that for all $(x_i^j)_{i \leqslant n, j \leqslant m}, (y_i^j)_{i \leqslant n, j \leqslant m}$ in X and $(\lambda_i^j)_{i \leqslant n, j \leqslant m}$ in $\mathbb{R}$,

$$\Big\| \Big(\sum_{i=1}^n \Big| \sum_{j=1}^m \lambda_i^j (T(x_i^j) - T(y_i^j)) \Big|^p \Big)^{1/p} \Big\| \leqslant K \Big(\sum_{i=1}^n \Big| \sum_{j=1}^m \lambda_i^j d(x_i^j, y_i^j) \Big|^p \Big)^{1/p}. \tag{2}$$

The definition of Chávez-Domínguez is a bit weaker; we get it if we make $m = 1$ in inequality (2). This is the reason we use the term "strongly" in our definition. We can prove a result that is similar to the one given in [3, Theorem 3.3].

LEMMA 1. *Let $T \in Lip_0(X, Y(\mu))$ be a Lipschitz operator. If T is Lipschitz strongly p-convex, then $T_L : Æ(X) \to Y(\mu)$ is p-convex.*

Proof. The proof is straightforward. Assume that T is Lipschitz strongly p-convex and use (2). Consider the molecules $m_i \in \mathscr{M}(X)$, that can be written as $m_i = \sum_{j=1}^{r_i} \lambda_i^j m_{x_i^j, y_i^j}$ for $i = 1, \ldots, n$. Write $m = \max\{r_i : i = 1, \ldots, n\}$ and complete the sums to m terms by adding $\lambda_i^j = 0$ for $r_i < j \leqslant m$ for each i.

$$\begin{aligned}
\Big\| \Big(\sum_{i=1}^n |T_L(m_i)|^p \Big)^{1/p} \Big\| &= \Big\| \Big(\sum_{i=1}^n \Big| T_L\Big(\sum_{j=1}^m \lambda_i^j m_{x_i^j, y_i^j} \Big) \Big|^p \Big)^{1/p} \Big\| \\
&= \Big\| \Big(\sum_{i=1}^n \Big| \sum_{j=1}^m \lambda_i^j (T(x_i^j) - T(y_i^j)) \Big|^p \Big)^{1/p} \Big\| \\
&\leqslant K \Big(\sum_{i=1}^n \Big| \sum_{j=1}^m \lambda_i^j d(x_i^j, y_i^j) \Big|^p \Big)^{1/p} \\
&\leqslant K \Big(\sum_{i=1}^n \Big(\sum_{j=1}^m |\lambda_i^j| d(x_i^j, y_i^j) \Big)^p \Big)^{1/p}.
\end{aligned}$$

Since this computation can be done for each set of representations of the molecules m_i and the left hand side of the inequality does not depend on the representations, we get that

$$\Big\| \Big(\sum_{i=1}^{n} |T_L(m_i)|^p \Big)^{1/p} \Big\| \leqslant K \Big(\sum_{i=1}^{n} \big\| m_i \big\|_{Æ(X)}^p \Big)^{1/p}.$$

Thus, T_L is p-convex. □

THEOREM 3. *Let $Y(\mu)$ be a Banach function space and let $T \in Lip_0(X,Y(\mu))$ be a Lipschitz operator. If $Y(\mu)$ is p-concave and T is Lipschitz strongly p-convex ($1 \leqslant p < \infty$), then there is $g \in Mult(L^p(\mu);Y(\mu))$ and a Lipschitz map $S : X \to L^p(\mu)$ (with $S(0) = 0$) such that $T = M_g \circ S$.*

Proof. By Lemma 1, $T_L : Æ(X) \to Y(\mu)$ is p-convex. Since $Y(\mu)$ is p-concave, it is in particular order continuous, and by the Maurey-Rosenthal Theorem [4, Corollary 2], there exists a positive multiplication operator $M_g : L^p(\mu) \to Y(\mu)$ and a continuous operator $u : Æ(X) \to L^p(\mu)$ such that $T_L = M_g \circ u$. Defining $S := u \circ \delta_X$ we complete the proof. □

