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ON SOME CLASSICAL TRACE INEQUALITIES AND A NEW HILBERT–SCHMIDT NORM INEQUALITY

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Abstract. Let A be a positive semidefinite matrix and B be a Hermitian matrix. Using some classical trace inequalities, we prove, among other inequalities, that

$$\left\|A^sB+BA^{1-s}\right\|_2 \leqslant \left\|A^tB+BA^{1-t}\right\|_2$$

for $\frac{1}{2} \leqslant s \leqslant t \leqslant 1$. We conjecture that this inequality is also true for all unitarily invariant norms, and we affirmatively settle this conjecture for the case $s=\frac{1}{2}$ and $t=1$.

1. Introduction

Throughout this paper, all matrices are assumed to be $n\times n$ complex matrices. In their investigation of trace inequalities for multiple products of powers of two positive semidefinite matrices, T. Ando, F. Hiai, and K. Okubo [1] proved that if A and B are positive semidefinite matrices, then

$$\operatorname{tr}\left(A^{\frac{1}{2}}B\right)^2 \leqslant \operatorname{tr}\ A^tBA^{1-t}B \leqslant \operatorname{tr}\ AB^2 \tag{1}$$

for $0\leqslant t\leqslant 1$. See Corollary 2.2 in [1].

The inequalities (1) can be generalized by proving that the inequality

$$\operatorname{tr}\ A^sBA^{1-s}B \leqslant \operatorname{tr}\ A^tBA^{1-t}B \tag{2}$$

holds for $\frac{1}{2}\leqslant s\leqslant t\leqslant 1$, where A is a positive semidefinite matrix and B is a Hermitian matrix.

To accomplish this, we consider the function $f(t)=\operatorname{tr}\ A^tBA^{1-t}B$ for $0\leqslant t\leqslant 1$. Note that $f(t)=f(1-t)$, and so $f(t)$ is symmetric about $t=\frac{1}{2}$. The Cauchy Schwarz inequality (see [2, p. 96]) says that for any two matrices X and Y, we have

$$|\operatorname{tr}\ XY| \leqslant (\operatorname{tr}\ X^*X)^{\frac{1}{2}}(\operatorname{tr}\ Y^*Y)^{\frac{1}{2}}.$$

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Using this inequality, we can prove that $f(t)$ is logarithmically convex (and hence it is convex) for $0 \leqslant t \leqslant 1$. In fact, if $0 \leqslant s,t \leqslant 1$, then

$$\begin{aligned} f\left(\frac{s+t}{2}\right) &= \operatorname{tr} A^{\frac{s+t}{2}} B A^{1-\left(\frac{s+t}{2}\right)} B \\ &= \operatorname{tr}\left(A^{\frac{t}{2}} B A^{\frac{1-t}{2}}\right)\left(A^{\frac{1-s}{2}} B A^{\frac{s}{2}}\right) \\ &\leqslant \left(\operatorname{tr} A^{\frac{1-t}{2}} B A^{t} B A^{\frac{1-t}{2}}\right)^{\frac{1}{2}}\left(\operatorname{tr} A^{\frac{s}{2}} B A^{1-s} B A^{\frac{s}{2}}\right)^{\frac{1}{2}} \\ &= \left(\operatorname{tr} A^{t} B A^{1-t} B\right)^{\frac{1}{2}}\left(\operatorname{tr} A^{s} B A^{1-s} B\right)^{\frac{1}{2}} \\ &= (f(t))^{\frac{1}{2}}(f(s))^{\frac{1}{2}} \\ &\leqslant \frac{1}{2}(f(s)+f(t)). \end{aligned}$$

Thus, $f(t)$ is decreasing for $0 \leqslant t \leqslant \frac{1}{2}$, increasing for $\frac{1}{2} \leqslant t \leqslant 1$, attains its minimum at $t=\frac{1}{2}$, and attains its maximum at $t=0$ and $t=1$.

Another proof of the inequality (2) can be concluded from Lemma 2 in [7]. We remark here that the inequality (2) is equivalent to saying that

$$\operatorname{tr} A^{\alpha} B A^{\beta} B \leqslant \operatorname{tr} A^{\gamma} B A^{\delta} B$$

for $\alpha,\beta,\gamma,\delta \geqslant 0$ with $\alpha+\beta=\gamma+\delta$ and

$$\max\{\alpha,\beta\} \leqslant \max\{\gamma,\delta\}.$$

Related classical trace inequalities, based on log convexity results, can be found in [8], [13], and [14].

