Mathematical
Inequalities
& Applications
Volume 21, Number 4 (2018), 1111–1123 doi:10.7153/mia-2018-21-76

# TWO MAPPINGS IN CONNECTION TO FEJÉR INEQUALITY WITH APPLICATIONS

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(*Communicated by S. Varošanec*)

*Abstract.* By the use of two $h$-convex mappings $H_g$ and $F_g$, some results and refinements related to the $h$-convex version of Fejér inequality are established. Also some applications for obtained inequalities in connection with Beta function of Euler are given.

## 1. Introduction

The following integral inequalities

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \leqslant \int_a^b f(x)g(x)dx \leqslant \frac{f(a)+f(b)}{2}\int_a^b g(x)dx, \tag{1}$$

where $f:[a,b]\to\mathbb{R}$ is convex and $g:[a,b]\to[0,+\infty)$ is integrable and symmetric to $x=\frac{a+b}{2}\Big(g(x)=g(a+b-x), \forall x\in[a,b]\Big)$, known in the literature as Fejér inequality, has been proved in 1906 by L. Fejér [8].

In 2006, the concept of $h$-convex functions related to the nonnegative real functions has been introduced in [16] by S. Varošanec, although it was not a complete generalization of the concept of convexity. The class of $h$-convex functions is including a large class of nonnegative functions such as nonnegative convex functions, Godunova-Levin functions [9], s-convex functions in the second sense [2] and P-functions [7].

DEFINITION 1. [16] Let $h:[0,1]\to\mathbb{R}^+$ be a function such that $h\not\equiv 0$. We say that $f:I\to\mathbb{R}^+$ is a h-convex function, if for all $x,y\in I$, $\lambda\in[0,1]$ we have

$$f\big(\lambda x+(1-\lambda)y\big)\leqslant h(\lambda)f(x)+h(1-\lambda)f(y). \tag{2}$$

Also the function $h$ is said to be supermultiplicative if

$$h(xy)\geqslant h(x)h(y),$$

for all $x,y\in[0,1]$.

*Mathematics subject classification* (2010): 26D15, 26A51, 52A01.
*Keywords and phrases*: $h$-convex function, Fejér inequality.


Paper MIA-21-76

The Fejér inequality related to $h$-convex functions has been introduced in [1] by M. Bombardelli et al. as the following without the assumption that $h$ is nonnegative.

THEOREM 1. *Let $f:[a,b]\to\mathbb{R}$ be $h$-convex, $w:[a,b]\to\mathbb{R}$, $w\geqslant 0$, symmetric with respect to $\frac{a+b}{2}$ with nonzero integral. Then*

$$\frac{1}{2h(\frac{1}{2})}f\left(\frac{a+b}{2}\right)\int_a^b w(t)dt \leqslant \int_a^b f(t)w(t)dt \\ \leqslant (b-a)[f(a)+f(b)]\int_0^1 h(t)w\big(ta+(1-t)b\big)dt. \tag{3}$$

For other inequalities in connection to Fejér inequality see [1, 6, 10, 11, 13, 14, 15] and references therein.

In this paper, by the use of two $h$-convex mappings $H_g$ (4) and $F_g$ (13), we establish some inequalities and refinements related to the left part of (3). Also some applications for obtained results in connection with Beta function of Euler are given.

## 2. Main results

### 2.1. The mapping $H_g$

The mapping $H_g:[0,1]\to\mathbb{R}$ defined by

$$H_g(t):=\int_a^b f\left(tu+(1-t)\frac{a+b}{2}\right)g(u)du, \tag{4}$$

has been introduced in [6] and some basic properties and applications related to the Fejér inequality in convex version have been obtained where symmetric function $g$ enjoyed the density property on $[a,b]$, i.e.

$$\int_a^b g(u)du = 1.$$

This mapping reduces to $H(t)$ in the classical case if we consider $g(u)=\frac{1}{b-a}$ (see [5]).

The following theorem is $h$-convex version of Theorem 84 in [6] without density condition for $g$.

