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TURÁN TYPE INEQUALITIES FOR q–MITTAG–LEFFLER AND q–WRIGHT FUNCTIONS

KHALED MEHREZ

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Abstract. Our aim in this paper is to derive several Turán type inequalities for the q-Mittag Leffler and q-Wright functions. Moreover, we prove the monotonicity of ratios for sections of series of q-Mittag Leffler and q-Wright functions, the results is also closely connected with Turán type inequalities. In order to obtain some of the main results we apply the methods developed in the case of classical Mittag–Leffler and Wright functions. At the end of the paper we pose two open problems, which may be of interest for further research.

1. Introduction

The special functions of mathematical physics are found to be very useful for finding solutions of initial and boundary–value problems governed by partial differential equations and fractional differential equations. Several special functions, called recently special functions of fractional calculus, play a very important and interesting role as solutions of fractional order differential equations, such as the Mittag–Leffler function and the Wright functions. Recently, this special functions plays an important role in analysis where it is used in the theory of integral transforms and representations of complex-valued functions [3, 4], fractional calculus [12, 7, 15], and other areas [14, 17]. Because of this their properties worth to be studied also from the point of view of analytic inequalities. For a long list of applications concerning inequalities involving Mittag–Leffler and Wright functions of we refer to the papers [8], [9], [10] and to the references therein. An important result which initiated a new field of research on inequalities for special functions was proved by Paul Turán, it is,

$$[P_n(x)]^2 - P_{n-1}(x)P_{n+1}(x) \geqslant 0,$$

where $x \in [0,1]$, $n \in \mathbb{N}$ and $P_n(x)$ stands for the classical Legendre polynomial. For more literature on Turán inequality for various orthogonal polynomials and special functions, we refer the reader to the details given in [1, 2, 11] and references therein. The Turán type inequalities now have an extensive literature and some of the results have been applied successfully to different problems in information theory, economic theory, biophysics, probability and statistics. Since Turán inequality was investigated

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for the orthogonal polynomials having hypergeometric representation, it is worth studying the validity of such inequality for various special functions as well. In [8, 9, 10], the Turán type inequalities for the classical and generalized Mittag–Leffler functions and the Wright function were discussed. In this paper, we would like to present the q-version of some results obtained in [8, 9, 10] for the classical Mittag–Leffler functions and the Wright function.

In our present investigation, we shall need the following notations and definitions. First of all, for $q \in (0,1)$, $a \in \mathbb{C}$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ ($\mathbb{N}$ being the set of positive integers), the q-shifted factorials are defined by

$$(a;q)_0 = 1, \ \ (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \ \ (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

The q-gamma function $\Gamma_q(z)$ is defined by

$$\Gamma_q(z) = (1-q)^{1-z}\prod_{n=0}^{\infty}\frac{1-q^{n+1}}{1-q^{n+z}}, \ \ q \in (0,1), \ \ z \in \mathbb{C}, \tag{1}$$

and

$$\Gamma_q(z) = (q-1)^{1-z}q^{\frac{z(z-1)}{2}}\prod_{n=0}^{\infty}\frac{1-q^{-(n+1)}}{1-q^{-(n+z)}}, \ \ q > 1, \ \ z \in \mathbb{C}. \tag{2}$$

From the previous definitions, for a positive z and $q \geqslant 1$, we get

$$\Gamma_q(z) = (q-1)^{1-z}q^{\frac{(z-1)(z-2)}{2}}\Gamma_{\frac{1}{q}}(z). \tag{3}$$

The q-digamma function $\psi_q(x)$ is defined as the logarithmic derivative of the q-gamma function

$$\psi_q(x) = \frac{d}{dz}(\log\Gamma_q(z)) = -\log(1-q) + \log(q)\sum_{k=1}^{\infty}\frac{q^{kx}}{1-q^k}, \tag{4}$$

for $q \in (0,1)$ and from (3) we obtain for $q > 1$ and $x > 0$,

$$\psi_q(x) = -\log(1-q) + \log(q)\left[x - \frac{1}{2} - \sum_{k=1}^{\infty}\frac{q^{-kx}}{1-q^{-k}}\right]. \tag{5}$$

For further details about the q-calculus, one may refer to the books by Gasper and Rahman [6].

The present sequel to some of the aforementioned investigations is organized as follows. In section 2, we present some Turán type inequalities for the q-Mittag–Leffler functions. Moreover, we prove monotonicity of ratios for sections of series of q-Mittag-Leffler functions, the result is also closely connected with Turán–type inequalities. In section 3, we prove several Turán type inequalities for the q-Wright functions. In addition, we derive the monotonicity of ratios for sections of series of q-Wright functions, as consequence we obtain the Turán type inequalities for the remainder of series of q-Wright functions. Finally, in Section 4, we would like to comment the main results and we present certain open problems, which may be of interest for further research.

2. Turán type inequalities for the q-Mittag–Leffler functions

In this section, we consider the q-Mittag–Leffler function [16]:

$$E_{\alpha,\beta}(q;z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma_q(\alpha k+\beta)}, \quad \alpha,\beta,z \in \mathbb{C}; \;\; \Re(\alpha) > 0. \tag{6}$$

The q-Mittag–Leffler function contains many known functions as its special cases. For example, we have

$$\begin{cases} E_{0,\beta}(q;z) = \frac{\Gamma_q(\beta)}{1-z}, & E_{1,1}(q;z) = e(q;z) \\ E_{1,2}(q;z) = \frac{e(q;z)-1}{z}, & E_{1,3}(q;z) = \frac{e(q;z)-z-1}{z^2} \\ E_{1,4}(q;z) = \frac{(1+q)(e(q;z)-z-1)-z^2}{(1+q)z^3} & \end{cases} \tag{7}$$

where $e(q;z)$ one of the q-analogues of the classical exponential function e^z given by

$$e(q;z) = \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty}, \quad |z| < 1,$$

Another widely-studied q-analogue of the classical exponential function e^z is given by

$$E(q;z) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} \frac{z^n}{(q;q)_n} = \prod_{k=0}^{\infty}(1+zq^k), \;\; z \in \mathbb{C}.$$

Our first main result is asserted by the following theorem.

THEOREM 1. *Let $\alpha,\beta > 0$ and $q \in (0,q_\beta)$, such that $f_\beta(q_\beta) = 0$ where f_β is defined on $(0,1)$, by $f_\beta(x) = x^{\beta+1}+x-1$. Then the following Turán type inequality*

$$\Big(E_{\alpha,\beta+1}(q;z)\Big)^2 - E_{\alpha,\beta}(q;z)E_{\alpha,\beta+2}(q;z) \geqslant 0, \tag{8}$$

hold for all $z > 0$.

