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NEW INEQUALITIES FOR OPERATOR CONCAVE FUNCTIONS INVOLVING POSITIVE LINEAR MAPS

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Abstract. The purpose of this paper is to present some general inequalities for operator concave functions which include some known inequalities as a particular case. Among other things, we prove that if $A \in \mathscr{B}(\mathscr{H})$ is a positive operator such that $mI \leqslant A \leqslant MI$ for some scalars $0 < m < M$ and Φ is a normalized positive linear map on $\mathscr{B}(\mathscr{H})$, then

$$\begin{aligned}\left(\frac{M+m}{2\sqrt{Mm}}\right)^r &\geqslant \left(\frac{\frac{1}{\sqrt{Mm}}\Phi(A)+\sqrt{Mm}\Phi\left(A^{-1}\right)}{2}\right)^r\\ &\geqslant \frac{\frac{1}{(Mm)^{\frac{r}{2}}}\Phi(A)^r+(Mm)^{\frac{r}{2}}\Phi\left(A^{-1}\right)^r}{2}\\ &\geqslant \Phi(A)^r\sharp\Phi\left(A^{-1}\right)^r,\end{aligned}$$

where $0 \leqslant r \leqslant 1$, which nicely extend the operator Kantorovich inequality.

1. Introduction

In this paper we consider operator monotone and convex functions defined on the half real line $(0,\infty)$. Let $\mathscr{B}(\mathscr{H})$ be the algebra of all bounded linear operators on a complex Hilbert space and I denote the identity operator. If A is an operator then we denote $Sp(A)$ its spectrum. An operator A is called *positive* if $\langle Ax,x\rangle \geqslant 0$ for all $x \in \mathscr{H}$, and we then write $A \geqslant 0$. By $B \geqslant A$ we mean that $B-A$ is positive, while $B > A$ means that $B-A$ is strictly positive. A mapping Φ on $\mathscr{B}(\mathscr{H})$ is said to be *positive* if $\Phi(A) \geqslant 0$ for each $A \geqslant 0$ and is called *normalized* if Φ preserves the identity operator.

For any strictly positive operator $A,B \in \mathscr{B}(\mathscr{H})$ and $v \in [0,1]$, we write

$$A\nabla_v B := (1-v)A+vB \quad \text{and} \quad A\sharp_v B := A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^v A^{\frac{1}{2}}.$$

For the case $v=\frac{1}{2}$, we write ∇ and $\sharp$, respectively. The operator arithmetic-geometric mean inequality (in short, AM-GM inequality) asserts that $A\sharp_v B \leqslant A\nabla_v B$, for any positive operators $A,B \in \mathscr{B}(\mathscr{H})$ and any $v \in [0,1]$. A real valued function f defined on

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an interval J is said to be *operator convex* (resp. *operator concave*) if $f(A\nabla_v B) \leqslant f(A)\nabla_v f(B)$ (resp. $f(A\nabla_v B) \geqslant f(A)\nabla_v f(B)$) for all self-adjoint operators A, B with spectra in J and all $v \in [0,1]$. A continuous real valued function f defined on an interval J is called *operator monotone* (more precisely, *operator monotone increasing*) if $B \geqslant A$ implies that $f(B) \geqslant f(A)$, and *operator monotone decreasing* if $B \geqslant A$ implies $f(B) \leqslant f(A)$ for all self-adjoint operators A, B with spectra in J.

During the past decades several formulations, extensions or refinements of the Kantorovich inequality [7] in various settings have been introduced by many mathematicians; see [6, 8, 9, 11] and references therein.

Let $A \in \mathscr{B}(\mathscr{H})$ be a positive operator such that $mI \leqslant A \leqslant MI$ for some scalars $0 < m < M$ and Φ be a normalized positive linear map on $\mathscr{B}(\mathscr{H})$, then

$$\Phi\left(A^{-1}\right)\sharp\Phi(A) \leqslant \frac{M+m}{2\sqrt{Mm}}. \tag{1}$$

In addition

$$\Phi(A)\sharp\Phi(B) \leqslant \frac{M+m}{2\sqrt{Mm}}\Phi(A\sharp B), \tag{2}$$

whenever $m^2A \leqslant B \leqslant M^2A$ and $0 < m < M$. The first inequality goes back to Nakamoto and Nakamura in the 1996's [12], the second is more general and has been proved only in 2009 by Lee [5] (its matrix version).