THEOREM 2. *If $f:[a,b]\to\mathbb{R}$ is a $h$-convex function with $h(\frac{1}{2})>0$ and $g:[a,b]\to[0,\infty)$ is a symmetric function, then:*

(i) *$H_g$ is $h$-convex on $[0,1]$.*

(ii) *For $t=0$ and $t=1$,*

$$H_g(0)=f\left(\frac{a+b}{2}\right)\int_a^b g(u)du \quad \text{and} \quad H_g(1)=\int_a^b f(u)g(u)du.$$

(iii) *For any* $t \in (0,1]$,

$$\frac{1}{2h(\frac{1}{2})} f\left(\frac{a+b}{2}\right) \int_a^b g(u)du \leqslant H_g(t), \tag{5}$$

*and for any* $t \in (0,1)$,

$$H_g(t) \leqslant \left[h(t) + 2h\left(\frac{1}{2}\right)h(1-t)\right] \int_a^b f(u)g(u)du. \tag{6}$$

(iv) *There exist bounds,*

$$\inf_{t\in[0,1]} H_g(t) \geqslant \min\left\{\frac{1}{2h(\frac{1}{2})}, 1\right\} H_g(0),$$

*and*

$$\sup_{t\in[0,1]} H_g(t) \leqslant \max\left\{\sup_{t\in[0,1)} \left[h(t) + 2h\left(\frac{1}{2}\right)h(1-t)\right], 1\right\} H_g(1).$$

(v) *If* $h$ *is nonnegative and supermultiplicative, then for any* $0 < t_1 < t_2 < 1$ *with* $h(t_2) \neq 0$ *we have*

$$H_g(t_1) \leqslant \alpha H_g(t_2),$$

*where* $\alpha = \frac{2h(\frac{1}{2})h(t_2-t_1)+h(t_1)}{h(t_2)}$.

*Proof.* (i) It follows from $h$-convexity of $f$ that

$$\begin{aligned}
H_g(\alpha t_1 + \beta t_2) &= \int_a^b f\left(\left[\alpha t_1 + \beta t_2\right]u + \left[1 - \alpha t_1 - \beta t_2\right]\frac{a+b}{2}\right) g(u)du \\
&= \int_a^b f\left(\alpha\left[t_1 u + (1-t_1)\frac{a+b}{2}\right] + \beta\left[t_2 u + (1-t_2)\frac{a+b}{2}\right]\right) g(u)du \\
&\leqslant h(\alpha)\int_a^b f\left(t_1 u + (1-t_1)\frac{a+b}{2}\right) g(u)du \\
&\quad + h(\beta)\int_a^b f\left(t_2 u + (1-t_2)\frac{a+b}{2}\right) g(u)du \\
&= h(\alpha)H_g(t_1) + h(\beta)H_g(t_2),
\end{aligned}$$

provided that $\alpha + \beta = 1$.

(ii) It is obvious.

(iii) For inequality (5), consider the change of variable $x = tu + (1-t)\frac{a+b}{2}$ $(t > 0)$ in (4). Then

$$H_g(t) = \frac{1}{t} \int_{ta+(1-t)\frac{a+b}{2}}^{tb+(1-t)\frac{a+b}{2}} f(x) g\left(\frac{x + (t-1)\frac{a+b}{2}}{t}\right) dx, \tag{7}$$

where

$$t = \frac{\left(tb + (1-t)\frac{a+b}{2}\right) - \left(ta + (1-t)\frac{a+b}{2}\right)}{b-a}.$$

On the other hand since $g$ is symmetric to $\frac{a+b}{2}$ and

$$\frac{a+b}{2} = \frac{\left(ta + (1-t)\frac{a+b}{2}\right) + \left(tb + (1-t)\frac{a+b}{2}\right)}{2},$$

then $g$ remains symmetric on interval $\left[ta + (1-t)\frac{a+b}{2}, tb + (1-t)\frac{a+b}{2}\right]$ and so from Theorem 5 in [1] we have

$$\begin{aligned} &\frac{1}{2h(\frac{1}{2})} f\left(\frac{a+b}{2}\right) \int_{ta+(1-t)\frac{a+b}{2}}^{tb+(1-t)\frac{a+b}{2}} g\left(\frac{x+(t-1)\frac{a+b}{2}}{t}\right) dx \\ &= \frac{1}{2h(\frac{1}{2})} f\left(\frac{ta + (1-t)\frac{a+b}{2} + tb + (1-t)\frac{a+b}{2}}{2}\right) \\ &\times \int_{ta+(1-t)\frac{a+b}{2}}^{tb+(1-t)\frac{a+b}{2}} g\left(\frac{x+(t-1)\frac{a+b}{2}}{t}\right) dx \\ &\leqslant \int_{ta+(1-t)\frac{a+b}{2}}^{tb+(1-t)\frac{a+b}{2}} f(x) g\left(\frac{x+(t-1)\frac{a+b}{2}}{t}\right) dx. \end{aligned} \tag{8}$$