Proof. By applying the Cauchy product, we find that

$$E_{\alpha,\beta+2}(q;z)E_{\alpha,\beta}(q;z) = \sum_{k=0}^{\infty}\sum_{j=0}^{k} \frac{z^k}{\Gamma_q(\alpha j+\beta)\Gamma_q(\alpha(k-j)+\beta+2)} \tag{9}$$

and

$$\Big(E_{\alpha,\beta+1}(q;z)\Big)^2 = \sum_{k=0}^{\infty}\sum_{j=0}^{k} \frac{z^k}{\Gamma_q(\alpha j+\beta+1)\Gamma_q(\alpha(k-j)+\beta+1)}. \tag{10}$$

In views of (9) and (10) and the functional equation:

$$\Gamma_q(x+1) = \frac{1-q^x}{1-q}\Gamma_q(x), \tag{11}$$

we get

$$\begin{aligned}
&E_{\alpha,\beta}(q;z)E_{\alpha,\beta+2}(q;z)-E^2_{\alpha,\beta+1}(q;z)\\
&=\sum_{k=0}^{\infty}\sum_{j=0}^{k}\left(\frac{1}{\Gamma_q(\alpha j+\beta)\Gamma_q(\alpha(k-j)+\beta+2)}-\frac{1}{\Gamma_q(\alpha j+\beta+1)\Gamma_q(\alpha(k-j)+\beta+1)}\right)z^k\\
&=(1-q)q^{\beta}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{\left(q^{\alpha(k-j)+1}-q^{\alpha j}\right)}{\Gamma_q(\alpha j+\beta+1)\Gamma_q(\alpha(k-j)+\beta+2)}z^k\\
&=(1-q)q^{\beta}\sum_{k=0}^{\infty}\sum_{j=0}^{k}T^{(1)}_{j,k}(\alpha,\beta;q)z^k,
\end{aligned}\tag{12}$$

where

$$T^{(1)}_{j,k}(\alpha,\beta;q)=\frac{q^{\alpha(k-j)+1}-q^{\alpha j}}{\Gamma_q(\alpha j+\beta+1)\Gamma_q(\alpha(k-j)+\beta+2)}.\tag{13}$$

Case 1. Let n be an even positive integer. Then

$$\begin{aligned}
\sum_{j=0}^{k}T^{(1)}_{j,k}(\alpha,\beta;q)&=\sum_{j=0}^{\frac{k}{2}-1}T^{(1)}_{j,k}(\alpha,\beta;q)+\sum_{j=\frac{k}{2}+1}^{k}T^{(1)}_{j,k}(\alpha,\beta;q)+T^{(1)}_{k/2,k}(\alpha,\beta;q)\\
&=\sum_{j=0}^{[\frac{k-1}{2}]}\left(T^{(1)}_{j,k}(\alpha,\beta;q)+T^{(1)}_{k-j,k}(\alpha,\beta;q)\right)\\
&\quad+\frac{q^{\frac{\alpha k}{2}}(q-1)}{\Gamma_q(\alpha k/2+\beta+1)\Gamma_q(\alpha k/2+\beta+2)},
\end{aligned}\tag{14}$$

where, as usual, $[k]$ denotes the greatest integer part of $k\in\mathbb{R}$.

Case 2. Let n be an odd positive integer. Then, just as in Case 1, we get

$$\begin{aligned}
\sum_{j=0}^{k}T^{(1)}_{j,k}(\alpha,\beta;q)&=\sum_{j=0}^{[\frac{k-1}{2}]}\left(T^{(1)}_{j,k}(\alpha,\beta;q)+T^{(1)}_{k-j,k}(\alpha,\beta;q)\right)\\
&\quad+\frac{q^{\frac{\alpha k}{2}}(q-1)}{\Gamma_q(\alpha k/2+\beta+1)\Gamma_q(\alpha k/2+\beta+2)}.
\end{aligned}\tag{15}$$

Thus, by combining Case 1 and Case 2, we have

$$\begin{aligned}
E_{\alpha,\beta}(q;z)E_{\alpha,\beta+2}(q;z)-E^2_{\alpha,\beta+1}(q;z)&=\sum_{k=0}^{\infty}\sum_{j=0}^{[\frac{k-1}{2}]}(T^{(1)}_{j,k}(\alpha,\beta;q)+T^{(1)}_{k-j,k}(\alpha,\beta;q))z^k\\
&\quad+\frac{q^{\frac{\alpha k}{2}}(q-1)}{\Gamma_q(\alpha k/2+\beta+1)\Gamma_q(\alpha k/2+\beta+2)}.
\end{aligned}\tag{16}$$

Simplifying, we find that

$$
\begin{aligned}
&T_{j,k}^{(1)}(\alpha,\beta;q)+T_{k-j,k}^{(1)}(\alpha,\beta;q)\\
&=\frac{q^{\alpha(k-j)+1}-q^{\alpha j}}{\Gamma_q(\alpha j+\beta+1)\Gamma_q(\alpha(k-j)+\beta+2)}+\frac{q^{\alpha j+1}-q^{\alpha(k-j)}}{\Gamma_q(\alpha(k-j)+\beta+1)\Gamma_q(\alpha j+\beta+2)}\\
&=\frac{1-q}{\Gamma_q(\alpha j+\beta+1)\Gamma_q(\alpha(k-j)+\beta+1)}\left[\frac{q^{\alpha(k-j)+1}-q^{\alpha j}}{1-q^{\alpha(k-j)+\beta+1}}+\frac{q^{\alpha j+1}-q^{\alpha(k-j)}}{1-q^{\alpha j+\beta+1}}\right]\\
&=\frac{(1-q)A_{j,k}(\alpha,\beta;q)}{\Gamma_q(\alpha j+\beta+2)\Gamma_q(\alpha(k-j)+\beta+2)}
\end{aligned}
\tag{17}
$$

where is $A_{j,k}(\alpha,\beta;q)$ is defined by

$$
\begin{aligned}
A_{j,k}(\alpha,\beta;q)&=(q^{\alpha(k-j)+1}-q^{\alpha j})(1-q^{\alpha j+\beta+1})+(q^{\alpha j+1}-q^{\alpha(k-j)})(1-q^{\alpha(k-j)+\beta+1})\\
&=(q^{\alpha(k-j)+1}-q^{\alpha(k-j)})+(q^{\alpha j+1}-q^{\alpha j})+q^{\beta+1}(q^{2\alpha j}+q^{2\alpha(k-j)})-2q^{\alpha k+\beta+2}\\
&\leqslant(q^{\alpha(k-j)+1}-q^{\alpha(k-j)})+(q^{\alpha j+1}-q^{\alpha j})+q^{\beta+1}(q^{\alpha j}+q^{\alpha(k-j)})-2q^{\alpha k+\beta+2}\\
&=(q-1)(q^{\alpha j}+q^{\alpha(k-j)})+q^{\beta+1}(q^{\alpha j}+q^{\alpha(k-j)})-2q^{\alpha k+\beta+2}\\
&=(q-1+q^{\beta+1})(q^{\alpha j}+q^{\alpha(k-j)})-2q^{\alpha k+\beta+2}\\
&=f_\beta(q)(q^{\alpha j}+q^{\alpha(k-j)})-2q^{\alpha k+\beta+2}.
\end{aligned}
\tag{18}
$$

Then $A_{j,k}(\alpha,\beta;q)\leqslant 0$, for all $\alpha,\beta>0$ and $q\in(0,q_\beta)$, which yields that

$$T_{j,k}^{(1)}(\alpha,\beta;q)+T_{k-j,k}^{(1)}(\alpha,\beta;q)\leqslant 0,$$

for all $\alpha,\beta>0$ and $q\in(0,q_\beta)$. By means of (16), we deduce the Turán type inequality (8). The proof of Theorem 1 is completes. ☐

Taking in (8) the value $\beta=1$ and using the relations (7), we obtain the Turán type inequalities for the q-Mittag–Leffler function.