In Sec. 2, we first extend (2), then as an application, we obtain a generalization of (1). In Sec. 3, we use elementary operations and give some inequalities related to the Bellman type.

2. Some operator inequalities involving positive linear maps

We prove the following new result, from which (2) directly follows:

THEOREM 1. *Let $A, B \in \mathscr{B}(\mathscr{H})$ be two strictly positive operators such that $m_1^2 I \leqslant A \leqslant M_1^2 I$, $m_2^2 I \leqslant B \leqslant M_2^2 I$ for some positive scalars $m_1 < M_1$, $m_2 < M_2$, and let Φ be a normalized positive linear map on $\mathscr{B}(\mathscr{H})$. If f is an operator monotone, then*

$$\begin{aligned} f\left(\left(\frac{M+m}{2}\right)\Phi(A\sharp B)\right) &\geqslant f\left(\frac{Mm\Phi(A)+\Phi(B)}{2}\right) \\ &\geqslant \frac{f(Mm\Phi(A))+f(\Phi(B))}{2} \\ &\geqslant f(Mm\Phi(A))\sharp f(\Phi(B)), \end{aligned}$$

where $m = \frac{m_2}{M_1}$ and $M = \frac{M_2}{m_1}$.

Proof. According to the assumption, we have

$$mI \leqslant \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}} \leqslant MI,$$

it follows that

$$(M+m)\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}} \geqslant MmI + A^{-\frac{1}{2}}BA^{-\frac{1}{2}}.$$

The above inequality then implies

$$\left(\frac{M+m}{2}\right) A\sharp B \geqslant \frac{MmA+B}{2}.$$

Using the hypotheses made about Φ,

$$\left(\frac{M+m}{2}\right)\Phi(A\sharp B) \geqslant \frac{Mm\Phi(A)+\Phi(B)}{2}.$$

Thus we have

$$\begin{aligned} f\left(\left(\frac{M+m}{2}\right)\Phi(A\sharp B)\right) &\geqslant f\left(\frac{Mm\Phi(A)+\Phi(B)}{2}\right) \quad \text{(since } f \text{ is operator monotone)} \\ &\geqslant \frac{f(Mm\Phi(A))+f(\Phi(B))}{2} \quad \text{(by [2, Theorem 2.3])} \\ &\geqslant f(Mm\Phi(A))\sharp f(\Phi(B)) \quad \text{(by AM-GM inequality),} \end{aligned}$$

which is the statement of the theorem. □

We complement Theorem 1 by proving the following.

THEOREM 2. *Let $A,B \in \mathscr{B}(\mathscr{H})$ be two strictly positive operators such that $m_1^2 I \leqslant A \leqslant M_1^2 I$, $m_2^2 I \leqslant B \leqslant M_2^2 I$ for some scalars $m_1 < M_1$, $m_2 < M_2$, and let Φ be a normalized positive linear map on $\mathscr{B}(\mathscr{H})$. If g is an operator monotone decreasing, then*

$$\begin{aligned} g\left(\left(\frac{M+m}{2}\right)\Phi(A\sharp B)\right) &\leqslant g\left(\frac{Mm\Phi(A)+\Phi(B)}{2}\right) \\ &\leqslant \left\{\frac{g(Mm\Phi(A))^{-1}+g(\Phi(B))^{-1}}{2}\right\}^{-1} \\ &\leqslant g(Mm\Phi(A))\sharp g(\Phi(B)), \end{aligned}$$

where $m = \frac{m_2}{M_1}$ and $M = \frac{M_2}{m_1}$.

Proof. Since g is operator monotone decreasing on $(0,\infty)$, so $\frac{1}{g}$ is operator monotone on $(0,\infty)$. Now by applying Theorem 1 for $f = \frac{1}{g}$, we have

$$\begin{aligned} g\left(\left(\frac{M+m}{2}\right)\Phi(A\sharp B)\right)^{-1} &\geqslant g\left(\frac{Mm\Phi(A)+\Phi(B)}{2}\right)^{-1} \\ &\geqslant \frac{g(Mm\Phi(A))^{-1}+g(\Phi(B))^{-1}}{2} \\ &\geqslant g(Mm\Phi(A))^{-1}\sharp g(\Phi(B))^{-1}. \end{aligned}$$