The relations (7) and (8) imply that

$$H_g(t) \geqslant \frac{1}{2h(\frac{1}{2})} f\left(\frac{a+b}{2}\right) \frac{1}{t} \int_{ta+(1-t)\frac{a+b}{2}}^{tb+(1-t)\frac{a+b}{2}} g\left(\frac{x+(t-1)\frac{a+b}{2}}{t}\right) dx. \tag{9}$$

Now using the change of variable $u = \frac{x+(t-1)\frac{a+b}{2}}{t}$ in (9) we get desired inequality:

$$H_g(t) \geqslant \frac{1}{2h(\frac{1}{2})} f\left(\frac{a+b}{2}\right) \int_a^b g(u) du.$$

The case that $t = 1$, follows from inequality

$$\frac{1}{2h(\frac{1}{2})} f\left(\frac{a+b}{2}\right) \int_a^b g(u) du \leqslant \int_a^b f(u) g(u) du,$$

obtained from Theorem 5 in [1].

For inequality (6), using the $h$-convexity of $f$ we have

$$H_g(t) \leqslant h(t) \int_a^b f(u) g(u) du + h(1-t) f\left(\frac{a+b}{2}\right) \int_a^b g(u) du. \tag{10}$$

Now from Theorem 5 in [1] and inequality (10) we get

$$\begin{aligned} H_g(t) &\leqslant h(t) \int_a^b f(u) g(u) du + h(1-t) 2h\left(\frac{1}{2}\right) \int_a^b f(u) g(u) du \\ &= \left[h(t) + h(1-t) 2h\left(\frac{1}{2}\right)\right] \int_a^b f(u) g(u) du. \end{aligned}$$

(iv) It is a consequence of (iii).

(v) According to Proposition 16 in [16], assertions $(i)$ and $(iii)$, if we Consider $0<t_1<t_2<1$ and $h(t_2)\neq 0$ then

$$\begin{aligned} h(t_2)H_g(t_1) &\leqslant h(t_2-t_1)H_g(0)+h(t_1)H_g(t_2) \\ &\leqslant 2h\left(\frac{1}{2}\right)h(t_2-t_1)H_g(t_2)+h(t_1)H_g(t_2) \\ &= \left[2h\left(\frac{1}{2}\right)h(t_2-t_1)+h(t_1)\right]H_g(t_2). \end{aligned}$$

Then

$$H_g(t_1)\leqslant \frac{2h(\frac{1}{2})h(t_2-t_1)+h(t_1)}{h(t_2)}H_g(t_2). \quad \square$$

If in Theorem 2, we consider $h(t)=t$ and $g(u)=\frac{1}{b-a}$ for $a<b$ we recapture the following result.

COROLLARY 1. (Theorem 71 in [6]) (see also [3, 5]) *For a given convex mapping* $f:[a,b]\to\mathbb{R}$, *let* $H:[0,1]\to\mathbb{R}$ *be defined by*

$$H(t):=\frac{1}{b-a}\int_a^b f\left(tu+(1-t)\frac{a+b}{2}\right)du.$$

*Then*

(i) $H$ *is convex on* $[0,1]$.

(ii) *One has the bounds:*

$$\inf_{t\in[0,1]} H(t)=H(0)=f\left(\frac{a+b}{2}\right),$$

*and*

$$\sup_{t\in[0,1]} H(t)=H(1)=\frac{1}{b-a}\int_a^b f(u)du.$$

(iii) $H$ *increases monotonically on* $[0,1]$.

COROLLARY 2. *In Theorem 2, for* $0\leqslant a\leqslant b$ *consider*

$$\begin{cases} f(u)=u^r,\ r\in(-\infty,-1)\cup(-1,0]\cup[1,\infty); \\ h(t)=t^s, \quad s\leqslant 1; \\ g\equiv 1. \end{cases}$$

*From Example 7 in* [16], $f$ *is* $h$*-convex and then from inequalities (5) and (6) we have*

$$\begin{aligned} &2^{s-1}\left(\frac{a+b}{2}\right)^r(b-a) \qquad (11) \\ &\leqslant \frac{1}{t(r+1)}\left[\left(\frac{(1-t)a+(1+t)b}{2}\right)^{r+1}-\left(\frac{(1+t)a+(1-t)b}{2}\right)^{r+1}\right] \\ &\leqslant \left[t^s+2^{1-s}(1-t)^s\right]\left(\frac{b^{r+1}-a^{r+1}}{r+1}\right), \end{aligned}$$