COROLLARY 1. *Let $\alpha>0$ and $q\in(0,\frac{\sqrt{5}-1}{2})$, Then the q-Mittag Leffler function $E_{\alpha,1}(q;z)$ satisfies the following Turán type inequality*

$$\Big(E_{\alpha,2}(q;z)\Big)^2-E_{\alpha,1}(q;z)E_{\alpha,3}(q;z)\geqslant 0, \tag{19}$$

for all $z>0$.

THEOREM 2. *Let $\alpha,\beta>0$ and $q\in(0,1/2)$. Then the following Turán type inequality*

$$E_{\alpha,\beta+2}(q;z)E_{\alpha,\beta}(q;z)-\Big(E_{\alpha,\beta+1}(q;z)\Big)^2+E_{\alpha,\beta+1}(q;z)E_{\alpha,\beta+2}(q;z)\geqslant 0 \tag{20}$$

is true for all $z\in(0,\infty)$.

Proof. The relations (9), (10) and the Cauchy product gives

$$\Delta_{\alpha,\beta}(q;z) = E_{\alpha,\beta+2}(q;z)E_{\alpha,\beta}(q;z) - \Big(E_{\alpha,\beta+1}(q;z)\Big)^2 + E_{\alpha,\beta+1}(q;z)E_{\alpha,\beta+2}(q;z)$$
$$= \sum_{k=0}^{\infty}\sum_{j=0}^{k} T_{j,k}^{(2)}(\alpha,\beta;q)z^k, \tag{21}$$

where

$$T_{j,k}^{(2)}(\alpha,\beta;q) = \frac{q^{\beta}(1-q)\Big(q^{\alpha(k-j)+1}-q^{\alpha j}\Big)+1}{\Gamma_q(\alpha j+\beta+1)\Gamma_q(\alpha(k-j)+\beta+2)}.$$

Thus,

$$\Delta_{\alpha,\beta}(q;z) = \sum_{k=0}^{\infty}\sum_{j=0}^{[(k-1)/2}\Big(T_{j,k}^{(2)}(\alpha,\beta;q)+T_{k-j,k}^{(2)}(\alpha,\beta;q)\Big)z^k$$
$$+\frac{1-q^{\beta+\frac{\alpha k}{2}}(1-q)^2}{\Gamma_q(\alpha k/2+\beta+1)\Gamma_q(\alpha k/2+\beta+2)}. \tag{22}$$

By computation, we have

$$\begin{aligned} T_{j,k}^{(2)}(\alpha,\beta;q)+T_{k-j,k}^{(2)}(\alpha,\beta;q) &= \frac{q^{\beta}(1-q)\Big(q^{\alpha(k-j)+1}-q^{\alpha j}\Big)+1}{\Gamma_q(\alpha j+\beta+1)\Gamma_q(\alpha(k-j)+\beta+2)} \\ &\quad+\frac{q^{\beta}(1-q)\Big(q^{\alpha j+1}-q^{\alpha(k-j)}\Big)+1}{\Gamma_q(\alpha(k-j)+\beta+1)\Gamma_q(\alpha j+\beta+2)} \\ &= \frac{(1-q)B_k(\alpha,\beta;q)}{\Gamma_q(\alpha j+\beta+2)\Gamma_q(\alpha(k-j)+\beta+2)}, \end{aligned} \tag{23}$$

where

$$\begin{aligned} B_k(\alpha,\beta;q) &= q^{\beta}(1-q)[(q^{\alpha(k-j)+1}-q^{\alpha j})(1-q^{\alpha j+\beta+1}) \\ &\quad+(q^{\alpha j+1}-q^{\alpha(k-j)})(1-q^{\alpha(k-j)+\beta+1})] \\ &\quad+(1-q^{\alpha j+\beta+1})(1-q^{\alpha(k-j)+\beta+1}) \\ &= 2-q^{\beta+1}(q^{\alpha j}+q^{\alpha(k-j)})+q^{\beta}(1-q)[q^{\beta+1}(q^{2\alpha j} \\ &\quad+q^{2\alpha(k-j)})-(1-q)(q^{\alpha j}+q^{\alpha(k-j)})-2q^{\alpha k+\beta+2}] \\ &= 2-q^{\beta}(q+(1-q)^2)(q^{\alpha j}+q^{\alpha(k-j)}) \\ &\quad+q^{\beta}(1-q)\left[q^{\beta+1}(q^{2\alpha j}+q^{2\alpha(k-j)})-2q^{\alpha k+\beta+2}\right]. \end{aligned}$$

Since $\alpha,\beta>0$ and $q\in(0,1/2)$, then by the above equation we get

$$\begin{aligned}B(\alpha,\beta;q) &\geqslant 2-(q^{\alpha j}+q^{\alpha(k-j)})+q^{\beta}(1-q)\left[q^{\beta+1}(q^{2\alpha j}+q^{2\alpha(k-j)})-2q^{\alpha k+\beta+2}\right]\\ &=((1-q^{\alpha j})+(1-q^{\alpha(k-j)}))+q^{\beta}(1-q)\left[q^{\beta+1}(q^{2\alpha j}+q^{2\alpha(k-j)})-2q^{\alpha k+\beta+2}\right]\\ &\geqslant q^{\beta}(1-q)\left[q^{\beta+1}(q^{2\alpha j}+q^{2\alpha(k-j)})-2q^{\alpha k+\beta+2}\right].\end{aligned}$$

For $k-j>j$ (i.e., for $[(k-1)/2]\geqslant j$) and $q\in(0,1)$, we have $q^{2\alpha j}\geqslant q^{\alpha k}$ and consequently we have

$$\begin{aligned}B_k(\alpha,\beta;q) &\geqslant q^{\beta}(1-q)\left[q^{\beta+1}(q^{\alpha k}+q^{2\alpha(k-j)})-2q^{\alpha k+\beta+2}\right]\\ &=q^{2\beta+\alpha k+1}(1-q)(1-2q+q^{\alpha(k-2j)})\\ &\geqslant 0,\end{aligned}\tag{24}$$

for all $q\in(0,1/2)$. According to (22), (23) and (24) we get the Turán type inequality (20). □

THEOREM 3. *Let $1\neq q>0$, $\alpha>0$ and $\beta>0$. Then the following Turán type inequality*

$$E_{\alpha,\beta+2}(q;z)E_{\alpha,\beta}(q;z)-\left(\frac{1-q^{\beta}}{1-q^{\beta+1}}\right)\left(E_{\alpha,\beta+1}(q;z)\right)^2\geqslant 0,\tag{25}$$

hold for all $z\in(0,\infty)$.