Taking the inverse, we get

$$\begin{aligned}
g\left(\left(\frac{M+m}{2}\right)\Phi(A\sharp B)\right) &\leqslant g\left(\frac{Mm\Phi(A)+\Phi(B)}{2}\right)\\
&\leqslant \left\{\frac{g(Mm\Phi(A))^{-1}+g(\Phi(B))^{-1}}{2}\right\}^{-1}\\
&\leqslant \left\{g(Mm\Phi(A))^{-1}\sharp g(\Phi(B))^{-1}\right\}^{-1}\\
&= g(Mm\Phi(A))\sharp g(\Phi(B)),
\end{aligned}$$

proving the main assertion of the theorem. □

As a byproduct of Theorems 1 and 2, we have the following result.

COROLLARY 1. *Under the assumptions of Theorem 1.*

(i) If $0 \leqslant r \leqslant 1$*, then*

$$\begin{aligned}
\left(\frac{M+m}{2\sqrt{Mm}}\right)^r \Phi(A\sharp B)^r &\geqslant \left(\frac{Mm\Phi(A)+\Phi(B)}{2\sqrt{Mm}}\right)^r\\
&\geqslant \frac{(Mm)^r\Phi(A)^r+\Phi(B)^r}{2(Mm)^{\frac{r}{2}}}\\
&\geqslant \Phi(A)^r\sharp\Phi(B)^r.
\end{aligned}$$

The important special case

$$\frac{M+m}{2\sqrt{Mm}}\Phi(A\sharp B) \geqslant \frac{Mm\Phi(A)+\Phi(B)}{2\sqrt{Mm}} \geqslant \Phi(A)\sharp\Phi(B),$$

was observed by Moslehian et al. [11] *(see* [9, Theorem 2.5] *for much stronger result).*

(ii) If $-1 \leqslant r \leqslant 0$*, then*

$$\begin{aligned}
\left(\frac{M+m}{2\sqrt{Mm}}\right)^r \Phi(A\sharp B)^r &\leqslant \left(\frac{Mm\Phi(A)+\Phi(B)}{2\sqrt{Mm}}\right)^r\\
&\leqslant \frac{1}{(Mm)^{\frac{r}{2}}}\left\{\frac{(Mm)^{-r}\Phi(A)^{-r}+\Phi(B)^{-r}}{2}\right\}^{-1}\\
&\leqslant \Phi(A)^r\sharp\Phi(B)^r.
\end{aligned}$$

Our next result is a straightforward application of Theorems 1 and 2.

COROLLARY 2. *Let* $A \in \mathscr{B}(\mathscr{H})$ *be positive operator such that* $mI \leqslant A \leqslant MI$ *for some scalars* $0 < m < M$ *and* Φ *be a normalized positive linear map on* $\mathscr{B}(\mathscr{H})$.

(i) If f is an operator monotone, then

$$f\left(\frac{M+m}{2Mm}\right) \geqslant f\left(\frac{\frac{1}{Mm}\Phi(A)+\Phi\left(A^{-1}\right)}{2}\right)$$
$$\geqslant \frac{f\left(\frac{1}{Mm}\Phi(A)\right)+f\left(\Phi\left(A^{-1}\right)\right)}{2}$$
$$\geqslant f\left(\frac{1}{Mm}\Phi(A)\right)\sharp f\left(\Phi\left(A^{-1}\right)\right).$$

(ii) If g is an operator monotone decreasing, then

$$g\left(\frac{M+m}{2Mm}\right) \leqslant g\left(\frac{\frac{1}{Mm}\Phi(A)+\Phi\left(A^{-1}\right)}{2}\right)$$
$$\leqslant \left\{\frac{g\left(\frac{1}{Mm}\Phi(A)\right)^{-1}+g\left(\Phi\left(A^{-1}\right)\right)^{-1}}{2}\right\}^{-1}$$
$$\leqslant g\left(\frac{1}{Mm}\Phi(A)\right)\sharp g\left(\Phi\left(A^{-1}\right)\right).$$

In the same vein as in Corollary 1, we have the following consequences.