*for all* $t \in (0,1]$. *In more special case if we consider*

$$\begin{cases} f(u) = u^r, \ r \in [1,\infty); \\ h(t) = t, \\ g \equiv 1, \end{cases}$$

*then we get the following inequalities obtained in* [5].

$$\begin{aligned} &\left(\frac{a+b}{2}\right)^r (b-a) \\ &\leqslant \frac{1}{t(r+1)}\left[\left(\frac{a+b}{2} - t\left(\frac{b-a}{2}\right)\right)^{r+1} - \left(\frac{a+b}{2} - t\left(\frac{b-a}{2}\right)\right)^{r+1}\right] \\ &\leqslant \frac{b^{r+1} - a^{r+1}}{r+1}, \end{aligned} \tag{12}$$

*for all* $t \in (0,1]$.

REMARK 1. Assertion $(iii)$ in Theorem 2 can be stated as

$$\frac{1}{h(\frac{1}{2})} H_g(0) \leqslant H_g(t) \leqslant \left[h(t) + 2h\left(\frac{1}{2}\right)h(1-t)\right] H_g(1)$$

for all $t \in (0,1)$ which gives a refinement for the left part of (3). Also assertions $(i)$ and $(iii)$ of Theorem 2 together give generalized form of Theorem 12 in [16] for $t \in (0,1)$ in general case.

### 2.2. The mapping $F_g$

Now we consider the second mapping $F_g : [0,1] \to \mathbb{R}$ given by

$$F_g(t) := \int_a^b \int_a^b f\big(tx + (1-t)y\big) g(x) g(y) dxdy, \tag{13}$$

which has been introduced in [6], where the function $g$ assumed to be symmetric to $\frac{a+b}{2}$ with density property on $[a,b]$. Clearly, it reduces to $F$ in the classical case when $g(u) = \frac{1}{b-a}$ (see [5]). The following theorem involved some results related to the mapping $F_g$ when $f$ is $h$-convex without density property for $g$.

THEOREM 3. *If* $f : [a,b] \to \mathbb{R}$ *is* $h$*-convex with* $h(\frac{1}{2}) > 0$ *and* $g : [a,b] \to [0,\infty)$ *a symmetric function, then*

(i) $F_g$ *is* $h$*-convex on* $[0,1]$.

(ii) *For any* $t \in [0,1]$ *we have*

$$F_g(t) = F_g(1-t).$$

*Specially*

$$F_g(0)=F_g(1)=\int_a^b\int_a^b f(y)g(y)g(x)dxdy$$
$$=\int_a^b\int_a^b f(x)g(x)g(y)dxdy.$$

(iii) *For any* $t\in(0,1)$,

$$\frac{1}{2h(\frac{1}{2})}F_g\left(\frac{1}{2}\right)\leqslant F_g(t)\leqslant\big[h(t)+h(1-t)\big]F_g(0) \tag{14}$$
$$=\big[h(t)+h(1-t)\big]F_g(1).$$

*Also for* $t=0$ *and* $t=1$,

$$\frac{1}{2h(\frac{1}{2})}F_g\left(\frac{1}{2}\right)\leqslant F_g(0)=F_g(1).$$

(iv) *For any* $t\in[0,1]$,

$$\frac{1}{2h(\frac{1}{2})}f\left(\frac{a+b}{2}\right)\int_a^b\int_a^b g(x)g(y)dxdy\leqslant F_g(t). \tag{15}$$

(v) *If* $g$ *has density property, then for any* $t\in[0,1]$

$$F_g(t)\geqslant\frac{1}{2h(\frac{1}{2})}\max\{H_g(t),H_g(1-t)\}. \tag{16}$$

(vi) *There exist bounds,*

$$\inf_{t\in[0,1]}F_g(t)\geqslant\frac{1}{2h(\frac{1}{2})}F_g\left(\frac{1}{2}\right),$$

*and*

$$\sup_{t\in[0,1]}F_g(t)\leqslant\max\left\{\sup_{t\in(0,1)}\big[h(t)+h(1-t)\big],1\right\}F_g(1)$$
$$=\max\left\{\sup_{t\in(0,1)}\big[h(t)+h(1-t)\big],1\right\}F_g(0).$$

