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AN EXTENSION OF HARTFIEL'S DETERMINANT INEQUALITY

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Abstract. Let A and B be $n\times n$ positive definite matrices, Hartfiel obtained a lower bound for $\det(A+B)$. In this paper, we first extend his result to $\det(A+B+C)$, where A,B and C are $n\times n$ positive definite matrices, and then show a generalization of this to the case of matrices whose numerical ranges are contained in a sector.

1. Introduction

Let A,B be $n\times n$ positive semidefinite matrice, it is well known that

$$\det(A+B)\geqslant \det A+\det B. \tag{1}$$

In [5], Haynsworth proved the following refinement of (1),

$$\det(A+B)\geqslant \Big(1+\sum_{k=1}^{n-1}\frac{\det B_k}{\det A_k}\Big)\det A+\Big(1+\sum_{k=1}^{n-1}\frac{\det A_k}{\det B_k}\Big)\det B. \tag{2}$$

where A_k, B_k, $k=1,\cdots,n-1$, denote the k-th leading principal submatrices of positive definite matrices A and B, respectively.

Later, Hartfiel [4] obtained an improvement of (2) as follows:

$$\begin{aligned}\det(A+B)\geqslant&\Big(1+\sum_{k=1}^{n-1}\frac{\det B_k}{\det A_k}\Big)\det A+\Big(1+\sum_{k=1}^{n-1}\frac{\det A_k}{\det B_k}\Big)\det B\\&+(2^n-2n)\sqrt{\det A\det B}.\end{aligned} \tag{3}$$

where A_k,B_k are under the same condition as in (2). And the author also gave an interesting corollary:

$$\det(A+B)\geqslant \det A+\det B+(2^n-2)\sqrt{\det A\det B}. \tag{4}$$

In this paper, we fist extend Hartfiel's result by giving a lower bound for $\det(A+B+C)$, where A,B and C are $n\times n$ positive definite matrices. And then, we show some generalizations of the matrix form of the new determinant inequalities to a larger class of matrices, namely, matrices whose numerical ranges are contained in a sector.

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2. Auxiliary results

Let $\mathbb{M}_n$ be the set of $n\times n$ complex matrices. If A is positive semidefinite, we put $A\geqslant 0$, for two Hermitian matrices A, $B\in\mathbb{M}_n$, $A\geqslant B$ means $A-B$ is positive semidefinite. If A is positive definite, we put $A>0$.

For $A\in\mathbb{M}_n$, recall the Cartesian decomposition (see, e.g. [6, p. 7])

$$A=\Re A+i\Im A.$$

where

$$\Re A=\frac{1}{2}(A+A^*),\quad \Im A=\frac{1}{2i}(A-A^*).$$

We say $A\in\mathbb{M}_n$ is accretive-dissipative matrix if $\Re A$ and $\Im A$ are positive definie. For more details about this class of matrices, please refer to [3, 8, 9, 12].

The numerical range of $A\in\mathbb{M}_n$ is defined by

$$W(A)=\{x^*Ax|x\in\mathbb{C}^n,x^*x=1\}.$$

Also, for $\alpha\in[0,\frac{\pi}{2})$, let S_α be the sector in the complex plane given by

$$S_\alpha=\{z\in\mathbb{C}|\Re z>0,\ |\Im z|\leqslant(\Re z)\tan(\alpha)\}$$

Clearly, if $W(A)\subset S_0$, then A is positive definite. As $0\notin S_\alpha$, if $W(A)\subset S_\alpha$, we can get that A is necessarily nonsingular.

Relevant studies on matrices with numerical ranges in a sector can be found in [1, 2, 11, 7, 13].

Next, we present some lemmas which are useful for our proofs.

LEMMA 1. [6, Theorem 7.8.19] *Let $A\in\mathbb{M}_n$. If $\Re(A)>0$, then*

$$\det(\Re(A))\leqslant|\det(A)|.$$

LEMMA 2. [11, Lemma 2.6] *Let $A\in\mathbb{M}_n$ with $W(A)\subset S_\alpha$. Then*

$$\sec^n(\alpha)\det(\Re(A))\geqslant|\det(A)|.$$

LEMMA 3. [10, Theorem 1.1] *Let A, B, C be $n\times n$ positive definite matrices. Then*

$$\begin{aligned}\det(A+B+C)+\det(C)-\big(\det(A+C)+\det(B+C)\big)&\\ \geqslant\det(A+B)-\big(\det(A)+\det(B)\big).&\end{aligned}\tag{5}$$