The second inequality in (1) is a particular case of the inequality

$$\left|\operatorname{tr} A^{s} B^{t} A^{1-s} B^{1-t}\right| \leqslant \operatorname{tr} AB, \tag{3}$$

where A and B are positive semidefinite matrices and $0 \leqslant s,t \leqslant 1$.

In [1], T. Ando, F. Hiai, and K. Okubo proved that the inequality (3) holds for all non-negative real numbers s,t for which

$$\left|s-\frac{1}{2}\right|+\left|t-\frac{1}{2}\right| \leqslant \frac{1}{2}.$$

It is natural to ask what is the complete range of validity of the inequality (3). Plevnik [14] gave a counterexample to the inequality (3). He answered it in the negative for $s=\frac{4}{5}, t=\frac{1}{5}$.

Recently, M. Hayajneh, S. Hayajneh, and F. Kittaneh [11] generalized the inequality (3) by proving that the inequality

$$\left|\operatorname{tr} A^{w} B^{z} A^{1-w} B^{1-z}\right| \leqslant \operatorname{tr} AB, \tag{4}$$

holds for all complex numbers w, z for which

$$\left|\operatorname{Re} w - \frac{1}{2}\right| + \left|\operatorname{Re} z - \frac{1}{2}\right| \leqslant \frac{1}{2}.$$

A special case of the inequality (4) when $w = z$ is the inequality

$$\left|\operatorname{tr} A^z B^z A^{1-z} B^{1-z}\right| \leqslant \operatorname{tr} AB.$$

In [5], Bottazzi et al. have proved this inequality under the condition that

$$\frac{1}{4} \leqslant \operatorname{Re} z \leqslant \frac{3}{4}.$$

We mention here that the inequality (3) has been studied by several authors as in [3], [9], and [10].

Section 2 is devoted to proving the following Hilbert-Schmidt norm inequality as the first application of the inequality (2):

$$\left\|A^s B + B A^{1-s}\right\|_2 \leqslant \left\|A^t B + B A^{1-t}\right\|_2$$

for $\frac{1}{2} \leqslant s \leqslant t \leqslant 1$, where A is a positive semidefinite matrix and and B is a Hermitian matrix.

In Section 3, we prove the following trace inequality as the second application of the ineqaulity (2):

$$\operatorname{tr} A^t B A^{1-t} (\log A) B \leqslant \operatorname{tr} A^t (\log A) B A^{1-t} B,$$

where A is a positive definite matrix, B is a Hermitian matrix, and $\frac{1}{2} \leqslant t \leqslant 1$. As a consequence of this trace inequality, we prove that the inequality

$$\left\|A^t B + B A^{1-t} \log A\right\|_2 \leqslant \left\|A^t (\log A) B + B A^{1-t}\right\|_2$$

holds for $\frac{1}{2} \leqslant t \leqslant 1$, where A is a positive definite matrix with $\sigma(A) \subseteq \left[e^{-1}, 1\right] \cup [e, \infty)$ and B is a Hermitian matrix.

It would be interesting to investigate the following conjectures concerning the generalizations of our Hilbert-Schmidt norm inequalities to the wider class of unitarily invariant norms.

CONJECTURE 1. Let A be a positive semidefinite matrix and B be a Hermitian matrix. Then for $\frac{1}{2} \leqslant s \leqslant t \leqslant 1$ and for every unitarily invariant norm, we have

$$\left|\left|\left|A^s B + B A^{1-s}\right|\right|\right| \leqslant \left|\left|\left|A^t B + B A^{1-t}\right|\right|\right|.$$

CONJECTURE 2. Let A be a positive definite matrix such that $\sigma(A) \subseteq \left[e^{-1}, 1\right] \cup [e, \infty)$ and B be a Hermitian matrix. Then for $\frac{1}{2} \leqslant t \leqslant 1$ and for every unitarily invariant norm, we have

$$\left|\left|\left|A^t B + B A^{1-t} \log A\right|\right|\right| \leqslant \left|\left|\left|A^t (\log A) B + B A^{1-t}\right|\right|\right|.$$

In Section 4, we present further applications of the inequality (2). These applications contain trace inequalities involving means of two non-negative real numbers, which include a generalization of the Ando-Hiai-Okubo trace inequalities (1). We conclude the paper with a general trace inequality for products of positive definite matrices, which is related to the inequality (2).

