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THE BOUNDEDNESS AND ESSENTIAL NORM OF THE DIFFERENCE OF COMPOSITION OPERATORS FROM WEIGHTED BERGMAN SPACES INTO THE BLOCH SPACE

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Abstract. A new characterization for the boundedness of the difference of composition operators $C_\varphi - C_\psi$ from weighted Bergman spaces into the Bloch space in terms of the Bloch norm of the quantities $\varphi^n - \psi^n$, $n \in \mathbb{N}$, is given, as well as an asymptotic estimate for the essential norm of the operator.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the space of holomorphic functions in $\mathbb{D}$. For $a, z \in \mathbb{D}$, let $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$ be the Möbius transformation of $\mathbb{D}$ which interchanges 0 and a. For $z,\ w \in \mathbb{D}$, the pseudo-hyperbolic distance between z and w is given by

$$\rho(z,w) = |\sigma_w(z)| = \left|\frac{z-w}{1-\bar{w}z}\right|.$$

It is well known that $\rho(z,w) \leqslant 1$. We denote $\rho(\varphi(z), \psi(z))$ by $\rho(z)$.

For $0 < p < \infty$ and $\alpha > -1$, the weighted Bergman space $A^p_\alpha = A^p_\alpha(\mathbb{D})$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|^p_{A^p_\alpha} = (\alpha+1)\int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha dA(z) < \infty,$$

where dA denote the normalized Lebesgue area measure on $\mathbb{D}$ such that $A(\mathbb{D}) = 1$.

Recall that the classical Bloch space $\mathscr{B} = \mathscr{B}(\mathbb{D})$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_\beta = \sup_{z\in\mathbb{D}} (1-|z|^2)|f'(z)| < \infty.$$

Then the norm $\|f\|_{\mathscr{B}} = |f(0)| + \|f\|_\beta$ makes $\mathscr{B}$ a Banach space.

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Let φ be an analytic self-map of the unit disk $\mathbb{D}$. Associated with φ is the composition operator C_φ defined by

$$C_\varphi(f)(z) = f(\varphi(z)), \quad z \in \mathbb{D},$$

for $f \in H(\mathbb{D})$. We denote by $S(\mathbb{D})$ the set of all analytic self-maps of $\mathbb{D}$. The main subject in the study of composition operators is to describe operator theoretic properties of C_φ in terms of function theoretic properties of φ. We refer to books [3, 34] for the basic theory of composition operators on a wide variety of topics.

Recall that $L : X \to Y$ is compact if it maps bounded sets into relatively compact sets and X,Y are Banach spaces. The essential norm of a continuous linear operator L is its distance to compact operators, that is,

$$\|L\|_{e,X\to Y} = \inf\{\|L-K\|_{X\to Y} : K \text{ is compact}\},$$

where $\|\cdot\|_{X\to Y}$ is the operator norm. Clearly, L is compact if and only if $\|L\|_{e,X\to Y} = 0$.

There has been a considerable interest in estimating of essential norms of composition, weighted composition and other concrete operators involving the composition ones on spaces of analytic functions (see, e.g., [3, 8, 11, 13, 16, 18, 21, 22, 24, 25, 27, 29, 30, 33].

It is a simple consequence of the Schwarz-Pick inequality that any composition operator C_φ is bounded on $\mathscr{B}$. The compactness and essential norm of C_φ on $\mathscr{B}$ was studied in [12, 13, 31]. Wulan, Zheng and Zhu proved that $C_\varphi : \mathscr{B} \to \mathscr{B}$ is compact if and only if $\lim_{n\to\infty} \|\varphi^n\|_{\mathscr{B}} = 0$ in [32]. Soon after that, Zhao [33] obtained an exact value of the essential norm for $C_\varphi : \mathscr{B} \to \mathscr{B}$, i.e.

$$\|C_\varphi\|_{e,\mathscr{B}\to\mathscr{B}} = \frac{e}{2}\limsup_{n\to\infty} \|\varphi^n\|_{\mathscr{B}}.$$

In [8], it was proved that $C_\varphi : A^p_\alpha \to \mathscr{B}$, $1 \leqslant p < \infty$, $\alpha > -1$, $\varphi \in S(\mathbb{D})$, is bounded if and only if $\sup_{n\geqslant 1} n^{(\alpha+2)/p}\|\varphi^n\|_{\mathscr{B}} < \infty$ (for earlier characterizations, their extensions and related operators, see, e.g., [9, 10, 19, 20, 23, 26]). Moreover, under the assumption that $C_\varphi : A^p_\alpha \to \mathscr{B}$ is bounded, they showed that

$$\|C_\varphi\|_{e,A^p_\alpha\to\mathscr{B}} \approx \limsup_{n\to\infty} n^{(\alpha+2)/p}\|\varphi^n\|_{\mathscr{B}}.$$

Many authors have investigated in the last few decades the difference of composition operators on various analytic function spaces in order to study their topological structure. The study of the difference of two composition operators was started in [1, 15]. We refer to [2, 4, 5, 6, 7, 14, 16, 17, 24, 28] and related references therein for more information of the difference of composition operators between different spaces of analytic functions.