*Proof.* (i) It follows from $h$-convexity of $f$.
(ii) It is obvious.
(iii) For any $x,y\in[a,b]$ and $t\in(0,1)$ we have

$$f\left(\frac{x+y}{2}\right)=f\left(\frac{tx+(1-t)x+ty+(1-t)y}{2}\right) \tag{17}$$
$$\leqslant h\left(\frac{1}{2}\right)\Big[f\big(tx+(1-t)y\big)+f\big(ty+(1-t)x\big)\Big].$$

Multiplication by $g(x)g(y)$ and integration over $[a,b]\times[a,b]$ we get

$$\begin{aligned}\int_a^b\int_a^b f\Big(\frac{x+y}{2}\Big)g(x)g(y)dxdy &\leqslant h\Big(\frac{1}{2}\Big)\int_a^b\int_a^b f\big(tx+(1-t)y\big)g(x)g(y)dxdy\\ &\quad + h\Big(\frac{1}{2}\Big)\int_a^b\int_a^b f\big(ty+(1-t)x\big)g(x)g(y)dxdy\\ &= 2h\Big(\frac{1}{2}\Big)F_g(t),\end{aligned}$$

which proves the left side of (14).

For the right side of (14), using the $h$-convexity of $f$ we have

$$\begin{aligned}F_g(t) &\leqslant \int_a^b\int_a^b \big[h(t)f(x)g(x)g(y)+h(1-t)f(y)g(y)g(x)\big]dxdy \qquad (18)\\ &= [h(t)+h(1-t)]\int_a^b\int_a^b f(x)g(y)g(x)dxdy\\ &= [h(t)+h(1-t)]F_g(0)=[h(t)+h(1-t)]F_g(1).\end{aligned}$$

(iv) For any $t\in(0,1]$ and constant $y\in[a,b]$ define the function

$$F_g^y(t)=\int_a^b f\big(tx+(1-t)y\big)g(x)dx.$$

Using the change of variable $u=tx+(1-t)y$ we obtain

$$F_g^y(t)=\frac{1}{t}\int_{ta+(1-t)y}^{tb+(1-t)y} f(u)g\Big(\frac{u+(t-1)y}{t}\Big)du. \qquad (19)$$

Since $g$ is symmetric to $\frac{a+b}{2}$, then it remains symmetric on interval $\big[ta+(1-t)y,tb+(1-t)y\big]$ and so from Theorem 5 in [1] we have

$$\begin{aligned}F_g^y(t) &\geqslant \frac{1}{2h(\frac{1}{2})}f\Big(\frac{tb+(1-t)y+ta+(1-t)y}{2}\Big) \qquad (20)\\ &\quad\times\frac{1}{t}\int_{ta+(1-t)y}^{tb+(1-t)y} g\Big(\frac{u+(t-1)y}{t}\Big)du.\end{aligned}$$

Using the change of variable $x=\frac{u+(t-1)y}{t}$ in (20), for any $y\in[a,b]$ we have

$$F_g^y(t)\geqslant\frac{1}{2h(\frac{1}{2})}f\Big(\frac{a+b}{2}\Big)\int_a^b g(x)dx. \qquad (21)$$

Multiplying (21) by $g(y)$ and then integrating over $[a,b]$ with respect to $y$, we obtain

$$F_g(t)=\int_a^b F_g^y(t)g(y)dy\geqslant\frac{1}{2h(\frac{1}{2})}f\Big(\frac{a+b}{2}\Big)\int_a^b\int_a^b g(x)g(y)dxdy,$$

for any $t \in (0,1]$.

For $t=0$, using Theorem 5 in [1] we can obtain that

$$F_g(0)=\int_a^b\int_a^b f(y)g(x)g(y)dxdy=\int_a^b\Big[\int_a^b f(y)g(y)dy\Big]g(x)dx$$
$$\geqslant \frac{1}{2h(\frac{1}{2})}f\Big(\frac{a+b}{2}\Big)\int_a^b\int_a^b g(x)g(y)dxdy,$$

(v) From density of $g$, for any $t\in(0,1]$ we have

$$\frac{1}{t}\int_{ta+(1-t)y}^{tb+(1-t)y} g\left(\frac{u+(t-1)y}{t}\right)du=\int_a^b g(x)dx=1.$$