4. Applications: Lipschitz isomorphisms among subsets of measurable functions with different metrics

Factorization of operators among Banach function spaces are useful tools for the study of the properties of subspaces of these spaces. For example, Maurey-Rosenthal type factorizations of operators hold the key to getting the structure of reflexive subspaces of L^1-spaces. The Lipschitz version of these results – that are based in the arguments explained in the previous sections – can also be applied to subsets A of Banach function spaces, when these subsets A are considered as metric substructures. Therefore, we will center our attention on the application of our results to the analysis of the metric structure of subsets A of Banach function spaces. Certain norm inequalities among the elements of A will provide Lipschitz isomorphisms between metrics on A provided by different Banach function spaces. In particular, when considering the classical Banach function spaces $L^p(\mu)$ with respect to a finite measure μ, we will characterize in terms of norm inequalities, those subsets A of $L^q(\mu)$ for which there is a weight h so that $L^p(hd\mu)$ and $L^q(\mu)$ induce a Lipschitz automorphism on A ($1 < q < p < \infty$). To illustrate these results, we provide some examples that easily characterize the solutions of some integral inequalities.

We will develop our results in the following setting. Suppose that we have a Lipschitz copy – that is, a (bi)Lipschitz bijection from a metric space to a metric subspace of a fixed Banach function space $Y_1(\mu)$, and we want to know if this metric space can be found as a metric subspace of a weighted L^p space defined on the same measure space (Ω, Σ, μ). A direct application of Theorem 2 to the identity map gives the following result.

Let us introduce before a technical definition. We will say that a set of measurable functions A has measurable support if the union of all the supports of all the functions belonging to it is a measurable set. We will assume that μ is a finite measure.

THEOREM 4. *Let $A \subset Y_1(\mu)$ such that $0 \in A$ and has measurable support. Suppose that $Y_1(\mu)$ and $Y_2(\mu)\pi Y_1(\mu)'$ are order continuous and $Y_1(\mu)$ has the Fatou property. The following statements are equivalent:*

(i) There is a Lipschitz map $S : (A, d_{\|\cdot\|_{Y_1(\mu)}}) \to Y_2(\mu)$ with $S(0) = 0$ such that the inequality

$$\int \sum_{i=1}^{n} \left(x_i^1 - x_i^2\right) y_i' \, d\mu \leqslant \left\| \sum_{i=1}^{n} \left(S(x_i^1) - S(x_i^2)\right) y_i' \right\|_{Y_2(\mu)\pi Y_1(\mu)'}$$

holds for each $n \in \mathbb{N}$ and every $x_1^1, \ldots, x_n^1, x_1^2, \ldots, x_n^2 \in A$ and $y_1', \ldots, y_n' \in Y_1(\mu)'$.

(ii) There exists a function $g \in Mult(Y_2(\mu); Y_1(\mu))$ such that the multiplication map

$$M_{1/g} : (A, d_{\|\cdot\|_{Y_1(\mu)}}) \longrightarrow Y_2(\mu)$$

defines a Lipschitz isomorphism.

Moreover, if (i) (and so (ii)) happens, the equation $x = gS(x)$ (maybe up to a multiplicative constant) holds for each $x \in A$.

Proof. For the aim of clarity, we will write Y_1 and Y_2 instead of $Y_1(\mu)$ and $Y_2(\mu)$ in all the proof.