Proof. For convenience, let us write

$$\mathbb{E}_{\alpha,\beta}(q;z)=\sum_{k=0}^{\infty}\gamma_k(\alpha,\beta;q)z^k,\ \text{where}gamma_k(\alpha,\beta;q)=\Gamma_q(\beta)/\Gamma_q(\alpha k+\beta).$$

Thus,

$$\frac{\partial^2[\log\gamma_k(\alpha,\beta;q)]}{\partial\beta^2}=\psi'(\beta)-\psi'(\alpha k+\beta).$$

By using (4) it is easy to see that the q-digamma function $\psi_q(x)$ is concave on $(0,\infty)$ for each $q\in(0,1)$. On the other hand, from the relation (3) we get

$$\psi_q(x)=\frac{2x-3}{2}\log q+\psi_{1/q}(x),$$

for $q>0$, and consequently the the q-digamma function $\psi_q(x)$ is concave on $(0,\infty)$ for all $1\neq q>0$. So, using the fact that sums of log–convex functions are log–convex too, we deduce that the function $\beta\mapsto\mathbb{E}_{\alpha,\beta}(q;z)$ is log–convex $(0,\infty)$ for all $q>0$. It follows that for $\beta_1,\beta_2>0$, $t\in[0,1]$, we have

$$\mathbb{E}_{\alpha,t\beta_1+(1-t)\beta_2}(q;z)\leqslant\left[\mathbb{E}_{\alpha,\beta_1+}(q;z)\right]^t\left[\mathbb{E}_{\alpha,\beta_2}(q;z)\right]^{1-t}.$$

Choosing $\beta_1 = \beta$, $\beta_2 = \beta + 2$ and $t = 1/2$, the above inequality reduces to the Turán type inequality (25). $\square$

THEOREM 4. *Let $n \in \mathbb{N}$, we define the function $E^n_{\alpha,\beta}(q;z)$ by*

$$E^n_{\alpha,\beta}(q;z) = E^n_{\alpha,\beta}(q;z) - \sum_{k=0}^{n} \frac{z^k}{\Gamma_q(\alpha k + \beta)}, \tag{26}$$

for all $q \in (0,1)$ and $\alpha, \beta, z > 0$. Then, the following Turán type inequality

$$\left(E^{n+1}_{\alpha,\beta}(q;z)\right)^2 - E^n_{\alpha,\beta}(q;z)E^{n+2}_{\alpha,\beta}(q;z) \geqslant 0, \tag{27}$$

hold true for all $q \in (0,1)$ and $\alpha, \beta, z > 0$. Furthermore, the following Turán type inequality

$$\left(E_{\alpha,\beta+(n+2)\alpha}(q;z)\right)^2 - E_{\alpha,\beta+(n+1)\alpha}(q;z)E_{\alpha,\beta+(n+3)\alpha}(q;z) \geqslant 0, \tag{28}$$

holds true for all $q \in (0,1)$ and $\alpha, \beta, z > 0$.

Proof. By using the definition of the function $E^n_{\alpha,\beta}(q;z)$, we get

$$E^n_{\alpha,\beta}(q;z) = E^{n+1}_{\alpha,\beta}(q;z) + \frac{z^{n+1}}{\Gamma_q(\alpha(n+1)+\beta)}, \tag{29}$$

and

$$E^{n+2}_{\alpha,\beta}(q;z) = E^{n+1}_{\alpha,\beta}(q;z) - \frac{z^{n+2}}{\Gamma_q(\alpha(n+2)+\beta)}, \tag{30}$$

which yields that

$$\begin{aligned}
&\left(E^{n+1}_{\alpha,\beta}(q;z)\right)^2 - E^n_{\alpha,\beta}(q;z)E^{n+2}_{\alpha,\beta}(q;z)\\
&= E^{n+1}_{\alpha,\beta}(q;z)\left(\frac{z^{n+2}}{\Gamma_q(\alpha(n+2)+\beta)} - \frac{z^{n+1}}{\Gamma_q(\alpha(n+1)+\beta)}\right)\\
&\quad + \frac{z^{2n+3}}{\Gamma_q(\alpha(n+2)+\beta)\Gamma_q(\alpha(n+1)+\beta)}\\
&= \sum_{k=n+2}^{\infty} \frac{z^{k+n+2}}{\Gamma_q(\alpha(n+2)+\beta)\Gamma_q(\alpha k+\beta)} - \sum_{k=n+3}^{\infty} \frac{z^{k+n+1}}{\Gamma_q(\alpha(n+1)+\beta)\Gamma_q(\alpha k+\beta)}\\
&= \sum_{k=n+3}^{\infty}\left(\frac{z^{k+n+1}}{\Gamma_q(\alpha(n+2)+\beta)\Gamma_q(\alpha(k-1)+\beta)} - \frac{z^{k+n+1}}{\Gamma_q(\alpha(n+1)+\beta)\Gamma_q(\alpha k+\beta)}\right)\\
&= \sum_{k=n+3}^{\infty} \frac{C_k(\alpha,\beta;q)z^{k+n+1}}{\Gamma_q(\alpha(n+2)+\beta)\Gamma_q(\alpha(k-1)+\beta)\Gamma_q(\alpha(n+1)+\beta)\Gamma_q(\alpha k+\beta)},
\end{aligned}$$

where $(C_k(\alpha,\beta;q))_{k\geqslant n+3}$ is defined by

$$C_k(\alpha,\beta;q)=\Gamma_q(\alpha(n+1)+\beta)\Gamma_q(\alpha k+\beta)-\Gamma_q(\alpha(n+2)+\beta)\Gamma_q(\alpha(k-1)+\beta). \quad (31)$$

By using the fact that the q-gamma function $\Gamma_q(x)$ is log–convex on $(0,\infty)$, we deduce that the function $x\mapsto\frac{\Gamma_q(x+a)}{\Gamma_q(x)}$ is increasing on $(0,\infty)$, when $a>0$. Thus implies the following inequality

$$\frac{\Gamma_q(x+a)}{\Gamma_q(x)}\leqslant\frac{\Gamma_q(x+a+b)}{\Gamma_q(x+b)} \quad (32)$$

holds for all $a,b>0$. Now, let $x=\alpha(n+1)+\beta$, $a=\alpha>0$, $b=\alpha(k-n-2)>0$ in (32), we get $C_k(\alpha,\beta;q)\geqslant 0$, for each $k\geqslant n+3$, $q\in(0,1)$ and $\alpha,\beta>0$. The desired inequality (27) is thus established. Next, we prove the inequality (28). It is clear from The definitions of the functions $E_{\alpha,\beta}(q;z)$ and $E^n_{\alpha,\beta}(q;z)$ we obtain

$$E^n_{\alpha,\beta}(q;z)=z^{n+1}E_{\alpha,\beta+\alpha(n+1)}(q;z).$$

The above relation and the Turán type inequality (27) gives the inequality (28), which evidently completes the proof of Theorem 4. □

In the proof of the next Theorem, we require the following two lemmas:

LEMMA 1. *Let* (a_n) *and* (b_n) $(n=0,1,2\ldots)$ *be real numbers, such that* $b_n>0$, $n=0,1,2,\ldots$ *and* $\left(\frac{a_n}{b_n}\right)_{n\geqslant 0}$ *is increasing (decreasing), then* $\left(\frac{a_0+\ldots+a_n}{b_0+\ldots+b_n}\right)_n$ *is also increasing (decreasing).*

The second lemma is about the monotonicity of two power series, see [13] for more details.