Let $\varphi, \psi \in S(\mathbb{D})$ and $1 \leqslant p < \infty$, $\alpha > -1$. We define

$$\mathscr{D}_\varphi(z) := \frac{(1-|z|^2)\varphi'(z)}{(1-|\varphi(z)|^2)^{\frac{2+\alpha+p}{p}}}, \qquad \mathscr{D}_\psi(z) := \frac{(1-|z|^2)\psi'(z)}{(1-|\psi(z)|^2)^{\frac{2+\alpha+p}{p}}}.$$

In [35], Zhu and Yang studied the boundedness and compactness of the difference of two composition operators from A^p_α to $\mathscr{B}$. For example, they showed that $C_\varphi - C_\psi : A^p_\alpha \to \mathscr{B}$ is bounded if and only if $\sup_{z\in\mathbb{D}} |\mathscr{D}_\varphi(z) - \mathscr{D}_\psi(z)| < \infty$ and

$$\sup_{z\in\mathbb{D}} |\mathscr{D}_\varphi(z)|\rho(z) < \infty \quad (\text{or} \ \sup_{z\in\mathbb{D}} |\mathscr{D}_\psi(z)|\rho(z) < \infty).$$

Motivated by the results in [8] and [35], in the present paper, we study the boundedness, compactness and essential norm of the operator $C_\varphi - C_\psi : A^p_\alpha \to \mathscr{B}$ by using the sequence $n^{(\alpha+2)/p}\|\varphi^n - \psi^n\|_{\mathscr{B}}$, $n \in \mathbb{N}$.

For two quantities P and Q which may depend on φ and ψ, we use the abbreviation $P \lesssim Q$ whenever there is a positive constant c (independent of φ and ψ) such that $P \leqslant cQ$. We write $P \approx Q$, if $P \lesssim Q \lesssim P$.

2. Boundedness of $C_\varphi - C_\psi : A^p_\alpha \to \mathscr{B}$

In this section we characterize the boundedness of the operator $C_\varphi - C_\psi : A^p_\alpha \to \mathscr{B}$. In order to prove the main result in this section, we need the following lemmas.

LEMMA 2.1. [9] *Let $1 \leqslant p < \infty$, $\alpha > -1$. If $f \in A^p_\alpha$, then*

$$|f(z)| \leqslant \frac{C\|f\|_{A^p_\alpha}}{(1-|z|^2)^{\frac{2+\alpha}{p}}} \quad \textit{and} \quad |f'(z)| \leqslant \frac{C\|f\|_{A^p_\alpha}}{(1-|z|^2)^{\frac{2+\alpha+p}{p}}}.$$

LEMMA 2.2. [35] *Let $1 \leqslant p < \infty$, $\alpha > -1$. Then for all $z, w \in \mathbb{D}$,*

$$\sup_{\|f\|_{A^p_\alpha} \leqslant 1} |(1-|z|^2)^{\frac{2+\alpha+p}{p}} f'(z) - (1-|w^2|)^{\frac{2+\alpha+p}{p}} f'(w)| \lesssim \rho(z,w).$$

For any $a \in \mathbb{D}$, we define the following two families of test functions:

$$f_a(z) = \int_0^z \frac{(1-|a|^2)^{\frac{2+\alpha+p}{p}}}{(1-\overline{a}u)^{2\frac{2+\alpha+p}{p}}} du,$$

$$g_a(z) = \int_0^z \frac{(1-|a|^2)^{\frac{2+\alpha+p}{p}}}{(1-\overline{a}u)^{2\frac{2+\alpha+p}{p}}} \cdot \sigma_a(u) du, \ \ z \in \mathbb{D}.$$

LEMMA 2.3. *Let $\varphi, \psi \in S(\mathbb{D})$, $1 \leqslant p < \infty$, $\alpha > -1$. Then*
(i)

$$\sup_{a\in\mathbb{D}} \|(C_\varphi - C_\psi) f_a\|_{\mathscr{B}} \lesssim \sup_{n\in\mathbb{N}} n^{\frac{2+\alpha}{p}} \|\varphi^n - \psi^n\|_{\mathscr{B}}.$$

(ii)

$$\sup_{a\in\mathbb{D}} \|(C_\varphi - C_\psi) g_a\|_{\mathscr{B}} \lesssim \sup_{n\in\mathbb{N}} n^{\frac{2+\alpha}{p}} \|\varphi^n - \psi^n\|_{\mathscr{B}}.$$

Proof. It is easy to see that $\|g_a\|_{A^p_\alpha} \leqslant \|f_a\|_{A^p_\alpha} \lesssim 1$. Moreover,

$$f_a(z) = (1-|a|^2)^t \sum_{k=0}^{\infty} \frac{\Gamma(k+2t)}{(k+1)!\Gamma(2t)} \bar{a}^k z^{k+1}, \quad z \in \mathbb{D}.$$

Here $t = \frac{2+\alpha+p}{p}$. Then by Stirling's formula we get

$$f_a(z) \approx (1-|a|^2)^t \sum_{k=0}^{\infty} k^{2t-2} \bar{a}^k z^{k+1}, \quad z \in \mathbb{D}.$$