(i) $\Rightarrow$ (ii). Suppose that there is such a Lipschitz map S. Considering the metric space $X = (A, d_{\|\cdot\|_{Y_1}})$, by Theorem 2 there is a function $g \in Mult(Y_2; Y_1)$ such that $x = gS(x)$ for all $x \in A$. In particular, we can assume that g is non-zero in any subset of positive measure of the measurable support of all the functions in the subspace generated by A, that coincides with union of the supports of all the functions in A. We can assume that $g = 1$ outside this support. Since we are giving a factorization of the identity map, this implies that $S(x) = x/g$ for all $x \in A$. Moreover, since S is Lipschitz, we obtain that there is a constant $K > 0$ such that if $x_1, x_2 \in A$,

$$d_{\|\cdot\|_{Y_2}}(x_1/g, x_2/g) = d_{\|\cdot\|_{Y_2}}(S(x_1), S(x_2)) \leqslant K d_{\|\cdot\|_{Y_1}}(x_1, x_2),$$

and, for $f_1, f_2 \in \frac{A}{g} \subseteq Y_2$,

$$d_{\|\cdot\|_{Y_1}}(gf_1, gf_2) \leqslant \|g\|_{Mult(Y_2;Y_1)} \cdot \|f_1 - f_2\|_{Y_2} = \|g\|_{Mult(Y_2;Y_1)} \cdot d_{\|\cdot\|_{Y_2}}(f_1, f_2).$$

That is, if $x_1 = gf_1 \in A$ and $x_2 = gf_2 \in A$, we obtain that

$$d_{\|\cdot\|_{Y_1}}(x_1, x_2) \leqslant \|g\|_{Mult(Y_2;Y_1)} \cdot d_{\|\cdot\|_{Y_2}}(x_1/g, x_2/g).$$

Consequently, the multiplication $1/g$ defines a bijective map that gives a Lipschitz isomorphism, as we wanted to prove.

(ii) $\Rightarrow$ (i). Note first that by the factorization we can assume w.l.o.g that g cannot be equal to 0 in any set of positive measure. Take $n \in \mathbb{N}$ and finite sets

$x_1^1,\dots,x_n^1,x_1^2,\dots,x_n^2 \in Y_1$ and $y_1',\dots,y_n' \in Y_1'$. Then, if we define the operator S as $S(x) := \|g\|_{Mult(Y_2;Y_1)} \cdot x/g$, we obtain

$$\begin{aligned}\sum_{i=1}^n \int (x_i^1 - x_i^2)y_i'\,d\mu &= \int g\Big(\sum_{i=1}^n (x_i^1/g - x_i^2/g)y_i'\Big)\,d\mu \\ &\leqslant \|g\|_{Mult(Y_2;Y_1)} \cdot \Big\| \sum_{i=1}^n (x_i^1/g - x_i^2/g)y_i' \Big\|_{Y_2\pi Y_1'} \\ &= \Big\| \sum_{i=1}^n (S(x_i^1) - S(x_i^2))y_i' \Big\|_{Y_2\pi Y_1'}.\end{aligned}$$

Since clearly $S(0)=0$, these computations show (i) and the proof is finished. □

EXAMPLE 1. The first non-trivial example – despite its simplicity – is given by *the Lipschitz map defined by an appropriate multiplication operator.* Let $Y(\mu)$ be an order continuous Banach function space over the finite measure μ and suppose that it has the Fatou property. Let $A \subset Y(\mu)$ such that $0 \in A$ with measurable support and take $Y_1(\mu) = Y_2(\mu) = Y(\mu)$ in Theorem 4. Note that in this case $Y_1(\mu)\pi Y_1(\mu)' = L^1(\mu)$ ([11]). Assume that the measurable function h defines a multiplication operator $M_h : (A, d_{\|\cdot\|_{Y_1(\mu)}}) \to Y_1(\mu)$ – if $h \in B_{L^\infty(\mu)}$ – in such a way that the inequality

$$\int \sum_{i=1}^n \left(x_i^1 - x_i^2\right) y_i'\,d\mu \leqslant \Big\| \sum_{i=1}^n h\left(x_i^1 - x_i^2\right) y_i' \Big\|_{L^1(\mu)} = \int \Big| \sum_{i=1}^n \left(x_i^1 - x_i^2\right) y_i' \Big|\,|h|\,d\mu \tag{3}$$