LEMMA 2. *Let* $\{a_n\}_{n\geqslant 0}$ *and* $\{b_n\}_{n\geqslant 0}$ *be two sequences of real numbers, and let the power series* $f(x)=\sum_{n\geqslant 0}a_nx^n$ *and* $g(x)=\sum_{n\geqslant 0}b_nx^n$ *be convergent for* $|x|<r$. *If* $b_n>0$ *for* $n\geqslant 0$ *and if the sequence* $\{a_n/b_n\}_{n\geqslant 0}$ *is (strictly) increasing (decreasing), then the function* $x\mapsto f(x)/g(x)$ *is (strictly) increasing (decreasing) on* $(0,r)$.

The idea of the proof of this interesting result is taken from [9].

THEOREM 5. *Let* n *be a positive integer, let* $\alpha,\beta>0$ *and let either* $q\in(0,1)$. *We define the function* $K^n_{\alpha,\beta}(q;z)$ *by*

$$K^n_{\alpha,\beta}(q;z)=\frac{E^n_{\alpha,\beta}(q;z)E^{n+2}_{\alpha,\beta}(q;z)}{\left(E^{n+1}_{\alpha,\beta}(q;z)\right)^2},\quad z>0. \quad (33)$$

Then, the function $z\mapsto K^n_{\alpha,\beta}(q;z)$ *is increasing on* $(0,\infty)$. *Moreover, the following Turán type inequality*

$$E^n_{\alpha,\beta}(q;z)E^{n+2}_{\alpha,\beta}(q;z)-\frac{\Gamma_q^2(\alpha(n+2)+\beta)}{\Gamma_q(\alpha(n+1)+\beta)\Gamma_q(\alpha(n+3)+\beta)}\cdot\left(E^{n+1}_{\alpha,\beta}(q;z)\right)^2\geqslant 0, \quad (34)$$

is valid for $z \in (0,\infty)$. *The constant* $\frac{\Gamma_q^2(\alpha(n+2)+\beta)}{\Gamma_q(\alpha(n+1)+\beta)\Gamma_q(\alpha(n+3)+\beta)}$ *in inequality (34) is sharp.*

Proof. The Cauchy product rule gives

$$K^n_{\alpha,\beta}(q;z) = \sum_{k=0}^{\infty}\sum_{j=0}^{k} u_j(\alpha,\beta;q)z^k \Big/ \sum_{k=0}^{\infty}\sum_{j=0}^{k} v_j(\alpha,\beta;q)z^k,$$

where

$$u_j = \frac{1}{\Gamma_q(\alpha(j+n+1)+\beta)\Gamma_q(\alpha(k-j+n+3)+\beta)}$$

and

$$v_j = \frac{1}{\Gamma_q(\alpha(j+n+2)+\beta)\Gamma_q(\alpha(k-j+n+2)+\beta)}.$$

Next, we define the sequence $(w_j = u_j/v_j)_{j\geqslant 0}$. Then

$$\frac{w_{j+1}}{w_j} = \left(\frac{\Gamma_q(\alpha(j+n+1)+\beta)\Gamma_q(\alpha(j+n+3)+\beta)}{\Gamma_q^2(\alpha(j+n+2)+\beta)}\right) \times \left(\frac{\Gamma_q(\alpha(k-j+n+3)+\beta)\Gamma_q(\alpha(k-j+n+1)+\beta)}{\Gamma_q^2(\alpha(k-j+n+2)+\beta)}\right). \tag{35}$$

Firstly, let $x = \alpha(j+n+1)+\beta$, $a = b = \alpha$ in (32), we obtain the following inequality

$$\Gamma_q(\alpha(j+n+1)+\beta)\Gamma_q(\alpha(j+n+3)+\beta) - \Gamma_q^2(\alpha(j+n+2)+\beta) \geqslant 0. \tag{36}$$

Secondly, let $x = \alpha(k-j+n+1)+\beta$, $a = b = \alpha$ in (32), we have

$$\Gamma_q(\alpha(k-j+n+1)+\beta)\Gamma_q(\alpha(k-j+n+3)+\beta) - \Gamma_q^2(\alpha(k-j+n+2)+\beta) \geqslant 0. \tag{37}$$

In view of inequalities (35) and (36) and (37), we deduce that the sequence $(w_j)_{j\geqslant 0}$ is increasing. Consequently $\sum_{j=0}^k u_j/\sum_{j=0}^k v_j$ is increasing, by means of Lemma 1. Thus the function $z \mapsto K^n_{\alpha,\beta}(q;z)$ is increasing on $(0,\infty)$, in view of Lemma 2. Moreover,

$$\lim_{z\longrightarrow 0} K^n_{\alpha,\beta}(q;z) = \frac{\Gamma_q^2(\alpha(n+2)+\beta)}{\Gamma_q(\alpha(n+1)+\beta)\Gamma_q(\alpha(n+3)+\beta)}.$$

This completes the proof. □

3. Turán type inequalities for q-Wright functions

Let k be a positive integer, let β be a complex number, and let either $\alpha > -\log(1-q)/\log(1-q^k)$ and $z \neq 0$ or $\alpha = -\log(1-q)/\log(1-q^k)$ and $|z| < 1$, the q-Wright function is defined by [5]

$$W_{\alpha,\beta}(q^k;z) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} \frac{z^n}{[n]_q!\Gamma_{q^k}(\alpha n+\beta)}, \tag{38}$$

where $[n]_q! = (q;q)_n/(1-q)^n$. For q tends to 1 the q-Wright function tends to the classical Wright function

$$W_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(\alpha n+\beta)}.$$

THEOREM 6. *Let k be a positive integer, let $\alpha,\beta > 0$ and let either $q \in (0, q_\beta^k)$. Then the following Turán type inequality*

$$\left(W_{\alpha,\beta+1}(q^k;z)\right)^2 - W_{\alpha,\beta}(q^k;z)W_{\alpha,\beta+2}(q^k;z) \geqslant 0, \tag{39}$$

hold true for all $z \in (0,\infty)$.

Proof. By again using the Cauchy product we find that

$$W_{\alpha,\beta}(q^k;z)W_{\alpha,\beta+2}(q^k;z) = \sum_{n=0}^{\infty}\sum_{j=0}^{n} \frac{q^{(j(j+1)+(k-j)(k-j+1))/2}z^n}{[j]_q![n-j]_q!\Gamma_{q^k}(\alpha j+\beta)\Gamma_{q^k}(\alpha(k-j)+\beta+2)}$$

and

$$\left(W_{\alpha,\beta+1}(q^k;z)\right)^2 = \sum_{n=0}^{\infty}\sum_{j=0}^{n} \frac{q^{(j(j+1)+(k-j)(k-j+1))/2}z^n}{[j]_q![n-j]_q!\Gamma_{q^k}(\alpha j+\beta+1)\Gamma_{q^k}(\alpha(k-j)+\beta+1)}.$$