2. A new Hilbert-Schmidt norm inequality

In this section, we affirmatively settle Conjecture 1 for the Hilbert-Schmidt norm. This application is a Hilbert-Schmidt norm inequality, which asserts that

$$\left\|A^sB+BA^{1-s}\right\|_2 \leqslant \left\|A^tB+BA^{1-t}\right\|_2$$

where A is a positive semidefinite matrix, B is a Hermitian matrix, and $\frac{1}{2} \leqslant s \leqslant t \leqslant 1$.

A useful lemma for our purpose is the following.

LEMMA 1. *Let A and C be any two positive semidefinite matrices. Then the function $g(t) = \operatorname{tr}\left(A^t + A^{1-t}\right)C$ is increasing for $\frac{1}{2} \leqslant t \leqslant 1$.*

Proof. Without loss of generality, we may assume that A is a positive definite matrix. The general case follows by a continuity argument.

By the spectral theorem, it is evident that the matrix $\left(A^t - A^{1-t}\right)\log A$ is a positive semidefinite matrix for $\frac{1}{2} \leqslant t \leqslant 1$. Since C is a positive semidefinite matrix, it follows that

$$\frac{d}{dt}g(t) = \operatorname{tr}\left(A^t\log A - A^{1-t}\log A\right)C \geqslant 0.$$

Therefore, $g(t)$ is increasing for $\frac{1}{2} \leqslant t \leqslant 1$. □

THEOREM 1. *Let A be a positive semidefinite matrix and B be a Hermitian matrix. Then*

$$\left\|A^sB+BA^{1-s}\right\|_2 \leqslant \left\|A^tB+BA^{1-t}\right\|_2 \quad \text{for} \quad \frac{1}{2} \leqslant s \leqslant t \leqslant 1.$$

In other words, the function $h(t) = \left\|A^tB+BA^{1-t}\right\|_2$ is increasing for $\frac{1}{2} \leqslant t \leqslant 1$.

Proof. Using the fact that for any matrix X, $\|X\|_2^2 = \operatorname{tr} X^*X$, we have

$$\begin{aligned}
(h(t))^2 &= \left\|A^tB+BA^{1-t}\right\|_2^2 \\
&= \operatorname{tr}\left(BA^t + A^{1-t}B\right)\left(A^tB+BA^{1-t}\right) \\
&= \operatorname{tr}\left(A^{2t}B^2 + A^{2(1-t)}B^2\right) + 2\operatorname{tr} A^tBA^{1-t}B \\
&= \operatorname{tr}\left(A^{2t}+A^{2(1-t)}\right)B^2 + 2\operatorname{tr} A^tBA^{1-t}B.
\end{aligned}$$

Replacing A by A^2 and taking $C = B^2$ in Lemma 1, we see that tr $\left(A^{2t} + A^{2(1-t)}\right) B^2$ is increasing for $\frac{1}{2} \leqslant t \leqslant 1$. Since tr $A^t B A^{1-t} B$ is increasing for $\frac{1}{2} \leqslant t \leqslant 1$, it follows that $h(t)$ is increasing for $\frac{1}{2} \leqslant t \leqslant 1$. This completes the proof of the theorem. □

The arithmetic-geometric mean inequality for unitarily invariant norms (see, e.g., [4] or [12]) says that if S and T are positive semidefinite matrices, then for every matrix X and every unitarily invariant norm, we have

$$2|||SXT||| \leqslant \left|\left|\left|S^2 X + X T^2\right|\right|\right|.$$

Using the triangle inequality, the self-adjointness of unitarily invariant norms, and the arithmetic-geometric mean inequality for unitarily invariant norms, we have

$$\begin{aligned}\left|\left|\left|A^{\frac{1}{2}} B + B A^{\frac{1}{2}}\right|\right|\right| &\leqslant \left|\left|\left|A^{\frac{1}{2}} B\right|\right|\right| + \left|\left|\left|B A^{\frac{1}{2}}\right|\right|\right| \\ &= 2\left|\left|\left|A^{\frac{1}{2}} B\right|\right|\right| \\ &\leqslant |||AB + B|||,\end{aligned}$$

where A is a positive semidefinite matrix and B is a Hermitian matrix. This affirmatively settles Conjecture 1 for the case $s = \frac{1}{2}$ and $t = 1$.

3. Related inequalities

The following trace inequality is the second application of the inequality (2).