So from inequality (20) we get

$$F_g(t)=\int_a^b\int_a^b f\big(tx+(1-t)y\big)g(x)g(y)dxdy=\int_a^b F_g^y(t)g(y)dy$$
$$\geqslant \frac{1}{2h(\frac{1}{2})}\int_a^b f\left(t\frac{a+b}{2}+(1-t)y\right)g(y)dy=\frac{1}{2h(\frac{1}{2})}H_g(t).$$

In the case that $t=0$ we have

$$F_g(0)=\int_a^b\int_a^b f(y)g(x)g(y)dxdy=\int_a^b f(y)g(y)dy$$
$$\geqslant \frac{1}{2h(\frac{1}{2})}\int_a^b f\Big(\frac{a+b}{2}\Big)\int_a^b g(y)dy=\frac{1}{2h(\frac{1}{2})}H_g(0).$$

Also it is not hard to see that $F_g(t)$ is symmetric to $t=\frac{1}{2}$. So from assertion $(ii)$ we obtain

$$F_g(t)\geqslant \frac{1}{2h(\frac{1}{2})}\max\Big\{H_g(t),H_g(1-t)\Big\}.$$

(vi) It immediately follows from relation (14). □

If in Theorem 3, we consider $h(t)=t$ and $g(u)=\frac{1}{b-a}$ for $a<b$ we recapture the following result.

COROLLARY 3. (Theorem 74 in [6]) (see also [3, 4]) *Let* $f:[a,b]\to\mathbb{R}$ *be a convex function and* $F:[0,1]\to\mathbb{R}$,

$$F(t):=\frac{1}{(b-a)^2}\int_a^b\int_a^b f\big(tx+(1-t)y\big)dxdy.$$

*Then*

(i) $F$ *is convex on* $[0,1]$.
(ii) *For any* $t\in[0,1]$ *we have*

$$F(t)=F(1-t).$$

(iii) *The following inequality holds:*

$$f\Big(\frac{a+b}{2}\Big) \leqslant F\Big(\frac{1}{2}\Big).$$

(iv) *For any* $t \in [0,1]$*,*

$$F(t) \geqslant H(t).$$

(v) *We have the bounds:,*

$$\inf_{t\in[0,1]} F(t) = F\Big(\frac{1}{2}\Big) = \frac{1}{(b-a)^2}\int_a^b\int_a^b f\Big(\frac{x+y}{2}\Big)dxdy,$$

*and*

$$\sup_{t\in[0,1]} F(t) = F(0) = F(1) = \frac{1}{b-a}\int_a^b f(x)dx.$$

REMARK 2. Assertions $(i)-(iii)$ and $(v)$ in Theorem 3 together, give generalized form of Theorem 14 and Remark 15 in [16] for $t \in [0,1]$ in general case.

COROLLARY 4. *In Theorem 3, for* $0 \leqslant a < b$ *consider*

$$\begin{cases} f(u) = u^r,\ r \in (-\infty,-2)\cup(-2,-1)\cup(-1,0]\cup[1,\infty); \\ h(t) = t^s, \quad s \in (0,1); \\ g \equiv 1. \end{cases}$$

*Then*

$$\begin{aligned} &2^{s-1}\Big(\frac{a+b}{2}\Big)^r(b-a)^2 \\ &\leqslant \frac{1}{t(1-t)(r+1)(r+2)} \\ &\quad\times\Big[b^{r+2} - \big(tb+(1-t)a\big)^{r+2} - \big(ta+(1-t)b\big)^{r+2} + a^{r+2}\Big] \\ &\leqslant \big[t^s + (1-t)^s\big](b-a)\frac{b^{r+1}-a^{r+1}}{r+1}, \end{aligned} \tag{22}$$

*for all* $t \in (0,1)$*. In (22), if we consider* $h(t) = t$*, then we get*

$$\begin{aligned} &\Big(\frac{a+b}{2}\Big)^r(b-a)^2 \\ &\leqslant \frac{1}{t(1-t)(r+1)(r+2)} \\ &\quad\times\Big[b^{r+2} - \big(tb+(1-t)a\big)^{r+2} - \big(ta+(1-t)b\big)^{r+2} + a^{r+2}\Big] \\ &\leqslant (b-a)\frac{b^{r+1}-a^{r+1}}{r+1} \end{aligned}$$

*for all* $t \in (0,1]$. *Furthermore in point* $t = \frac{1}{2}$ *we have*

$$\left(\frac{a+b}{2}\right)^r (b-a)^2 \leqslant \frac{4}{(r+1)(r+2)}\left[b^{r+2} - 2\left(\frac{a+b}{2}\right)^{r+2} + a^{r+2}\right] \leqslant (b-a)\frac{b^{r+1}-a^{r+1}}{r+1},$$

*which was obtained in* [7].