Therefore,

$$\begin{aligned}
\|(C_\varphi - C_\psi) f_a\|_{\mathscr{B}} &\lesssim (1-|a|^2)^t \sum_{k=1}^{\infty} k^{2t-2} |a|^k \|\varphi^{k+1} - \psi^{k+1}\|_{\mathscr{B}} \\
&\lesssim (1-|a|^2)^t \sum_{k=1}^{\infty} k^{t-1} |a|^k (k+1)^{t-1} \|\varphi^{k+1} - \psi^{k+1}\|_{\mathscr{B}} \\
&\lesssim (1-|a|^2)^t \sum_{k=1}^{\infty} k^{t-1} |a|^k \sup_{n\in\mathbb{N}} n^{\frac{2+\alpha}{p}} \|\varphi^n - \psi^n\|_{\mathscr{B}} \\
&\lesssim \sup_{n\in\mathbb{N}} n^{\frac{2+\alpha}{p}} \|\varphi^n - \psi^n\|_{\mathscr{B}}.
\end{aligned}$$

Since $a \in \mathbb{D}$ is arbitrary, we see that (i) holds.

Similarly, we have

$$\begin{aligned}
g_a(z) &= (1-|a|^2)^t \int_0^z \left(\sum_{k=0}^{\infty} \frac{\Gamma(k+2t)}{k!\Gamma(2t)} \bar{a}^k u^k \right) \left(a - (1-|a|^2) \sum_{k=0}^{\infty} \bar{a}^k u^{k+1} \right) du \\
&= a f_a(z) - (1-|a|^2)^{t+1} \int_0^z \sum_{k=1}^{\infty} \left(\sum_{l=0}^{k-1} \frac{\Gamma(l+2t)}{l!\Gamma(2t)} \right) \bar{a}^{k-1} u^k du.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\|(C_\varphi - C_\psi) g_a\|_{\mathscr{B}} \\
&\lesssim \|(C_\varphi - C_\psi) f_a\|_{\mathscr{B}} + (1-|a|^2)^{t+1} \sum_{k=1}^{\infty} k^{2t-1} |a|^{k-1} \|\varphi^{k+1} - \psi^{k+1}\|_{\mathscr{B}} \\
&\lesssim \|(C_\varphi - C_\psi) f_a\|_{\mathscr{B}} + (1-|a|^2)^{t+1} \sum_{k=1}^{\infty} k^t |a|^{k-1} \sup_{n\geqslant 2} n^{t-1} \|\varphi^n - \psi^n\|_{\mathscr{B}} \\
&\lesssim \sup_{n\in\mathbb{N}} n^{\frac{2+\alpha}{p}} \|\varphi^n - \psi^n\|_{\mathscr{B}},
\end{aligned}$$

for any $a \in \mathbb{D}$. Therefore (ii) holds. The proof is complete. □

By modifying the proof of Corollary 4.20 in [34], we can easily obtain the following result.

LEMMA 2.4. *Let $1 \leqslant p < \infty$, $\alpha > -1$. Then*

$$\|z^n\|_{A^p_\alpha} \approx \frac{1}{n^{\frac{2+\alpha}{p}}}.$$

THEOREM 2.1. *Let $1 \leqslant p < \infty$ and $-1 < \alpha < \infty$. Let φ and ψ be analytic self-maps of $\mathbb{D}$. Then $C_\varphi - C_\psi : A^p_\alpha \to \mathscr{B}$ is bounded if and only if*

$$\sup_{n\in\mathbb{N}} n^{\frac{2+\alpha}{p}} \|\varphi^n - \psi^n\|_{\mathscr{B}} < \infty. \tag{1}$$

Proof. First, we assume that (1) holds. For any $z \in \mathbb{D}$, we have

$$\begin{aligned}
&\|(C_\varphi - C_\psi) f_{\varphi(z)}\|_{\mathscr{B}} \\
\geqslant\ & |((C_\varphi - C_\psi) f_{\varphi(z)})'(z)|(1-|z|^2) \\
=\ & \left|f'_{\varphi(z)}(\varphi(z))\varphi'(z) - f'_{\varphi(z)}(\psi(z))\psi'(z)\right|(1-|z|^2) \\
=\ & \left|\mathscr{D}_\varphi(z) - \frac{(1-|\varphi(z)|^2)^{\frac{2+\alpha+p}{p}}(1-|\psi(z)|^2)^{\frac{2+\alpha+p}{p}}}{(1-\overline{\varphi(z)}\psi(z))^{\frac{2(2+\alpha+p)}{p}}}\mathscr{D}_\psi(z)\right| \\
\geqslant\ & |\mathscr{D}_\varphi(z)| - \frac{(1-|\varphi(z)|^2)^{\frac{2+\alpha+p}{p}}(1-|\psi(z)|^2)^{\frac{2+\alpha+p}{p}}}{|1-\overline{\varphi(z)}\psi(z)|^{\frac{2(2+\alpha+p)}{p}}}|\mathscr{D}_\psi(z)|
\end{aligned}$$

and

$$\begin{aligned}
\|(C_\varphi - C_\psi) g_{\varphi(z)}\|_{\mathscr{B}} &\geqslant \left|g'_{\varphi(z)}(\varphi(z))\varphi'(z) - g'_{\varphi(z)}(\psi(z))\psi'(z)\right|(1-|z|^2) \\
&= \frac{(1-|\varphi(z)|^2)^{\frac{2+\alpha+p}{p}}(1-|\psi(z)|^2)^{\frac{2+\alpha+p}{p}}}{|1-\overline{\varphi(z)}\psi(z)|^{\frac{2(2+\alpha+p)}{p}}}|\mathscr{D}_\psi(z)|\rho(z).
\end{aligned}$$