THEOREM 2. *Let A be a positive definite matrix and B be a Hermitian matrix. Then for $\frac{1}{2} \leqslant t \leqslant 1$, we have*

$$\operatorname{tr} A^t B A^{1-t} (\log A) B \leqslant \operatorname{tr} A^t (\log A) B A^{1-t} B.$$

Proof. Consider $f(t) = \operatorname{tr} A^t B A^{1-t} B$. Then we have

$$\begin{aligned}\frac{d}{dt} f(t) &= \operatorname{tr}\left(\frac{d}{dt}\left(A^t B\right) A^{1-t} B + A^t B \frac{d}{dt}\left(A^{1-t} B\right)\right) \\ &= \operatorname{tr}\left(-A^t B A^{1-t} (\log A) B + A^t (\log A) B A^{1-t} B\right).\end{aligned}$$

Since the function $f(t) = \operatorname{tr} A^t B A^{1-t} B$ is increasing for $\frac{1}{2} \leqslant t \leqslant 1$, it follows that $\frac{d}{dt} f(t) \geqslant 0$. Thus,

$$\operatorname{tr} A^t B A^{1-t} (\log A) B \leqslant \operatorname{tr} A^t (\log A) B A^{1-t} B.$$

This completes the proof of the theorem. □

Letting $t = 1$ in Theorem 2, we have the following corollary.

COROLLARY 1. *Let A be a positive definite matrix and B be a Hermitian matrix. Then*

$$\operatorname{tr} AB(\log A)B \leqslant \operatorname{tr} A(\log A)B^2. \tag{5}$$

It should be mentioned here that the inequality (5) can also be concluded from Theorem 1.2 in [6].

The following norm inequality is a another consequence of Theorem 2.

THEOREM 3. *Let A be a positive definite matrix such that $\sigma(A) \subseteq \left[e^{-1}, 1\right] \cup [e, \infty)$ and B be a Hermitian matrix. Then for $\frac{1}{2} \leqslant t \leqslant 1$, we have*

$$\left\|A^t B + BA^{1-t}(\log A)\right\|_2 \leqslant \left\|A^t(\log A)B + BA^{1-t}\right\|_2.$$

Proof. We can see that the square of the right-hand side of the desired norm inequality is equal to

$$\operatorname{tr}\left(A^{2t}(\log A)^2 B^2 + A^{2(1-t)}B^2\right) + 2\operatorname{tr} A^t(\log A)BA^{1-t}B$$

and the square of the left-hand side is equal to

$$\operatorname{tr}\left(A^{2t}B^2 + A^{2(1-t)}(\log A)^2 B^2\right) + 2\operatorname{tr} A^t BA^{1-t}(\log A)B.$$

Note that $\operatorname{tr} A^t BA^{1-t}(\log A)B \leqslant \operatorname{tr} A^t(\log A)BA^{1-t}B$ by Theorem 2. Thus, it is enough to show that

$$\operatorname{tr}\left(A^{2t}B^2 + A^{2(1-t)}(\log A)^2 B^2\right) \leqslant \operatorname{tr}\left(A^{2t}(\log A)^2 B^2 + A^{2(1-t)}B^2\right). \tag{6}$$

By the spectral theorem, it is evident that $\sigma(A) \subseteq \left[e^{-1}, 1\right] \cup [e, \infty)$ implies that the matrix $\left(A^{2t} - A^{2(1-t)}\right)\left((\log A)^2 - I\right)$ is a positive semidefinite matrix. Since B^2 is also positive semidefinite, it follows that

$$\operatorname{tr}\left(A^{2t} - A^{2(1-t)}\right)\left((\log A)^2 - I\right)B^2 \geqslant 0.$$

This gives the inequality (6).

Thus,

$$\begin{aligned}
&\operatorname{tr}\left(A^{2t}(\log A)^2 B^2 + A^{2(1-t)}B^2\right) + 2\operatorname{tr} A^t(\log A)BA^{1-t}B \\
&\geqslant \operatorname{tr}\left(A^{2t}B^2 + A^{2(1-t)}(\log A)^2 B^2\right) + 2\operatorname{tr} A^t BA^{1-t}(\log A)B.
\end{aligned}$$

Hence, the desired norm inequality is valid for $\frac{1}{2} \leqslant t \leqslant 1$. □

Note that if we set $t = \frac{1}{2}$ in Theorem 3, the inequality becomes equality, but if we set $t = 1$, we get the following inequality.