## 3. Applications for the Beta function

In this section as an application we find some relations between the results obtained in Theorem 2 and Theorem 3 and the Beta function of Euler. Consider the Beta function of Euler, that is,

$$B(p,q) := \int_0^1 t^{p-1}(1-t)^{q-1}dt, \ \ p, \ q > -1,$$

and

$$H_B^r(t,p) = \int_0^1 \left(tu + \frac{1-t}{2}\right)^r u^{p-1}(1-u)^{p-1}du,$$

where $t \in [0,1]$, $p > -1$, $r \geqslant 1$. Also for all $t \in [0,1]$ define the following functions

$$\begin{cases} f(t) = \left(tu + \frac{1-t}{2}\right)^r, & r \geqslant 1, u \geqslant 0; \\ h(t) = t^k, & k \leqslant 1; \\ g(t) = t^{p-1}(1-t)^{p-1}, \ p > -1. \end{cases}$$

According to Example 7 in [16], the function $f$ is $h$-convex. Also the function $g$ is symmetric to $t = \frac{1}{2}$. Then from Theorem 2, the function $H_B^r(.,p)$ is $h$-convex on $[0,1]$ and

$$\frac{1}{2(\frac{1}{2})^k}\left(\frac{1}{2}\right)^r \int_0^1 u^{p-1}(1-u)^{p-1}du \leqslant H_B^r(t,p) \leqslant \left[t^k + 2(\frac{1}{2})^k(1-t)^k\right]\int_0^1 u^r u^{p-1}(1-u)^{p-1}du,$$

which implies that

$$2^{k-r-1}B(p,p) \leqslant H_B^r(t,p) \leqslant \left[t^k + 2^{1-k}(1-t)^k\right]B(r+p,p), \tag{23}$$

for all $t \in [0,1]$, $r \geqslant 1$, $k \leqslant 1$ and $p > -1$.

Now define the function

$$F_B^r(t,p) = \int_0^1\int_0^1 \left(tx + (1-t)y\right)^r x^{p-1}y^{p-1}(1-x)^{p-1}(1-y)^{p-1}dxdy, \tag{24}$$

where $t \in [0,1]$, $r \geqslant 1$ and $p > -1$ (also see [7]).

With assumptions

$$\begin{cases} f(t) = \big(tx + (1-t)y\big)^r, & r \geqslant 1, x, y \geqslant 0; \\ h(t) = t^k, & k \leqslant 1; \\ g(t) = t^{p-1}(1-t)^{p-1}, & p > -1, \end{cases}$$

for all $t \in [0,1]$, from Example 7 in [16], the function $f$ is $h$-convex. Therefore from Theorem 3, the function $F_B^r(.,p)$ is $h$-convex on $[0,1]$ and symmetric to $t = \frac{1}{2}$. Also we have the following inequalities:

$$\begin{aligned} &\frac{1}{2(\frac{1}{2})^k} \int_0^1 \int_0^1 \left(\frac{x+y}{2}\right)^r x^{p-1} y^{p-1} (1-x)^{p-1} (1-y)^{p-1} dxdy \\ &\leqslant F_B^r(t,p) \leqslant \Big[h(t) + h(1-t)\Big] \int_0^1 \int_0^1 x^r x^{p-1} y^{p-1} (1-x)^{p-1} (1-y)^{p-1} dxdy \\ &= \Big[h(t) + h(1-t)\Big] \int_0^1 \int_0^1 y^r x^{p-1} y^{p-1} (1-x)^{p-1} (1-y)^{p-1} dxdy, \end{aligned}$$

which implies that

$$2^{k-r-1} B^2(p,p) \leqslant F_B^r(t,p) \leqslant \Big[t^k + (1-t)^k\Big] B(r+p,p) B(p,p), \tag{25}$$

for all $t \in [0,1]$, $r \geqslant 1$, $k \leqslant 1$ and $p > -1$.