Hence

$$\begin{aligned}
|\mathscr{D}_\varphi(z)|\rho(z) &\leqslant \|(C_\varphi - C_\psi) f_{\varphi(z)}\|_{\mathscr{B}}\rho(z) + \|(C_\varphi - C_\psi) g_{\varphi(z)}\|_{\mathscr{B}} \\
&\leqslant \|(C_\varphi - C_\psi) f_{\varphi(z)}\|_{\mathscr{B}} + \|(C_\varphi - C_\psi) g_{\varphi(z)}\|_{\mathscr{B}},
\end{aligned}$$

and consequently

$$\begin{aligned}
\sup_{z\in\mathbb{D}} |\mathscr{D}_\varphi(z)|\rho(z) &\leqslant \sup_{z\in\mathbb{D}} \|(C_\varphi - C_\psi) f_{\varphi(z)}\|_{\mathscr{B}} + \sup_{z\in\mathbb{D}} \|(C_\varphi - C_\psi) g_{\varphi(z)}\|_{\mathscr{B}} \\
&\leqslant \sup_{a\in\mathbb{D}} \|(C_\varphi - C_\psi) f_a\|_{\mathscr{B}} + \sup_{a\in\mathbb{D}} \|(C_\varphi - C_\psi) g_a\|_{\mathscr{B}}.
\end{aligned} \tag{2}$$

Similarly,

$$|\mathscr{D}_\psi(z)|\rho(z) \leqslant \|(C_\varphi - C_\psi) f_{\psi(z)}\|_{\mathscr{B}} + \|(C_\varphi - C_\psi) g_{\psi(z)}\|_{\mathscr{B}}. \tag{3}$$

By Lemma 2.2, we have

$$
\begin{aligned}
&\|(C_\varphi - C_\psi) f_{\varphi(z)}\|_{\mathscr{B}} \\
\geqslant &\left|\mathscr{D}_\varphi(z) - \frac{(1-|\varphi(z)|^2)^{\frac{2+\alpha+p}{p}}(1-|\psi(z)|^2)^{\frac{2+\alpha+p}{p}}}{(1-\overline{\varphi(z)}\psi(z))^{\frac{2(2+\alpha+p)}{p}}}\mathscr{D}_\psi(z)\right| \\
\geqslant &|\mathscr{D}_\varphi(z) - \mathscr{D}_\psi(z)| - \left|1 - \frac{(1-|\varphi(z)|^2)^{\frac{2+\alpha+p}{p}}(1-|\psi(z)|^2)^{\frac{2+\alpha+p}{p}}}{(1-\overline{\varphi(z)}\psi(z))^{\frac{2(2+\alpha+p)}{p}}}\right||\mathscr{D}_\psi(z)| \\
\geqslant &|\mathscr{D}_\varphi(z) - \mathscr{D}_\psi(z)| \\
&-\left|(1-|\varphi(z)|^2)^{\frac{2+\alpha+p}{p}} f'_{\varphi(z)}(\varphi(z)) - (1-|\psi(z)|^2)^{\frac{2+\alpha+p}{p}} f'_{\varphi(z)}(\psi(z))\right||\mathscr{D}_\psi(z)| \\
\gtrsim &|\mathscr{D}_\varphi(z) - \mathscr{D}_\psi(z)| - |\mathscr{D}_\psi(z)|\rho(z).
\end{aligned}
$$

Thus, by (3) we get

$$
\begin{aligned}
&|\mathscr{D}_\varphi(z) - \mathscr{D}_\psi(z)| \\
\lesssim &\|(C_\varphi - C_\psi) f_{\varphi(z)}\|_{\mathscr{B}} + |\mathscr{D}_\psi(z)|\rho(z) \\
\lesssim &\|(C_\varphi - C_\psi) f_{\varphi(z)}\|_{\mathscr{B}} + \|(C_\varphi - C_\psi) f_{\psi(z)}\|_{\mathscr{B}} + \|(C_\varphi - C_\psi) g_{\psi(z)}\|_{\mathscr{B}}.
\end{aligned}
$$

Hence

$$
\sup_{z\in\mathbb{D}} |\mathscr{D}_\varphi(z) - \mathscr{D}_\psi(z)| \lesssim \sup_{a\in\mathbb{D}} \|(C_\varphi - C_\psi) f_a\|_{\mathscr{B}} + \sup_{a\in\mathbb{D}} \|(C_\varphi - C_\psi) g_a\|_{\mathscr{B}}. \tag{4}
$$

Combining (2), (4), and Lemma 2.3, we have

$$
\begin{aligned}
&\sup_{z\in\mathbb{D}} |\mathscr{D}_\varphi(z)|\rho(z) + \sup_{z\in\mathbb{D}} |\mathscr{D}_\varphi(z) - \mathscr{D}_\psi(z)| \\
\lesssim &\sup_{a\in\mathbb{D}} \|(C_\varphi - C_\psi) f_a\|_{\mathscr{B}} + \sup_{a\in\mathbb{D}} \|(C_\varphi - C_\psi) g_a\|_{\mathscr{B}} \\
\lesssim &\sup_{n\in\mathbb{N}} n^{\frac{2+\alpha}{p}} \|\varphi^n - \psi^n\|_{\mathscr{B}} < \infty.
\end{aligned}
$$

By Theorem 1 of [35], we see that $C_\varphi - C_\psi : A^p_\alpha \to \mathscr{B}$ is bounded.