COROLLARY 2. *Let A be a positive definite matrix such that $\sigma(A) \subseteq \left[e^{-1}, 1\right] \cup [e, \infty)$ and B be a Hermitian matrix. Then*

$$\|AB + B(\log A)\|_2 \leqslant \|A(\log A)B + B\|_2.$$

4. Further applications

In this section, we give more applications of the inequality (2). These applications contain trace inequalities involving means of two non-negative real numbers, which include a generalization of the Ando-Hiai-Okubo trace inequalities (1). Here, we assume that A is a positive semidefinite matrix, B is a Hermitian matrix, $a,b \geqslant 0$, and $\frac{1}{2} \leqslant r \leqslant 1$.

REMARK 1. Let $f(a,b)$ and $g(a,b)$ be means of a and b. Then

$$\operatorname{tr} A^{g\left(r,\frac{1}{2}\right)}BA^{1-g\left(r,\frac{1}{2}\right)}B \leqslant \operatorname{tr} A^{r}BA^{1-r}B \leqslant \operatorname{tr} A^{f(r,1)}BA^{1-f(r,1)}B. \tag{7}$$

In fact, since $f(a,b)$ and $g(a,b)$ are means of a and b and $\frac{1}{2} \leqslant r \leqslant 1$, it follows by the internality property that

$$\frac{1}{2} \leqslant g\left(r,\frac{1}{2}\right) \leqslant r \leqslant f(r,1) \leqslant 1.$$

Therefore, using the inequality (2), we have

$$\operatorname{tr} A^{g\left(r,\frac{1}{2}\right)}BA^{1-g\left(r,\frac{1}{2}\right)}B \leqslant \operatorname{tr} A^{r}BA^{1-r}B \leqslant \operatorname{tr} A^{f(r,1)}BA^{1-f(r,1)}B.$$

The following example is derived from the inequality (1).

EXAMPLE 1. Let $f(a,b) = \max\{a,b\}$ and $g(a,b) = \min\{a,b\}$ in the inequality (1). Then

$$\operatorname{tr}\left(A^{\frac{1}{2}}B\right)^{2} \leqslant \operatorname{tr} A^{r}BA^{1-r}B \leqslant \operatorname{tr} AB^{2}. \tag{8}$$

The inequalities (8) yeild the inequalities (1) when B is a positive semidefinite matrix.

Another related trace inequality is

$$\operatorname{tr} A^{\alpha}BA^{\beta}B \leqslant \frac{1}{2}\operatorname{tr}\left(A^{\alpha+\eta}BA^{\beta-\eta}B + A^{\alpha-\eta}BA^{\beta+\eta}B\right), \tag{9}$$

where A is a positive semidefinite matrix, B is a Hermitian matrix and $\alpha,\beta \geqslant \eta \geqslant 0$.

To prove the inequality (9), let $C = BA^{\frac{\beta+\eta}{2}} - A^{\eta}BA^{\frac{\beta-\eta}{2}}$ and $R = A^{\alpha-\eta}$. Since $\operatorname{tr} RCC^{*} \geqslant 0$, it follows that

$$\operatorname{tr} A^{\alpha-\eta}\left(BA^{\frac{\beta+\eta}{2}} - A^{\eta}BA^{\frac{\beta-\eta}{2}}\right)\left(A^{\frac{\beta+\eta}{2}}B - A^{\frac{\beta-\eta}{2}}BA^{\eta}\right) \geqslant 0,$$

which is equivalent to the inequality (9).

It is interesting to see that the inequality (9) gives another proof of the convexity of the function $f(t)$. To see this, replace A by $A^{\frac{1}{\alpha+\beta}}$ in the inequality (9) and set $s = \frac{\alpha+\eta}{\alpha+\beta}$, $t = \frac{\alpha-\eta}{\alpha+\beta}$ to get $f\left(\frac{s+t}{2}\right) \leqslant \frac{1}{2}\left(f(s)+f(t)\right)$.

REMARK 2. Since the function $f(t) = \operatorname{tr} A^{t}BA^{1-t}B$ is logarithmically convex (and hence it is convex) for $0 \leqslant t \leqslant 1$, it follows that $\frac{d^2}{dt^2}f(t) \geqslant 0$. Thus, for a positive definite matrix A and a Hermitian matrix B, we have the trace inequality

$$\operatorname{tr} A^{t}(\log A)BA^{1-t}(\log A)B \leqslant \frac{1}{2}\operatorname{tr}\left(A^{t}BA^{1-t}(\log A)^{2}B + A^{1-t}BA^{t}(\log A)^{2}B\right). \tag{10}$$