holds for each $n \in \mathbb{N}$ and every pair of finite sets $x_1^1,\dots,x_n^1,x_1^2,\dots,x_n^2 \in A$ and $y_1',\dots,y_n' \in Y_1(\mu)'$. Of course, M_h is a Lipschitz operator that preserves 0. Consequently, Corollary 4 gives a function $g \in L^\infty(\mu)$ such that the multiplication operator $M_{1/g}$ gives a Lipschitz isomorphism from A to $Y(\mu)$. By the equation $x = gS(x) = ghx$ for all $x \in A$, we have that g can be identified with $1/h$, at least in the support of the set A. In particular, $1/h$ must be bounded in the support of A, since g belongs to $L^\infty(\mu)$; it can be seen that otherwise the inequality (3) does not hold for some elements of $Y(\mu)$.

The following results show why Theorem 4 provides the key tool of the present paper. For instance, it provides the next corollary, that characterizes when a subset of $L^q(\mu)$ – considered as a metric space – can be identified metrically with a subset of a space $L^p(hd\mu)$ for a certain weight h.

COROLLARY 1. *Let $1 < q < p < \infty$ and let $1/r = 1/p + 1/q'$. Let $A \subseteq L^q(\mu)$ such that $0 \in A$ and has measurable support. The following statements are equivalent.*

(i) There is a Lipschitz operator $S : (A, d_{\|\cdot\|_{L^q(\mu)}}) \to L^p(\mu)$ such that $S(0) = 0$ and

$$\int \sum_{i=1}^n (x_i^1 - x_i^2)y_i'\,d\mu \leqslant \Big\| \sum_{i=1}^n \left(S(x_i^1) - S(x_i^2)\right)y_i' \Big\|_{L^r(\mu)}$$

holds for each $n \in \mathbb{N}$ and each pair of finite sets $x_1^1,\dots,x_n^1,x_1^2,\dots,x_n^2 \in A$ and $y_1',\dots,y_n' \in L^{q'}(\mu)$.

(ii) *There is a function* $g \in L^{r'}(\mu)$ *such that the identity between* $(A, d_{\|\cdot\|_{L^q(\mu)}})$ *and* $(A, d_{\|\cdot\|_{L^p(d\mu/g^p)}})$ *is a Lipschitz isomorphism.*

Moreover, if this holds we have that S *satisfy the equation* $S(x) = x/g$ *for all* $x \in A$ *(maybe up to a multiplicative constant).*

Proof. (i) $\Rightarrow$ (ii). Let us see how (ii) follows from Theorem 4 by taking $Y_1(\mu) = L^q(\mu)$ and $Y_2(\mu) = L^p(\mu)$. Note that, by the conditions on p and q, we have that $1/q = 1/p + 1/r'$, and so $Mult(L^p(\mu); L^q(\mu)) = L^{r'}(\mu)$ and $L^p(\mu)\pi L^{q'}(\mu) = L^r(\mu)$. All these spaces are order continuous and have the Fatou property. Call B to the support of A, that is, the union of all the supports of the functions of A. By Theorem 4, we have that for a certain $g \in L^{r'}(\mu)$ – that can be assumed to be positive – the operator $S(x) = x/g$, for $x \in A$, defines a Lipschitz isomorphism between $(A, d_{\|\cdot\|_{L^q(\mu|_B)}})$ and $(A, d_{\|\cdot\|_{L^p(\mu|_B)}})$. The map $y \rightsquigarrow gy$ defines an isometry from $L^p(\mu|_B)$ on $L^p(d\mu|_B/g^p)$. Therefore, we have that the identity is a Lipschitz isomorphism between $(A, d_{\|\cdot\|_{L^q(\mu)}})$ and $(A, d_{\|\cdot\|_{L^p(d\mu/g^p)}})$.