Conversely, suppose that $C_\varphi - C_\psi : A^p_\alpha \to \mathscr{B}$ is bounded. For any $n \in \mathbb{N}$, let $f_n(z) = z^n/\|z^n\|_{A^p_\alpha}$. Then $\|f_n\|_{A^p_\alpha} = 1$. Thus, by the boundedness of $C_\varphi - C_\psi$ and Lemma 2.4, we obtain

$$
\begin{aligned}
\infty > \|C_\varphi - C_\psi\|_{A^p_\alpha \to \mathscr{B}} \geqslant \|(C_\varphi - C_\psi) f_n\|_{\mathscr{B}} &= \frac{\|\varphi^n - \psi^n\|_{\mathscr{B}}}{\|z^n\|_{A^p_\alpha}} \\
&\gtrsim n^{\frac{2+\alpha}{p}} \|\varphi^n - \psi^n\|_{\mathscr{B}},
\end{aligned}
$$

which implies that (1) holds. The proof is complete. □

3. Essential norm estimates

An estimate for the essential norm of $C_\varphi - C_\psi$ from A^p_α to $\mathscr{B}$ will be given in this section by using $n^{\frac{2+\alpha}{p}}\|\varphi^n - \psi^n\|_{\mathscr{B}}$. We state some auxiliary results firstly.

LEMMA 3.1. [31] *Let X,Y be two Banach spaces of analytic functions on $\mathbb{D}$. Suppose that the following statements hold.*

(1) The point evaluation functionals on Y are continuous.

(2) The closed unit ball of X is a compact subset of X in the topology of uniform convergence on compact sets.

(3) $T : X \to Y$ is continuous when X and Y are given the topology of uniform convergence on compact sets.

Then, T is a compact operator if and only if given a bounded sequence $\{f_n\}$ in X such that $f_n \to 0$ uniformly on compact sets, then the sequence $\{Tf_n\}$ converges to zero in the norm of Y.

LEMMA 3.2. *Let $1 \leqslant p < \infty$ and $-1 < \alpha < \infty$. Let φ and ψ be analytic self-maps of $\mathbb{D}$ such that $C_\varphi - C_\psi : A^p_\alpha \to \mathscr{B}$ is bounded. Then the following inequalities hold.*

(i)

$$\limsup_{|a|\to 1}\|(C_\varphi - C_\psi)f_a\|_{\mathscr{B}} \lesssim \limsup_{n\to\infty} n^{\frac{2+\alpha}{p}}\|\varphi^n - \psi^n\|_{\mathscr{B}}$$

(ii)

$$\limsup_{|a|\to 1}\|(C_\varphi - C_\psi)g_a\|_{\mathscr{B}} \lesssim \limsup_{n\to\infty} n^{\frac{2+\alpha}{p}}\|\varphi^n - \psi^n\|_{\mathscr{B}}.$$

Proof. Let $t = \frac{2+\alpha+p}{p}$. For any $N \in \mathbb{N}$, from the proof of Lemma 2.3, we obtain

$$\begin{aligned}
&\limsup_{|a|\to 1}\|(C_\varphi - C_\psi)f_a\|_{\mathscr{B}}\\
\leqslant &\limsup_{|a|\to 1}(1-|a|^2)^t\sum_{k=0}^{N}\frac{\Gamma(k+2t)}{(k+1)!\Gamma(2t)}(k+1)^{1-t}|a|^k(k+1)^{\frac{2+\alpha}{p}}\|\varphi^{k+1} - \psi^{k+1}\|_{\mathscr{B}}\\
&+\limsup_{|a|\to 1}(1-|a|^2)^t\sum_{k=N+1}^{\infty}\frac{\Gamma(k+2t)}{(k+1)!\Gamma(2t)}(k+1)^{1-t}|a|^k\\
&\times\sup_{n\geqslant N+1}(n+1)^{\frac{2+\alpha}{p}}\|\varphi^{n+1} - \psi^{n+1}\|_{\mathscr{B}}\\
\lesssim &\sup_{n\geqslant N+2} n^{\frac{2+\alpha}{p}}\|\varphi^n - \psi^n\|_{\mathscr{B}},
\end{aligned}$$

which implies that *(i)* holds.

Similarly, from the proof of Lemma 2.3 and *(i)*, we have

$$\limsup_{|a|\to 1}\|(C_\varphi - C_\psi)g_a\|_{\mathscr{B}} \lesssim \limsup_{|a|\to 1}\|(C_\varphi - C_\psi)f_a\|_{\mathscr{B}} + \sup_{n\geqslant N+2} n^{\frac{2+\alpha}{p}}\|\varphi^n - \psi^n\|_{\mathscr{B}}$$
$$\lesssim \sup_{n\geqslant N+2} n^{\frac{2+\alpha}{p}}\|\varphi^n - \psi^n\|_{\mathscr{B}}.$$