Letting $t=\frac{1}{2}$ in the inequality (10), we obtain the inequality

$$\operatorname{tr}\left(A^{\frac{1}{2}}(\log A)B\right)^2 \leqslant \operatorname{tr} A^{\frac{1}{2}}BA^{\frac{1}{2}}(\log A)^2 B.$$

Letting $t=0$ or $t=1$ in the inequality (10), we obtain the inequality

$$\operatorname{tr}\,(\log A)BA(\log A)B \leqslant \frac{1}{2}\operatorname{tr}\left(AB(\log A)^2B + BA(\log A)^2B\right).$$

REMARK 3. It should be mentioned here that the functions $g(t)$ given in Lemma 1 and $h(t)$ given in Theorem 1 are also logarithmically convex (and hence they are convex) for $0\leqslant t\leqslant 1$, symmetric about $t=\frac{1}{2}$, decreasing for $0\leqslant t\leqslant\frac{1}{2}$, increasing for $\frac{1}{2}\leqslant t\leqslant 1$, attain their minima at $t=\frac{1}{2}$, and attain their maxima at $t=0$ and $t=1$.

We conclude the paper with a general trace inequality, from which we obtain a trace inequality related to those given in the previous sections.

THEOREM 4. *Let T be a positive definite matrix, X,Y be positive semidefinite matrices, and B be a Hermitian matrix. Then*

$$\operatorname{tr}\left(T^{\frac{1}{2}}YT^{-\frac{1}{2}}BXB+T^{-\frac{1}{2}}YT^{\frac{1}{2}}BXB\right) \leqslant \operatorname{tr}\left(T^{-\frac{1}{2}}YT^{-\frac{1}{2}}BX^{\frac{1}{2}}TX^{\frac{1}{2}}B+T^{\frac{1}{2}}YT^{\frac{1}{2}}BX^{\frac{1}{2}}T^{-1}X^{\frac{1}{2}}B\right).$$

If, in addition, T commutes with X and Y, then

$$\operatorname{tr} YBXB \leqslant \frac{1}{2}\operatorname{tr}\left(YT^{-1}BXTB + YTBXT^{-1}B\right).$$

Proof. Let $C=BX^{\frac{1}{2}}T^{\frac{1}{2}}-TBX^{\frac{1}{2}}T^{-\frac{1}{2}}$ and $R=T^{-\frac{1}{2}}YT^{-\frac{1}{2}}$. Since $\operatorname{tr} RCC^*\geqslant 0$, it follows that

$$\operatorname{tr}\,T^{-\frac{1}{2}}YT^{-\frac{1}{2}}\left(BX^{\frac{1}{2}}T^{\frac{1}{2}}-TBX^{\frac{1}{2}}T^{-\frac{1}{2}}\right)\left(T^{\frac{1}{2}}X^{\frac{1}{2}}B-T^{-\frac{1}{2}}X^{\frac{1}{2}}BT\right)\geqslant 0,$$

which is equivalent to

$$\operatorname{tr}\left(T^{\frac{1}{2}}YT^{-\frac{1}{2}}BXB+T^{-\frac{1}{2}}YT^{\frac{1}{2}}BXB\right) \leqslant \operatorname{tr}\left(T^{-\frac{1}{2}}YT^{-\frac{1}{2}}BX^{\frac{1}{2}}TX^{\frac{1}{2}}B+T^{\frac{1}{2}}YT^{\frac{1}{2}}BX^{\frac{1}{2}}T^{-1}X^{\frac{1}{2}}B\right).$$

This completes the proof of the theorem. □

Based on Theorem 4, we have the following trace inequality, which is closely related to the one given in the inequality (9). In this inequality, the positivity of the matrix A is strengthend, while the positivity of the exponents is released.

COROLLARY 3. *Let A be a positive definite matrix and B be a Hermitian matrix. Then for the real numbers α,β,η, we have*

$$\operatorname{tr} A^{\alpha}BA^{\beta}B \leqslant \frac{1}{2}\operatorname{tr}\left(A^{\alpha+\eta}BA^{\beta-\eta}B + A^{\alpha-\eta}BA^{\beta+\eta}B\right).$$

Proof. The result follows immediately by replacing X,Y,T by A^{β}, A^{α}, A^{η}, respectively in Theorem 4. □

Note that if we restrict the values of α,β,η in Corollary 3 such that $\alpha,\beta\geqslant\eta\geqslant 0$ and if we use a continuity argument, then we retain the inequality (9).

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