COROLLARY 3. *Under the assumptions of Corollary 2.*

(i) If $0 \leqslant r \leqslant 1$, then

$$\left(\frac{M+m}{2\sqrt{Mm}}\right)^r \geqslant \left(\frac{\frac{1}{\sqrt{Mm}}\Phi(A)+\sqrt{Mm}\Phi\left(A^{-1}\right)}{2}\right)^r$$
$$\geqslant \frac{\frac{1}{(Mm)^{\frac{r}{2}}}\Phi(A)^r+(Mm)^{\frac{r}{2}}\Phi\left(A^{-1}\right)^r}{2}$$
$$\geqslant \Phi(A)^r \sharp \Phi\left(A^{-1}\right)^r.$$

For the special case in which $r=1$, we have

$$\frac{M+m}{2\sqrt{Mm}} \geqslant \frac{\frac{1}{\sqrt{Mm}}\Phi(A)+\sqrt{Mm}\Phi\left(A^{-1}\right)}{2} \geqslant \Phi(A)\sharp\Phi\left(A^{-1}\right).$$

(ii) If $-1 \leqslant r \leqslant 0$, then

$$\left(\frac{M+m}{2\sqrt{Mm}}\right)^r \leqslant \left(\frac{\frac{1}{\sqrt{Mm}}\Phi(A)+\sqrt{Mm}\Phi\left(A^{-1}\right)}{2}\right)^r$$
$$\leqslant \left\{\frac{(Mm)^r\Phi(A)^{-r}+\Phi\left(A^{-1}\right)^{-r}}{2(Mm)^{\frac{r}{2}}}\right\}^{-1}$$
$$\leqslant \Phi(A)^r \sharp \Phi\left(A^{-1}\right)^r.$$

3. Operator Bellman inequality with negative parameter

Let $A,B \in \mathscr{B}(\mathscr{H})$ be two strictly positive operators and Φ be a normalized positive linear map on $\mathscr{B}(\mathscr{H})$. If f is an operator concave, then for any $v \in [0,1]$, the following inequality obtained in [10, Theorem 2.1]:

$$\Phi(f(A))\nabla_v\Phi(f(B)) \leqslant f(\Phi(A\nabla_v B)). \tag{3}$$

In the same paper, as an operator version of Bellman inequality [3], the authors showed that

$$\Phi((I-A)^r\nabla_v(I-B)^r) \leqslant \Phi(I-A\nabla_v B)^r, \tag{4}$$

where A,B are two operator contractions (in the sense that $\|A\|,\ \|B\| \leqslant 1$) and $r,v \in [0,1]$.

Under the convexity assumption on f, (4) can be reversed:

THEOREM 3. *Let $A,B \in \mathscr{B}(\mathscr{H})$ be two contraction operators and Φ be a normalized positive linear map on $\mathscr{B}(\mathscr{H})$. Then*

$$\Phi(I-A\nabla_v B)^r \leqslant \Phi((I-A)^r\nabla_v(I-B)^r), \tag{5}$$

for any $v \in [0,1]$ and $r \in [-1,0]\cup[1,2]$.

Proof. If f is operator convex, we have

$$\begin{aligned} f(\Phi(A\nabla_v B)) &\leqslant \Phi(f(A\nabla_v B)) \quad \text{(by Choi-Davis-Jensen inequality [4, p. 62])}\\ &\leqslant \Phi(f(A)\nabla_v f(B)) \quad \text{(by operator convexity of } f\text{)}. \end{aligned}$$

The function $f(t) = t^r$ is operator convex on $(0,\infty)$ for $r \in [-1,0]\cup[1,2]$ (see [4, Chapter 1]). It can be verified that $f(t) = (1-t)^r$ is operator convex on $(0,1)$ for $r \in [-1,0]\cup[1,2]$. This implies the desired result (5). □

However, we are looking for something stronger than (5). The principal object of this section is to prove the following:

THEOREM 4. *Let $A,B \in \mathscr{B}(\mathscr{H})$ be two contraction operators and Φ be a normalized positive linear map on $\mathscr{B}(\mathscr{H})$. Then*

$$\begin{aligned} \Phi(I-A\nabla_v B)^r &\leqslant \Phi(I-A)^r\sharp_v\Phi(I-B)^r\\ &\leqslant \Phi((I-A)^r\sharp_v(I-B)^r)\\ &\leqslant \Phi((I-A)^r\nabla_v(I-B)^r), \end{aligned}$$

where $v \in [0,1]$ and $r \in [-1,0]$.