The proof is complete. □

THEOREM 3.1. *Let $1 \leqslant p < \infty$ and $\alpha > -1$. Let φ and ψ be analytic self-maps of $\mathbb{D}$. If $C_\varphi, C_\psi : A^p_\alpha \to \mathscr{B}$ are bounded, then*

$$\|C_\varphi - C_\psi\|_{e,A^p_\alpha\to\mathscr{B}} \approx \limsup_{n\to\infty} n^{\frac{2+\alpha}{p}}\|\varphi^n - \psi^n\|_{\mathscr{B}}.$$

Proof. First, we consider the upper estimate. We adopt the method from [17]. For $h \in (0,1)$, let $K_h f(z) = f_h(z) = f(hz)$. Then $K_h : A^p_\alpha \to A^p_\alpha$ is compact with $\|K_h\|_{A^p_\alpha\to A^p_\alpha} \leqslant 1$. Let $\{h_n\} \subset (0,1)$ be a sequence such that $h_n \to 1$ as $n\to\infty$. Since each K_{h_n} is compact on A^p_α, $C_\varphi - C_\psi$ is bounded from A^p_α to $\mathscr{B}$, $(C_\varphi - C_\psi)K_{h_n}$ is also compact from A^p_α to $\mathscr{B}$. Then, we have

$$\begin{aligned}\|C_\varphi - C_\psi\|_{e,A^p_\alpha\to\mathscr{B}} &\leqslant \limsup_{n\to\infty}\|C_\varphi - C_\psi - (C_\varphi - C_\psi)K_{h_n}\|\\ &= \limsup_{n\to\infty}\|(C_\varphi - C_\psi)(I-K_{h_n})\|\\ &\leqslant \limsup_{n\to\infty}\sup_{\|f\|_{A^p_\alpha}\leqslant 1}\|(C_\varphi - C_\psi)(I-K_{h_n})f\|_{\mathscr{B}},\end{aligned}$$

which is bounded by

$$\begin{aligned}&\limsup_{n\to\infty}\sup_{\|f\|_{A^p_\alpha}\leqslant 1}|(I-K_{h_n})(f(\varphi(0)) - f(\psi(0)))| + \limsup_{n\to\infty}\sup_{\|f\|_{A^p_\alpha}\leqslant 1}\sup_{z\in\mathbb{D}}\\ &|((I-K_{h_n})f)'(\varphi(z))\varphi'(z) - ((I-K_{h_n})f)'(\psi(z))\psi'(z)|(1-|z|^2).\end{aligned} \tag{5}$$

Since $f - f_{h_n}$ uniformly on compact subsets of $\mathbb{D}$ as $n\to\infty$ and the sets $\{\varphi(0)\}$ and $\{\psi(0)\}$ are compact, we have

$$\limsup_{n\to\infty}\sup_{\|f\|_{A^p_\alpha}\leqslant 1}|(I-K_{h_n})(f(\varphi(0)) - f(\psi(0)))| = 0. \tag{6}$$

Let

$$P_n = \sup_{\|f\|_{A^p_\alpha}\leqslant 1}\sup_{z\in\mathbb{D}}|((I-K_{h_n})f)'(\varphi(z))\varphi'(z) - ((I-K_{h_n})f)'(\psi(z))\psi'(z)|(1-|z|^2),$$

$f \in A^p_\alpha$ with $\|f\|_{A^p_\alpha} \leqslant 1$ and fix an arbitrary $s \in (0,1)$. We set

$$\mathscr{Q}^f_n(z) := |((I-K_{h_n})f)'(\varphi(z))\varphi'(z) - ((I-K_{h_n})f)'(\psi(z))\psi'(z)|(1-|z|^2)$$

and

$$\mathbb{D}_1 := \{z \in \mathbb{D} : |\varphi(z)| > s, |\psi(z)| > s\}, \mathbb{D}_2 := \{z \in \mathbb{D} : |\varphi(z)| > s, |\psi(z)| \leqslant s\},$$

$$\mathbb{D}_3 := \{z \in \mathbb{D} : |\varphi(z)| \leqslant s, |\psi(z)| > s\}, \mathbb{D}_4 := \{z \in \mathbb{D} : |\varphi(z)| \leqslant s, |\psi(z)| \leqslant s\}.$$

Then

$$\begin{aligned}\limsup_{n\to\infty} P_n = \limsup_{n\to\infty} \sup_{\|f\|_{A^p_\alpha}\leqslant 1} \sup_{z\in\mathbb{D}} \mathscr{Q}_n^f &= \max_{1\leqslant i\leqslant 4} \limsup_{n\to\infty} \sup_{\|f\|_{A^p_\alpha}\leqslant 1} \sup_{z\in\mathbb{D}_i} \mathscr{Q}_n^f \\ &= \max_{1\leqslant i\leqslant 4} \{\limsup_{n\to\infty} P_n^{(i)}\},\end{aligned}$$

where $P_n^{(i)} = \sup_{\|f\|_{A^p_\alpha}\leqslant 1} \sup_{z\in\mathbb{D}_i} \mathscr{Q}_n^f$. In addition, we have