Furthermore since we have

$$\int_0^1 \frac{1}{B(p,p)} t^p (1-t)^{p-1} dt = 1,$$

then if we consider $g(t) = \frac{1}{B(p,p)} t^p (1-t)^{1-p}$, from inequality (16) we get

$$F_B^r(t,p) \geqslant 2^{k-1} \max\Big\{H_B^r(t,p), H_B^r(1-t,p)\Big\} B^2(p,p).$$

REMARK 3. Inequality (23) reduces to the convex version obtained in [6], if we consider $k = 1$,

$$2^{-r} B(p,p) \leqslant H_B^r(t,p) \leqslant B(r+p,p),$$

for all $t \in [0,1]$, $p > -1$, $r \geqslant 1$.

Also the convex version of inequality (25) can be stated as the following.

$$2^{-r} B^2(p,p) \leqslant F_B^r(t,p) \leqslant B(r+p,p) B(p,p),$$

for all $t \in [0,1]$, $p > -1$, $r \geqslant 1$.

Furthermore we have

$$F_B^r(t,p) \geqslant \max\Big\{H_B^r(t,p), H_B^r(1-t,p)\Big\} B^2(p,p),$$

for all $t \in [0,1]$, $p > -1$, $r \geqslant 1$.

## REFERENCES


[1] M. BOMBARDELLI AND S. VAROŠANEC, *Properties of h-convex functions related to the Hermite-Hadamard-Fejér inequalities*, Comput. Math. Appl. **58** (2009), 1869–1877.

[2] W. W. BRECKNER, *Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer funktionen in topologischen linearen Räumen*, Publ. Inst. Math. **23** (1978), 13–20.

[3] S. S. DRAGOMIR, *A mapping in connection to Hadamard's inequalities*, An. Öster. Akad. Wiss. Math.-Natur., (Wien), **128** (1991), 17–20.

[4] S. S. DRAGOMIR, *On Hadamard's inequalities for convex functions*, Mat. Balkanica **6** (1992), 215–222.

[5] S. S. DRAGOMIR, *Two Mappings in Connection to Hadamard's Inequalities*, J. Math. Anal. Appl. **167** (1992), 49–56.

[6] S. S. DRAGOMIR AND C. E. M. PEARCE, *Selected topics on Hermite-Hadamard inequalities and applications*, RGMIA Monographs, Victoria University, 2000, (online: `http://ajmaa.org/RGMIA/monographs.php/`).

[7] S. S. DRAGOMIR, J. PEČARIĆ, L. E. PERSSON, *Some inequalities of Hadamard type*, Soochow J. Math. **21** (1995), 335–341.

[8] L. FEJÉR, *Über die fourierreihen, II*, Math. Naturwise. Anz Ungar. Akad. Wiss. **24** (1906), 369–390.

[9] E. K. GODUNOVA, V. I. LEVIN, *Neravenstva dlja funkcii širokogo klassa, soderžaščego vypuklye, monotonnye i neko-torye drugie vidy funkcii*, in: Vyčislitel. Mat. i. Mat. Fiz. Mežvuzov. Sb. Nauč. Trudov, pp. 138–142, MGPI, Moskva, 1985.

[10] B. MICHERDA AND T. RAJBA, *On Some Hermite-Hadamard-Fejér inequalities for (k,h)-convex function*, Math. Inequal. Appl. **4** (2012), 931–940.

[11] J. PARK, *Inequalities of Hermite-Hadamard-Fejér type for convex functions via fractional integrals*, Int. J. Math. Anal. **8**, 59 (2014), 2927–2937.

[12] A. W. ROBERT AND D. E. VARBEG, *Convex functions*, Academic Press, 1973.

[13] M. ROSTAMIAN DELAVAR AND M. DE LA SEN, *On Generalization of Fejér Type Inequalities*, Communications in Mathematics and Applications, **8**, 1 (2017), 31–43.

[14] M. Z. SARIKAYA, *On new Hermite Hadamard Fejér type integral inequalities*, Stud. Univ. Babes-Bolyai Math. **57**, 3 (2012), 377–386.

[15] K.-L. TSENG, G.-S. YANG AND K.-C. HSU, *Some Inequalities for Differentiable Mappings and Applications to Fejér Inequality and Weighted Trapezoidal Formula*, Taiwanese J. Math. **15**, 4 (2011), 1737–1747.

[16] S. VAROŠANEC, *On h-convexity*, J. Math. Anal. Appl. **326** (2007), 303–311.


(Received December 18, 2017)

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