Thus,

$$\begin{aligned} &W_{\alpha,\beta}(q^k;z)W_{\alpha,\beta+2}(q^k;z) - \left(W_{\alpha,\beta+1}(q^k;z)\right)^2 \\ &= \sum_{n=0}^{\infty}\sum_{j=0}^{n} \frac{q^{(j(j+1)+n-j)(n-j+1))/2}}{[j]_q![n-j]_q!} T_{j,n}^{(1)}(\alpha,\beta,q^k)z^n \\ &= \sum_{n=0}^{\infty}\sum_{j=0}^{[(n-1)/2]} \frac{q^{\frac{j(j+1)+(n-j)(n-j+1)}{2}}}{[j]_q![n-j]_q!}\left(T_{j,n}^{(1)}(\alpha,\beta,q^k) + T_{n-j,n}^{(1)}(\alpha,\beta,q^k)\right)z^n \\ &\quad + \frac{q^{\frac{n(n+2\alpha k+2)}{4}}(q^k-1)z^n}{([n/2]_q!)^2\Gamma_{q^k}(\alpha n/2+\beta+1)\Gamma_{q^k}(\alpha n/2+\beta+2)}, \end{aligned} \tag{40}$$

where $T_{j,n}^{(1)}(\alpha,\beta,q)$ as defined in (13). In view of (17) and (18), we get

$$\begin{aligned} T_{j,n}^{(1)}(\alpha,\beta,q^k) + T_{n-j,n}^{(1)}(\alpha,\beta,q^k) &= \frac{(1-q^k)A_{j,n}(\alpha,\beta;q^k)}{\Gamma_{q^k}(\alpha j+\beta+2)\Gamma_{q^k}(\alpha(n-j)+\beta+2)} \\ &\leqslant f_\beta(q^k)(q^{\alpha kj} + q^{\alpha k(n-j)}) - 2q^{k(\alpha n+\beta+2)} \\ &\leqslant 0, \end{aligned} \tag{41}$$

for all $\alpha,\beta > 0$ and $q \in (0, q_\beta^k)$, which leads us readily to the required result. □

THEOREM 7. *Let k be a positive integer, let $\alpha,\beta>0$ and let either $q\in(0,1/2)$. Then the following Turán type inequality*

$$W_{\alpha,\beta}(q^k;z)W_{\alpha,\beta+2}(q^k;z)-\Big(W_{\alpha,\beta+1}(q^k;z)\Big)^2+W_{\alpha,\beta+1}(q^k;z)W_{\alpha,\beta+2}(q^k;z)\geqslant 0, \quad (42)$$

hold true for all $z\in(0,\infty)$.

Proof. From (40) and (24) we get

$$\begin{aligned}
&\Delta^1_{\alpha,\beta)}(q;z)\\
&=W_{\alpha,\beta}(q^k;z)W_{\alpha,\beta+2}(q^k;z)-\Big(W_{\alpha,\beta+1}(q^k;z)\Big)^2+W_{\alpha,\beta+1}(q^k;z)W_{\alpha,\beta+2}(q^k,z)\\
&=\sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{q^{(j(j+1)+(n-j)(n-j+1))/2}}{[j]_q![n-j]_q!}T^{(2)}_{j,n}(\alpha,\beta;q^k)z^n\\
&=\sum_{n=0}^{\infty}\sum_{j=0}^{[(n-1)/2]}\frac{q^{(j(j+1)+(n-j)(n-j+1))/2}}{[j]_q![n-j]_q!}\Big(T^{(2)}_{j,n}(\alpha,\beta;q^k)+(T^{(2)}_{n-j,n}(\alpha,\beta;q^k)\Big)z^n\\
&\quad+\frac{1-q^{k(\beta+\frac{\alpha n}{2})z^n}(1-q^k)^2}{([n/2]_q!)^2\Gamma_{q^k}(\alpha n/2+\beta+1)\Gamma^k_q(\alpha n/2+\beta+2)}\\
&=(1-q^k)\sum_{n=0}^{\infty}\sum_{j=0}^{[(n-1)/2]}\frac{q^{(j(j+1)+(n-j)(n-j+1))/2}B_k(\alpha,\beta;q^k)}{[j]_{q^k}![n-j]_{q^k}!\Gamma_{q^k}(\alpha j+\beta+2)\Gamma_{q^k}(\alpha(n-j)+\beta+2)}\\
&\quad+\frac{1-q^{k(\beta+\frac{\alpha n}{2})z^n}(1-q^k)^2}{([n/2]_q!)^2\Gamma_{q^k}(\alpha n/2+\beta+1)\Gamma_{q^k}(\alpha n/2+\beta+2)}\\
&\geqslant q^{2k\beta+k}(1-q^k)^2\sum_{n=0}^{\infty}\sum_{j=0}^{[(n-1)/2]}\frac{q^{(j(j+1)+(n-j)(n-j+1))/2}q^{\alpha kn}(1-2q^k+q^{k\alpha(n-2j)})}{[j]_q![n-j]_q!\Gamma_{q^k}(\alpha j+\beta+2)\Gamma_{q^k}(\alpha(n-j)+\beta+2)}\\
&\quad+\frac{1-q^{k(\beta+\frac{\alpha n}{2})z^n}(1-q^k)^2}{([n/2]_q!)^2\Gamma_{q^k}(\alpha n/2+\beta+1)\Gamma_{q^k}(\alpha n/2+\beta+2)}\\
&\geqslant 0,
\end{aligned}$$

for all $\alpha,\beta,z>0$, $q\in(0,1/2)$ and $k\in\mathbb{N}$. The proof of Theorem 7 is thus completed. □

THEOREM 8. *Let $\alpha,\beta,q>0$ and k be a positive integer. Then the function $\beta\mapsto\mathbb{W}_{\alpha,\beta}(q^k;z)=\Gamma_{q^k}(\beta)W_{\alpha,\beta}(z;q^k)$ is log-convex on $(0,\infty)$. In particular $\mathscr{W}_{\alpha,\beta}(z;q^k)$ satisfies the following Turán type inequality, that is for $\alpha,\beta,q>0$ and k be a positive integer we get*

$$W_{\alpha,\beta}(q^k;z)W_{\alpha,\beta+2}(q^k;z)-\frac{1-q^{k\beta}}{1-q^{k\beta+k}}\Big(W_{\alpha,\beta+1}(q^k;z)\Big)^2\geqslant 0,\ \ z>0. \quad (43)$$

Proof. Let us write $\mathbb{W}_{\alpha,\beta}(q^k;z)$ in the following form

$$\mathbb{W}_{\alpha,\beta}(q^k;z)=\sum_{n=0}^{\infty}\delta_n(\alpha,\beta;q)z^n,\ \text{ where }\ \delta_n(\alpha,\beta;q)=\frac{q^{n(n+1)+/2}\Gamma_{q^k}(\beta)}{[n]_q!\Gamma_{q^k}(\alpha n+\beta)}.$$