$$\begin{aligned}\mathscr{Q}_n^f(z) \leqslant\; & |((I-K_{h_n})f)'(\psi(z))|(1-|\psi(z)|^2)^{\frac{2+\alpha+p}{p}}|\mathscr{D}_\varphi(z)-\mathscr{D}_\psi(z)| \\ & +|((I-K_{h_n})f)'(\varphi(z))(1-|\varphi(z)|^2)^{\frac{2+\alpha+p}{p}} \\ & -((I-K_{h_n})f)'(\psi(z))(1-|\psi(z)|^2)^{\frac{2+\alpha+p}{p}}||\mathscr{D}_\varphi(z)| \\ \lesssim\; & |((I-K_{h_n})f)'(\psi(z))|(1-|\psi(z)|^2)^{\frac{2+\alpha+p}{p}}|\mathscr{D}_\varphi(z)-\mathscr{D}_\psi(z)|+|\mathscr{D}_\varphi(z)|\rho(z)\end{aligned}$$

and

$$\mathscr{Q}_n^f(z) \lesssim |((I-K_{h_n})f)'(\varphi(z))|(1-|\varphi(z)|^2)^{\frac{2+\alpha+p}{p}}|\mathscr{D}_\varphi(z)-\mathscr{D}_\psi(z)|+|\mathscr{D}_\psi(z)|\rho(z).$$

Using the fact that

$$\limsup_{n\to\infty} \|(I-K_{h_n})f\|_{A^p_\alpha} \leqslant \limsup_{n\to\infty} \|I-K_{h_n}\|_{A^p_\alpha\to A^p_\alpha}\|f\|_{A^p_\alpha} \leqslant 1,$$

we get

$$\begin{aligned}\limsup_{n\to\infty} P_n^{(1)} &\lesssim \limsup_{n\to\infty} \sup_{\|f\|_{A^p_\alpha}\leqslant 1} \sup_{z\in\mathbb{D}_1} \Big(|((I-K_{h_n})f)'(\varphi(z))|(1-|\varphi(z)|^2)^{\frac{2+\alpha+p}{p}} \\ &\qquad \times|\mathscr{D}_\varphi(z)-\mathscr{D}_\psi(z)|+|\mathscr{D}_\psi(z)|\rho(z)\Big) \\ &\lesssim \limsup_{n\to\infty} \sup_{\|f\|_{A^p_\alpha}\leqslant 1} \sup_{\substack{|\varphi(z)|>s \\ |\psi(z)|>s}} \|(I-K_{h_n})f\|_{A^p_\alpha}|\mathscr{D}_\varphi(z)-\mathscr{D}_\psi(z)| \\ &\qquad + \sup_{|\psi(z)|>s} |\mathscr{D}_\psi(z)|\rho(z) \\ &\lesssim \sup_{\substack{|\varphi(z)|>s \\ |\psi(z)|>s}} |\mathscr{D}_\varphi(z)-\mathscr{D}_\psi(z)| + \sup_{|\psi(z)|>s} |\mathscr{D}_\psi(z)|\rho(z).\end{aligned}$$

Thus,

$$\limsup_{n\to\infty} P_n^{(1)} \leqslant \limsup_{\substack{|\varphi(z)|\to 1 \\ |\psi(z)|\to 1}} |\mathscr{D}_\varphi(z)-\mathscr{D}_\psi(z)| + \limsup_{|\psi(z)|\to 1} |\mathscr{D}_\psi(z)|\rho(z).$$

Since C_φ and C_ψ are bounded and $\sup_{z\in\mathbb{D}}|\mathscr{D}_\varphi(z)-\mathscr{D}_\psi(z)|<\infty$, we obtain

$$\begin{aligned}
&\limsup_{n\to\infty} P_n^{(2)}\\
&\leqslant \limsup_{n\to\infty}\sup_{\|f\|_{A^p_\alpha}\leqslant 1}\sup_{z\in\mathbb{D}_2}\Big(|((I-K_{h_n})f)'(\psi(z))|(1-|\psi(z)|^2)^{\frac{2+\alpha+p}{p}}\\
&\quad\times|\mathscr{D}_\varphi(z)-\mathscr{D}_\psi(z)|+|\mathscr{D}_\varphi(z)|\rho(z)\Big)\\
&\lesssim \limsup_{n\to\infty}\sup_{\|f\|_{A^p_\alpha}\leqslant 1}\sup_{|\psi(z)|\leqslant s}|((I-K_{h_n})f)'(\psi(z))|(1-|\psi(z)|^2)^{\frac{2+\alpha+p}{p}}\\
&\quad\times|\mathscr{D}_\varphi(z)-\mathscr{D}_\psi(z)|+\sup_{|\varphi(z)|>s}|\mathscr{D}_\varphi(z)|\rho(z)\\
&\lesssim \sup_{|\varphi(z)|>s}|\mathscr{D}_\varphi(z)|\rho(z).
\end{aligned}$$

Since s is arbitrary, we have

$$\limsup_{n\to\infty} P_n^{(2)}\lesssim\limsup_{|\varphi(z)|\to 1}|\mathscr{D}_\varphi(z)|\rho(z).$$

Similarly, we obtain

$$\limsup_{n\to\infty} P_n^{(3)}\lesssim\limsup_{|\psi(z)|\to 1}|\mathscr{D}_\psi(z)|\rho(z).$$