LEMMA 4. [11, Proposition 2.1] *Let $A,B,C\in\mathbb{M}_n$, and partition A as $A=\begin{pmatrix}A_{11}&A_{12}\\A_{21}&A_{22}\end{pmatrix}$ with A_{11} square. If $W(A)$, $W(B)$, $W(C)\subset S_\alpha$, then $W(A+B+C)\subset S_\alpha$ and $W(A_{11})\subset S_\alpha$.*

3. Main results

In this section, we begin with extending inequality (3) and inequality (4) as follows:

THEOREM 1. *Let A, B, C be $n\times n$ positive definite matrices, A_k, B_k, C_k, $k=1,\cdots,n-1$, denote the k-th leading principal submatrices of A, B, C, respectively. Then*

$$\begin{aligned}\det(A+B+C)\geqslant&(1+\sum_{k=1}^{n-1}\frac{\det B_k+\det C_k}{\det A_k})\det A\\&+(1+\sum_{k=1}^{n-1}\frac{\det A_k+\det C_k}{\det B_k})\det B\\&+(1+\sum_{k=1}^{n-1}\frac{\det A_k+\det B_k}{\det C_k})\det C\\&+(2^n-2n)(\sqrt{\det AB}+\sqrt{\det AC}+\sqrt{\det BC}).\end{aligned}\tag{6}$$

Proof. According to Lemma 3, we have

$$\begin{aligned}\det(A+B+C)\geqslant&\det(A+B)+\det(A+C)+\det(B+C)\\&-\det A-\det B-\det C.\end{aligned}\tag{7}$$

Then we can obtain, by (3), that

$$\begin{aligned}\det(A+B+C)\geqslant&(1+\sum_{k=1}^{n-1}\frac{\det B_k}{\det A_k})\det A+(1+\sum_{k=1}^{n-1}\frac{\det A_k}{\det B_k})\det B\\&+(2^n-2n)\sqrt{\det AB}+(1+\sum_{k=1}^{n-1}\frac{\det C_k}{\det A_k})\det A\\&+(1+\sum_{k=1}^{n-1}\frac{\det A_k}{\det C_k})\det C+(2^n-2n)\sqrt{\det AC}\\&+(1+\sum_{k=1}^{n-1}\frac{\det C_k}{\det B_k})\det B+(1+\sum_{k=1}^{n-1}\frac{\det B_k}{\det C_k})\det C\\&+(2^n-2n)\sqrt{\det BC}-\det A-\det B-\det C\\\geqslant&(1+\sum_{k=1}^{n-1}\frac{\det B_k+\det C_k}{\det A_k})\det A+(1+\sum_{k=1}^{n-1}\frac{\det A_k+\det C_k}{\det B_k})\det B\\&+(1+\sum_{k=1}^{n-1}\frac{\det A_k+\det B_k}{\det C_k})\det C\\&+(2^n-2n)(\sqrt{\det AB}+\sqrt{\det AC}+\sqrt{\det BC}).\quad\square\end{aligned}$$

By inequality (4) and Lemma 3, the following interesting theorem can be obtained.

THEOREM 2. *Let A, B, C be $n\times n$ positive definite matrices. Then*

$$\begin{aligned}\det(A+B+C)\geqslant&\det A+\det B+\det C\\&+(2^n-2)(\sqrt{\det AB}+\sqrt{\det AC}+\sqrt{\det BC}).\end{aligned}\tag{8}$$

Now, we extend Theorem 1 and Theorem 2 to the case of matrices whose numerical ranges are contained in a sector.

THEOREM 3. *Let A, B, $C\in\mathbb{M}_n$, $W(A)$, $W(B)$, $W(C)\subset S_\alpha,\alpha\in[0,\frac{\pi}{2})$ and let A_k, B_k, C_k, $k=1,\cdots,n-1$, denote the k-th leading principal submatrices of A, B, C, respectively. Then*

$$\begin{aligned}\sec^n(\alpha)|\det(A+B+C)|\geqslant&(1+\sum_{k=1}^{n-1}\cos^k(\alpha)\frac{|\det B_k|+|\det C_k|}{|\det A_k|})|\det A|\\&+(1+\sum_{k=1}^{n-1}\cos^k(\alpha)\frac{|\det A_k|+|\det C_k|}{|\det B_k|})|\det B|\\&+(1+\sum_{k=1}^{n-1}\cos^k(\alpha)\frac{|\det A_k|+|\det B_k|}{|\det C_k|})|\det C|\\&+(2^n-2n)(\sqrt{|\det AB|}+\sqrt{|\det AC|}+\sqrt{|\det BC|}).\end{aligned}$$