Thus,

$$\frac{\partial^2\log\Big(\delta_n(\alpha,\beta;q)\Big)}{\partial^2\beta}=\psi'_{q^k}(\beta)-\psi'_{q^k}(\alpha n+\beta),$$

and the last expression is nonnegative by using the fact that the q-digamma function ψ'_{q^k} is decreasing on $(0,\infty)$ for each $q>0$ and k be a positive integer, and consequently the function $\beta\mapsto\mathscr{W}_{\alpha,\beta}(q^k;z)$ is log–convex on $(0,\infty)$ for each $q>0$ and $k\in\mathbb{N}$. Thus implies that for $\beta_1,\beta_2>0$, $t\in[0,1]$, we have

$$\mathbb{W}_{\alpha,t\beta_1+(1-t)\beta_2}(q^k;z)\leqslant\Big[\mathbb{W}_{\alpha,\beta_1+}(z;q^k)\Big]^t\Big[\mathbb{W}_{\alpha,\beta_2}(z;q^k)\Big]^{1-t}.$$

Choosing $\beta_1=\beta$, $\beta_2=\beta+2$ and $t=1/2$, the above inequality reduces to the Turán type inequality (43). □

THEOREM 9. *Let $n\in\mathbb{N}$, we define the function $W^n_{\alpha,\beta}(q;z)$ by*

$$W^n_{\alpha,\beta}(q^k;z)=W^n_{\alpha,\beta}(q^k;z)-\sum_{j=0}^{n}\frac{q^{j(j+1)/2}z^k}{[j]_{q^k}!\Gamma_{q^k}(\alpha j+\beta)},\tag{44}$$

for all $q\in(0,1)$, $k\in\mathbb{N}$ and $\alpha,\beta,z>0$ Then, the following Turán type inequality

$$\Big(W^{n+1}_{\alpha,\beta}(q^k;z)\Big)^2-W^n_{\alpha,\beta}(q^k;z)W^{n+2}_{\alpha,\beta}(q^k;z)\geqslant 0,\tag{45}$$

hold true for all $q\in(0,1)$, $k\in\mathbb{N}$ and $\alpha,\beta,z>0$.

Proof. By using the definition of the function $E^n_{\alpha,\beta}(q;z)$, we get

$$W^n_{\alpha,\beta}(q^k;z)=W^{n+1}_{\alpha,\beta}(q^k;z)+\frac{q^{(n+1)(n+2)/2}z^{n+1}}{[n+1]_q!\Gamma_q(\alpha(n+1)+\beta)},\tag{46}$$

and

$$W^{n+2}_{\alpha,\beta}(q^k;z)=W^{n+1}_{\alpha,\beta}(q^k;z)-\frac{q^{(n+2)(n+3)/2}z^{n+2}}{[n+2]_q!\Gamma_q(\alpha(n+2)+\beta)}.\tag{47}$$

Therefore,

$$
\begin{aligned}
&\left(W^{n+1}_{\alpha,\beta}(q^k;z)\right)^2 - W^{n}_{\alpha,\beta}(q^k;z)W^{n+2}_{\alpha,\beta}(q^k;z)\\
&= W^{n+1}_{\alpha,\beta}(q^k;z)\left(\frac{q^{(n+2)(n+3)/2}z^{n+2}}{[n+2]_q!\Gamma_q(\alpha(n+2)+\beta)} - \frac{q^{(n+1)(n+2)/2}z^{n+1}}{[n+1]_q!\Gamma_q(\alpha(n+1)+\beta)}\right)\\
&\quad + \frac{q^{(n+2)^2}z^{n+1}}{[n+1]_q![n+2]_q!\Gamma_q(\alpha(n+1)+\beta)\Gamma_q(\alpha(n+2)+\beta)}\\
&= \frac{q^{(n+2)(n+3)/2}z^{n+2}}{[n+2]_q!\Gamma_q(\alpha(n+2)+\beta)}\sum_{j=n+3}^{\infty}\frac{q^{j(j-1)/2}z^{k-1}}{[j-1]_q!\Gamma_{q^k}(\alpha j+\beta-\alpha)}\\
&\quad - \frac{q^{(n+1)(n+2)/2}z^{n+1}}{[n+1]_q!\Gamma_q(\alpha(n+1)+\beta)}\sum_{j=n+3}^{\infty}\frac{q^{j(j+1)/2}z^{k}}{[j]_q!\Gamma_{q^k}(\alpha j+\beta)}\\
&= \frac{1}{[n+1]_q!}\sum_{j=n+3}^{\infty}\frac{M_j(\alpha,\beta;q^k)z^{k+n+1}}{[j-1]_q!},
\end{aligned}
\tag{48}
$$

where $(M_j(\alpha,\beta;q^k))_{k\geqslant n+3}$ is defined by

$$
\begin{aligned}
&M_j(\alpha,\beta;q^k)\\
&= \frac{q^{((n+2)(n+3)+j(j-1))/2}}{[n+2]_q\Gamma_{q^k}(\alpha n+\beta+2\alpha)\Gamma_{q^k}(\alpha j+\beta-\alpha)} - \frac{q^{((n+1)(n+2)+j(j+1))/2}}{[j]_q\Gamma_{q^k}(\alpha n+\beta+\alpha)\Gamma_{q^k}(\alpha j+\beta)}\\
&\geqslant \frac{1}{[j]_q}\left(\frac{q^{((n+2)(n+3)+j(j-1))/2}}{\Gamma_{q^k}(\alpha n+\beta+2\alpha)\Gamma_{q^k}(\alpha j+\beta-\alpha)} - \frac{q^{((n+1)(n+2)+j(j+1))/2}}{\Gamma_{q^k}(\alpha n+\beta+\alpha)\Gamma_{q^k}(\alpha j+\beta)}\right)\\
&\geqslant \frac{q^{((n+1)(n+2)+j(j+1))/2}}{[j]_q}\\
&\quad\times\left(\frac{1}{\Gamma_{q^k}(\alpha n+\beta+2\alpha)\Gamma_{q^k}(\alpha j+\beta-\alpha)} - \frac{1}{\Gamma_{q^k}(\alpha n+\beta+\alpha)\Gamma_{q^k}(\alpha j+\beta)}\right)\\
&= \frac{q^{((n+1)(n+2)+j(j+1))/2}C_j(\alpha,\beta;q^k)}{[j]_q\Gamma_{q^k}(\alpha n+\beta+2\alpha)\Gamma_{q^k}(\alpha j+\beta-\alpha)\Gamma_{q^k}(\alpha n+\beta+\alpha)\Gamma_{q^k}(\alpha j+\beta)},
\end{aligned}
$$

where $C_j(\alpha,\beta;q^k)$ as defined in (31). Following the proof of Theorem 4 we see that the coefficient $C_j(\alpha,\beta;q^k)$ is nonnegative for all $j\geqslant n+3$. Hence $M_j(\alpha,\beta;q^k)\geqslant 0$, for all $j\geqslant n+3$ and $q\in(0,1)$, $\alpha,\beta>0$ and k be a positive integer. From thus fact and (48) we deduce the desired result. This ends the proof of Theorem 9. □

The next theorem is the q-version of [8, Theorem 3.4].