Finally, by Lemma 3.1, we obtain

$$\begin{aligned}
&\limsup_{n\to\infty} P_n^{(4)}=\limsup_{n\to\infty}\sup_{\|f\|_{A^p_\alpha}\leqslant 1}\sup_{z\in\mathbb{D}_4}\mathscr{Q}_n^f\\
&\leqslant \limsup_{n\to\infty}\sup_{\|f\|_{A^p_\alpha}\leqslant 1}\sup_{|\varphi(z)|\leqslant s}|((I-K_{h_n})f)'(\varphi(z))||\varphi'(z)|(1-|z|^2)\\
&\quad+\limsup_{n\to\infty}\sup_{\|f\|_{A^p_\alpha}\leqslant 1}\sup_{|\psi(z)|\leqslant s}|((I-K_{h_n})f)'(\psi(z))||\psi'(z)|(1-|z|^2)\\
&=0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\limsup_{n\to\infty} P_n &= \max\{\limsup_{n\to\infty} P_n^{(1)},\limsup_{n\to\infty} P_n^{(2)},\limsup_{n\to\infty} P_n^{(3)},\limsup_{n\to\infty} P_n^{(4)}\}\\
&\lesssim \limsup_{|\varphi(z)|\to 1}|\mathscr{D}_\varphi(z)|\rho(z)+\limsup_{|\psi(z)|\to 1}|\mathscr{D}_\psi(z)|\rho(z)\\
&\quad+\limsup_{\substack{|\varphi(z)|\to 1\\ |\psi(z)|\to 1}}|\mathscr{D}_\varphi(z)-\mathscr{D}_\psi(z)|. \qquad (7)
\end{aligned}$$

Set

$$\mathscr{A}_{fg}:=\limsup_{|a|\to 1}\|(C_\varphi-C_\psi)f_a\|_{\mathscr{B}}+\limsup_{|a|\to 1}\|(C_\varphi-C_\psi)g_a\|_{\mathscr{B}}.$$

From the proof of Theorem 2.1, we obtain

$$\limsup_{|\varphi(z)|\to 1} |\mathscr{D}_\varphi(z)|\rho(z) \lesssim \mathscr{A}_{fg}, \quad \limsup_{|\psi(z)|\to 1} |\mathscr{D}_\psi(z)|\rho(z) \lesssim \mathscr{A}_{fg}$$

and

$$\limsup_{\substack{|\varphi(z)|\to 1\\ |\psi(z)|\to 1}} |\mathscr{D}_\varphi(z) - \mathscr{D}_\psi(z)| \lesssim \mathscr{A}_{fg}.$$

By (5), (6), (7) and Lemma 3.2, we have

$$\begin{aligned}
&\|C_\varphi - C_\psi\|_{e,A^p_\alpha\to\mathscr{B}}\\
&\lesssim \limsup_{|\varphi(z)|\to 1} |\mathscr{D}_\varphi(z)|\rho(z) + \limsup_{|\psi(z)|\to 1} |\mathscr{D}_\psi(z)|\rho(z) + \limsup_{\substack{|\varphi(z)|\to 1\\ |\psi(z)|\to 1}} |\mathscr{D}_\varphi(z) - \mathscr{D}_\psi(z)|\\
&\lesssim \mathscr{A}_{fg} \lesssim \limsup_{n\to\infty} n^{\frac{2+\alpha}{p}} \|\varphi^n - \psi^n\|_{\mathscr{B}},
\end{aligned}$$

as desired.

Next, we consider the lower estimate. Let n be any positive integer. Set $y_n(z) = z^n/\|z^n\|_{A^p_\alpha}$. Then $\|y_n\|_{A^p_\alpha} = 1$ and by Lemma 2.4, $y_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. Let $K : A^p_\alpha \to \mathscr{B}$ be a compact operator. By Lemma 3.1 we have $\lim_{n\to\infty} \|Ky_n\|_{\mathscr{B}} = 0$. Hence,

$$\|C_\varphi - C_\psi - K\| \geqslant \limsup_{n\to\infty} \|(C_\varphi - C_\psi - K)y_n\|_{\mathscr{B}} \geqslant \limsup_{n\to\infty} \|(C_\varphi - C_\psi)y_n\|_{\mathscr{B}}.$$

By Lemma 2.4, we have

$$\begin{aligned}
\limsup_{n\to\infty} n^{\frac{2+\alpha}{p}} \|\varphi^n - \psi^n\|_{\mathscr{B}} &\lesssim \limsup_{n\to\infty} \frac{1}{\|z^n\|_{A^p_\alpha}} \|(C_\varphi - C_\psi)z^n\|_{\mathscr{B}}\\
&= \limsup_{n\to\infty} \|(C_\varphi - C_\psi)y_n\|_{\mathscr{B}}\\
&\leqslant \|C_\varphi - C_\psi\|_{e,A^p_\alpha\to\mathscr{B}}.
\end{aligned}$$

The proof is complete. □

From Theorem 3.1, we immediately get the following corollary.

COROLLARY 3.1. *Let $1 \leqslant p < \infty$ and $\alpha > -1$. Let φ and ψ be analytic self-maps of $\mathbb{D}$ such that C_φ, C_ψ are bounded from A^p_α to $\mathscr{B}$. Then $C_\varphi - C_\psi$ is compact from A^p_α to $\mathscr{B}$ if and only if*

$$\limsup_{n\to\infty} n^{\frac{2+\alpha}{p}} \|\varphi^n - \psi^n\|_{\mathscr{B}} = 0.$$

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