Theorem 4 for the spaces $L^q(\mu)$ and $L^p(\mu)$ and the last computations give (ii) $\Rightarrow$ (i). $\square$

EXAMPLE 2. Let us show some concrete examples of the dominations that can be considered when applying our results.

(1) *The* p*-th power as a Lipschitz operator defining an integral domination for the identity map.* Let $1 < q < p < \infty$ and μ a probability measure, and consider the power map $S : L^q(\mu) \to L^p(\mu)$ defined by $S(x) := |x|^{q/p}$. Take a subset $A \subset L^q(\mu)$ containing 0 such that S restricted to A is a Lipschitz operator, that is, there is a constant $K > 0$ such that

$$\big\| |x|^{q/p} - |y|^{q/p} \big\|_{L^p(\mu)} \leqslant K \|x - y\|_{L^q(\mu)}, \quad \text{for all } x, y \in A.$$

Assume also that the integral inequality

$$\int \sum_{i=1}^n (x_{i,1} - x_{i,2}) y_i' \, d\mu \leqslant \Big(\int \Big| \sum_{i=1}^n \big(|x_{i,1}|^{q/p} - |x_{i,2}|^{q/p}\big) y_i' \Big|^r d\mu \Big)^{1/r}$$

holds for each finite set $x_1^1, \dots, x_n^1, x_1^2, \dots, x_n^2 \in A$ and $y_1', \dots, y_n' \in L^{q'}(\mu)$, where $1/r = 1/p + 1/q'$.

By Corollary 1, we have that there is a function $g \in L^{r'}(\mu)$ such that $(A, d_{\|\cdot\|_{L^q(\mu)}})$ and $(A, d_{\|\cdot\|_{L^p(\mu/g^p)}})$ can be identified by means of a Lipschitz isomorphism. Moreover, by the last statement in this result we can write an explicit formula for the operator S when restricted to the set A: if $x \in A$ it must satisfy the equation $|x|^{q/p} = x/g$, which strongly restricts the structure of the sets A satisfying this property.

(2) *A logarithmic domination that implies Lipschitz isomorphism for a subset of measurable functions with topologies coming from different* L^p *-spaces.* Let $q=4/3$ and $p=4$, and so $q'=4$ and $r=2$. Consider a subset A of functions in $\frac{1}{2}B_{L^\infty[0,1]}$ considered as functions in $L^{4/3}[0,1]$ and containing the zero function. Consider also the metric subspace $(A,\|\cdot\|_{L^{4/3}[0,1]})$ of the metric space $(\frac{1}{2}B_{L^\infty[0,1]},\|\cdot\|_{L^{4/3}[0,1]})$. Take now the operator $R:(\frac{1}{2}B_{L^\infty[0,1]},\|\cdot\|_{L^{4/3}[0,1]})\to L^{4/3}[0,1]$ defined as

$$R(f):=\log(f+1),\quad f\in\frac{1}{2}B_{L^\infty[0,1]},$$

and notice that it is well-defined almost everywhere. Since for $-1/2\leqslant r\leqslant 1/2$ the function $r\mapsto\log(r+1)$ is differentiable with bounded derivative, we have that $|R(f_1)-R(f_2)|\leqslant 2|f_1-f_2|$ and so

$$\|R(f_1)-R(f_2)\|_{L^{4/3}[0,1]}\leqslant 2\|f_1-f_2\|_{L^{4/3}[0,1]},\quad f_1,f_2\in\frac{1}{2}B_{L^\infty[0,1]}.$$

Therefore, R is a Lipschitz operator.

Consider also any continuous linear map $P:L^{4/3}[0,1]\to L^4[0,1]$. Then we have that $S:=P\circ R:(\frac{1}{2}B_{L^\infty[0,1]},\|\cdot\|_{L^{4/3}[0,1]})\to L^4[0,1]$ is a Lipschitz operator too.