THEOREM 10. *Let n,k be a positive integer, let $\alpha,\beta>0$ and let either $q\in(0,1)$.*

We define the function $S^n_{\alpha,\beta}(q;z)$ *by*

$$S^n_{\alpha,\beta}(q^k;z)=\frac{W^n_{\alpha,\beta}(q^k;z)W^{n+2}_{\alpha,\beta}(q^k;z)}{\left(W^{n+1}_{\alpha,\beta}(q^k;z)\right)^2},\ \ z>0. \tag{49}$$

Then, the function $z\mapsto S^n_{\alpha,\beta}(q^k;z)$ *is increasing on* $(0,\infty)$. *Moreover, the following Turán type inequality*

$$W^n_{\alpha,\beta}(q;z)W^{n+2}_{\alpha,\beta}(q;z)-\left(\frac{q(1-q^{n+2})\Gamma_q^2(\alpha(n+2)+\beta)}{(1-q^{n+3})\Gamma_q(\alpha(n+1)+\beta)\Gamma_q(\alpha(n+3)+\beta)}\right)\times\left(W^{n+1}_{\alpha,\beta}(q;z)\right)^2\geqslant 0, \tag{50}$$

is valid for $z\in(0,\infty)$. *The constant* $\frac{q(1-q^{n+2})\Gamma_q^2(\alpha(n+2)+\beta)}{(1-q^{n+3})\Gamma_q(\alpha(n+1)+\beta)\Gamma_q(\alpha(n+3)+\beta)}$ *in inequality (50) is sharp.*

Proof. From the Cauchy series product we get

$$S^n_{\alpha,\beta}(q^k;z)=\sum_{m=0}^{\infty}\sum_{j=0}^{m}u^1_j(\alpha,\beta;q^k)z^n\Big/\sum_{j=0}^{m}v^1_j(\alpha,\beta;q^k)z^n,$$

where

$$u^1_j(\alpha,\beta;q^k)=\frac{q^{((j+n+1)(j+n+2)+(k-j+n+3)(k-j+n+4))/2}}{[j+n+1]_q![k-j+n+3]_q!\Gamma_{q^k}(\alpha(j+n+1)+\beta)\Gamma_{q^k}((\alpha(k-j+n+3)+\beta)}$$

and

$$v^1_j(\alpha,\beta;q^k)=\frac{q^{((j+n+2)(j+n+3)+(k-j+n+2)(k-j+n+3))/2}}{[j+n+2]_q![k-j+n+2]_q!\Gamma_{q^k}(\alpha(j+n+2)+\beta)\Gamma_{q^k}((\alpha(k-j+n+2)+\beta)}.$$

Now, we consider the sequence $(w^1_j=u^1_j/v^1_j)_{j\geqslant 0}$. By using the fact that the sequence $(w_j)_{j\geqslant 0}$ is increasing we have

$$\begin{aligned}\frac{w^1_{j+1}}{w^1_j}&=\frac{(1-q^{j+n+3})(1-q^{k-j+n+3})}{q^2(1-q^{j+n+2})(1-q^{k-j+n+2})}\cdot\left(\frac{w_{j+1}(\alpha,\beta,q^k)}{w_j(\alpha,\beta,q^k)}\right)\\&\geqslant\frac{(1-q^{j+n+3})(1-q^{k-j+n+3})}{(1-q^{j+n+2})(1-q^{k-j+n+2})}\cdot\left(\frac{w_{j+1}(\alpha,\beta,q^k)}{w_j(\alpha,\beta,q^k)}\right)\\&\geqslant\frac{w_{j+1}(\alpha,\beta,q^k)}{w_j(\alpha,\beta,q^k)}\\&\geqslant 1,\end{aligned} \tag{51}$$

for all $q\in(0,1),\alpha,\beta,z>0$ and k,n be a positive integer. So, the sequence $(w_j^1)_{j\geqslant 0}$ is increasing and consequently the sequence $\left(\sum_{j=0}^m u_j^1/\sum_{j=0}^m v_j^1\right)_{m\geqslant 0}$ is also increasing by means of Lemma 1. In view of Lemma 2, we deduce that the function $z\mapsto S_{\alpha,\beta}^n(q^k;z)$ is increasing on $(0,\infty)$. On the other hand we have

$$\lim_{z\longrightarrow 0} S_{\alpha,\beta}^n(q^k;z)=\frac{q(1-q^{n+2})\Gamma_q^2(\alpha(n+2)+\beta)}{(1-q^{n+3})\Gamma_q(\alpha(n+1)+\beta)\Gamma_q(\alpha(n+3)+\beta)},$$

which proves Theorem 10. □

4. Concluding Remarks

In this section we would like to comment the main results of this paper.

1. Open Problems: Motivated by the results of Section 2 and Section 3 we pose the following problems: find a generalization of the Turán type inequalities (8) and (39) for $q\in(0,1)$. In particular, proved the following Turán type inequalities for the Mittag–Leffler and Wright functions:

$$\Big(E_{\alpha,\beta+1}(z)\Big)^2-E_{\alpha,\beta}(z)E_{\alpha,\beta+2}(z)\geqslant 0 \tag{52}$$

and

$$\Big(W_{\alpha,\beta+1}(z)\Big)^2-W_{\alpha,\beta}(z)W_{\alpha,\beta+2}(z)\geqslant 0, \tag{53}$$

for all $\alpha,\beta,z>0$.

2. We note is another proof of the Turán type inequalities (25) and (43). Since the function $\beta\mapsto\Gamma_q(\beta)E_{\alpha,\beta}(q;z)$ is log–convex on $(0,\infty)$ for $z>0$, it follows that the function $\beta\mapsto(\Gamma_q(\beta+1)E_{\alpha,\beta+1}(q;z))/(\Gamma_q(\beta)E_{\alpha,\beta}(q;z))$ is increasing on $(0,\infty)$. Thus

$$\frac{\Gamma_q(\beta+2)E_{\alpha,\beta+2}(q;z)}{\Gamma_q(\beta+1)E_{\alpha,\beta+1}(q;z)}\geqslant\frac{\Gamma_q(\beta+1)E_{\alpha,\beta+1}(q;z)}{\Gamma_q(\beta)E_{\alpha,\beta}(q;z)},$$

which leads us readily to the required result. A similar argument for the Turán inequality (43).

3. Observe that if q tends to 1 in Theorem 3, then we get the following result: if $\alpha,\beta>0$. Thus, the following Turán type inequality

$$E_{\alpha,\beta}(z)E_{\alpha,\beta+2}(z)-\frac{\beta}{\beta+1}\Big(E_{\alpha,\beta+1}(z)\Big)^2\geqslant 0,$$

is valid for $z>0$. We note that this inequality was proved by Mehrez and Sitnik [9, Theorem 1, eq. 3].

4. Observe that when q tends to 1 the Turán type inequality (27) reduce to [9, Theorem 2, eq. 5]

$$\Big(E_{\alpha,\beta+1}^{n+1}(z)\Big)^2-E_{\alpha,\beta}^n(z)E_{\alpha,\beta+2}^{n+2}(z)\geqslant 0,$$

where n be a positive integer and $\alpha, \beta, z > 0$.

5. It is important to mention here that the Turán type inequalities (43) and (45) are in fact the q-version of the inequalities (3.1) and (3.6) in [8].

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Khaled Mehrez
Département de Mathématiques
Faculté des sciences de Tunis
Université Tunis-El Manar
Tunisia
and
Département de Mathématiques
Université de Kairouan
e-mail: k.mehrez@yahoo.fr;
mehrezkhaled23@yahoo.com