The proof is at the end of this section. The following lemma will play an important role in our proof.

LEMMA 1. *Let $A,B \in \mathscr{B}(\mathscr{H})$ be two strictly positive operators and Φ be a normalized positive linear map on $\mathscr{B}(\mathscr{H})$. If f is an operator monotone decreasing, then for any $v \in [0,1]$*

$$f(\Phi(A\nabla_v B)) \leqslant f(\Phi(A))\sharp_v f(\Phi(B)) \leqslant \Phi(f(A))\nabla_v \Phi(f(B)), \tag{6}$$

and

$$f(\Phi(A\nabla_v B)) \leqslant \Phi(f(A)\sharp_v f(B)) \leqslant \Phi(f(A))\nabla_v \Phi(f(B)). \tag{7}$$

More precisely,

$$f(\Phi(A\nabla_v B)) \leqslant f(\Phi(A))\sharp_v f(\Phi(B)) \leqslant \Phi(f(A)\sharp_v f(B)) \leqslant \Phi(f(A))\nabla_v \Phi(f(B)). \tag{8}$$

Proof. As Ando and Hiai mentioned in [2, (2.16)], the function f is an operator monotone decreasing if and only if

$$f(A\nabla_v B) \leqslant f(A)\sharp_v f(B). \tag{9}$$

We emphasize here that if f satisfies in (9), then is operator convex (this class of functions is called *operator log-convex*). It is easily verified that if $Sp(A), Sp(B) \subseteq J$, then $Sp(\Phi(A))$, $Sp(\Phi(B)) \subseteq J$. So we can replace A,B by $\Phi(A)$, $\Phi(B)$ in (9), respectively. Therefore we can write

$$\begin{aligned} f(\Phi(A\nabla_v B)) &\leqslant f(\Phi(A))\sharp_v f(\Phi(B)) \\ &\leqslant \Phi(f(A))\sharp_v \Phi(f(B)) \quad \text{(by Choi-Davis-Jensen inequality and} \\ &\qquad\qquad\qquad\qquad\qquad\quad \text{monotonicity property of mean)} \\ &\leqslant \Phi(f(A)\nabla_v f(B)) \quad \text{(by AM-GM inequality).} \end{aligned}$$

This completes the proof of the inequality (6). To prove the inequality (7), note that if $Sp(A)$, $Sp(B) \subseteq J$, then $Sp(A\nabla_v B) \subseteq J$. By computation

$$\begin{aligned} f(\Phi(A\nabla_v B)) &\leqslant \Phi(f(A\nabla_v B)) \quad \text{(by Choi-Davis-Jensen inequality)} \\ &\leqslant \Phi(f(A)\sharp_v f(B)) \quad \text{(by (9))} \\ &\leqslant \Phi(f(A))\sharp_v \Phi(f(B)) \quad \text{(by Ando's inequality [1, Theorem 3])} \\ &\leqslant \Phi(f(A)\nabla_v f(B)) \quad \text{(by AM-GM inequality),} \end{aligned}$$

proving the inequality (7). We know that if g is operator monotone on $(0,\infty)$, then g is operator concave. As before, it can be shown that

$$g(\Phi(A))\sharp_v g(\Phi(B)) \geqslant \Phi(g(A))\sharp_v \Phi(g(B)) \geqslant \Phi(g(A)\sharp_v g(B)).$$

Taking the inverse, we get

$$g(\Phi(A))^{-1}\sharp_v g(\Phi(B))^{-1} \leqslant \Phi(g(A)\sharp_v g(B))^{-1} \leqslant \Phi\left(g(A)^{-1}\sharp_v g(B)^{-1}\right).$$

If g is operator monotone, then $f = \frac{1}{g}$ is operator monotone decreasing, we conclude

$$f(\Phi(A))\sharp_v f(\Phi(B)) \leqslant \Phi(f(A)\sharp_v f(B)).$$

This proves (8). $\square$

We are now in a position to present a proof of Theorem 4.

Proof of Theorem 4. It is well-known that the function $f(t)=t^r$ on $(0,\infty)$ is operator monotone decreasing for $r\in[-1,0]$. It implies that the function $f(t)=(1-t)^r$ on $(0,1)$ is operator monotone decreasing too. By applying Lemma 1, we get the desired result. □

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