Let us show that the associated logarithmic domination implies by virtue of our results, a strong relation among the metric spaces $(A,d_{\|\cdot\|_{L^{4/3}}})$ and $(A,d_{\|\cdot\|_{L^4(w)}})$ for a certain weight function w. Indeed, suppose that for all finite sets $f_{1,1},\dots,f_{n,1}$, $f_{1,2},\dots,f_{n,2}\in A$ and $h_1,\dots,h_n\in L^4[0,1]$ the adapted version of the inequality in (i) of Corollary 1 holds, that is

$$\int_{[0,1]}\sum_{i=1}^n(f_i^1-f_i^2)\,h_i\,dx\leqslant\Big(\int_{[0,1]}\Big|\sum_{i=1}^n P\Big(\log\frac{(f_{i,1}+1)}{(f_{i,2}+1)}\Big)\,h_i\Big|^2\,dx\Big)^{1/2}.$$

Thus, by Corollary 1 we can say that for sets A satisfying these requirements, there is a weight $g\in L^2[0,1]$ such that the identity map between $(A,d_{\|\cdot\|_{L^{4/3}}})$ and $(A,d_{\|\cdot\|_{L^4(dx/g^4)}})$ is a Lipschitz isomorphism. Also, this situation forces the functions $f\in A$ to satisfy the equation $P\circ\log(f+1)=\frac{f}{g}$.

Let us finish the paper with the following result that concerns the metric structure of the space of measurable functions $L^0(\mu)$. Also in this case, whenever a subset A of measurable functions has a metric structure coming from a norm, our results can be applied to analyze the Lipschitz isomorphisms between subsets of two different (metric) spaces of (classes of) measurable functions. If (Ω,Σ,μ) is a finite measure space, consider a subset A of the space $L^0(\mu)$ of all the μ-a.e equal classes of functions endowed with its natural metric

$$d_0(f,g)=\int_\Omega\frac{|f(w)-g(w)|}{1+|f(w)-g(w)|}\,d\mu(w).$$

The idea is just to notice that the results are also valid when a metric isomorphism with a space of measurable function is taken instead of considering directly a subset of an L^q-space. The proof of the next result follows the lines of the corollary above, as a direct application of Theorem 4. We say that the metric d_0 is equivalent to another metric coming from a norm of a Banach function space $d_{\|\|_{X(\mu)}}$ in the set of measurable functions $A \subseteq X(\mu)$ if and only if there are positive constants q and Q such that

$$q\, d_0(x^1,x^2) \leqslant \|x^1 - x^2\|_{X(\mu)} \leqslant Q\, d_0(x^1,x^2), \quad x^1,x^2 \in A,$$

that is, the identity is Lipschitz in both directions.

COROLLARY 2. *Let $1 \leqslant q \leqslant p < \infty$ and let $1/r = 1/p + 1/q'$. Let $A \subseteq L^0(\mu)$ such that $0 \in A$ and has measurable support, and the metrics d_0 and $d_{\|\cdot\|_{L^q(\mu)}}$ are equivalent on it. The following statements are equivalent.*

(i) There is a Lipschitz operator $S : (A,d_0) \to L^p(\mu)$ such that $S(0) = 0$ and

$$\int \sum_{i=1}^{n} \left(x_i^1 - x_i^2\right) y_i' \, d\mu \leqslant \left\| \sum_{i=1}^{n} \left(S(x_i^1) - S(x_i^2)\right) y_i' \right\|_{L^r(\mu)}$$

holds for each $n \in \mathbb{N}$ and functions $x_1^1,\ldots,x_n^1,x_1^2,\ldots,x_n^2 \in A$ and $y_1',\ldots,y_n' \in L^{q'}(\mu)$.

(ii) There exists a function $g \in L^{r'}(\mu)$ such that the identity between (A,d_0) and $(A,d_{\|\cdot\|_{L^p(d\mu/g^p)}})$ is a Lipschitz isomorphism.

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