Proof. According to Lemma 4, we have $W(A_k)$, $W(B_k)$, $W(C_k)\subset S_\alpha$. Compute

$$\begin{aligned}|\det(A+B+C)|\geqslant&\det(\mathfrak{R}(A+B+C))\\\geqslant&(1+\sum_{k=1}^{n-1}\frac{\det\mathfrak{R}(B_k)+\det\mathfrak{R}(C_k)}{\det\mathfrak{R}(A_k)})\det\mathfrak{R}(A)\\&+(1+\sum_{k=1}^{n-1}\frac{\det\mathfrak{R}(A_k)+\det\mathfrak{R}(C_k)}{\det\mathfrak{R}(B_k)})\det\mathfrak{R}(B)\\&+(1+\sum_{k=1}^{n-1}\frac{\det\mathfrak{R}(A_k)+\det\mathfrak{R}(B_k)}{\det\mathfrak{R}(C_k)})\det\mathfrak{R}(C)\\&+(2^n-2n)(\sqrt{\det\mathfrak{R}(A)\det\mathfrak{R}(B)}+\sqrt{\det\mathfrak{R}(A)\det\mathfrak{R}(C)}\\&+\sqrt{\det\mathfrak{R}(B)\det\mathfrak{R}(C)})\\\geqslant&(1+\sum_{k=1}^{n-1}\cos^k(\alpha)\frac{|\det B_k|+|\det C_k|}{|\det A_k|})\cos^n_\alpha|\det A|\\&+(1+\sum_{k=1}^{n-1}\cos^k(\alpha)\frac{|\det A_k|+|\det C_k|}{|\det B_k|})\cos^n_\alpha|\det B|\\&+(1+\sum_{k=1}^{n-1}\cos^k(\alpha)\frac{|\det A_k|+|\det B_k|}{|\det C_k|})\cos^n_\alpha|\det C|\\&+(2^n-2n)\cos^n(\alpha)(\sqrt{|\det AB|}+\sqrt{|\det AC|}+\sqrt{|\det BC|}).\end{aligned}$$

where the first inequality above is by Lemma 1; the second is due to the inequality (6) and the last inequality holds by Lemma 1 and Lemma 2.

Multiplying both sides of the inequality by $\sec^n(\alpha)$ yields the desired inequality, which completes the proof. □

When $\alpha = 0$, Theorem 3 reduces to Theorem 1. Note that if A is accretive-dissipative, then $W(e^{-i\pi/4}A) \subset S_{\pi/4}$. Thus, we have the following corollary.

COROLLARY 1. *Suppose $A, B, C \in \mathbb{M}_n$ are accretive-dissipative, let A_k, B_k, C_k, $k = 1 \cdots n-1$, denote the k-th leading principal submatrices of A, B, C, respectively. Then*

$$\begin{aligned}2^{\frac{n}{2}}|\det(A+B+C)| \geqslant &(1+\sum_{k=1}^{n-1}\frac{1}{2^{k/2}}\frac{|\det B_k|+|\det C_k|}{|\det A_k|})|\det A|\\&+(1+\sum_{k=1}^{n-1}\frac{1}{2^{k/2}}\frac{|\det A_k|+|\det C_k|}{|\det B_k|})|\det B|\\&+(1+\sum_{k=1}^{n-1}\frac{1}{2^{k/2}}\frac{|\det A_k|+|\det B_k|}{|\det C_k|})|\det C|\\&+(2^n-2n)(\sqrt{|\det AB|}+\sqrt{|\det AC|}+\sqrt{|\det BC|}).\end{aligned}$$

For the generalization of Theorem 2, the following results can be obtained by Lemma 1 and Lemma 2.

THEOREM 4. *Let $A, B, C \in \mathbb{M}_n$, $W(A), W(B), W(C) \subset S_\alpha$, $\alpha \in [0, \frac{\pi}{2})$. Then*

$$\begin{aligned}\sec^n(\alpha)|\det(A+B+C)| \geqslant &|\det A|+|\det B|+|\det C|\\&+(2^n-2)(\sqrt{|\det AB|}+\sqrt{|\det AC|}+\sqrt{|\det BC|}).\end{aligned}$$

COROLLARY 2. *Let $A, B, C \in \mathbb{M}_n$ be accretive-dissipative. Then*

$$\begin{aligned}2^{\frac{n}{2}}|\det(A+B+C)| \geqslant &|\det A|+|\det B|+|\det C|\\&+(2^n-2)(\sqrt{|\det AB|}+\sqrt{|\det AC|}+\sqrt{|\det BC|}).\end{aligned}$$

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