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# MONOTONICITY PROPERTIES AND BOUNDS INVOLVING THE COMPLETE ELLIPTIC INTEGRALS OF THE FIRST KIND

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*Abstract.* In the article, we establish several monotonicity properties of the functions involving the complete elliptic integral of the first kind. As applications, we present sharp bounds for the complete elliptic integral of the first kind and the arithmetic-geometric mean.

## 1. Introduction

The well-known complete elliptic integrals $\mathscr{K}(r)$ and $\mathscr{E}(r)$ [15] of the first and second kinds are defined as

$$\mathscr{K}(r)=\int_0^{\pi/2}\frac{dt}{\sqrt{1-r^2\sin^2(t)}},\quad \mathscr{E}(r)=\int_0^{\pi/2}\sqrt{1-r^2\sin^2(t)}dt\quad (0<r<1),$$

$$\mathscr{K}\left(0^+\right)=\mathscr{E}\left(0^+\right)=\frac{\pi}{2},\quad \mathscr{E}\left(1^-\right)=1,\quad \mathscr{K}\left(1^-\right)=\infty.$$

Both $\mathscr{K}(r)$ and $\mathscr{E}(r)$ are the particular cases of the Gaussian hypergeometric function [57–59, 63, 64, 75, 83]

$$F(a,b;c;x)=\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n)}\frac{x^n}{n!}\quad (-1<x<1), \tag{1.1}$$

where $(a,0)=1$ for $a\neq 0$, $(a,n)=\Gamma(a+n)/\Gamma(a)$ and $\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt$ $(x>0)$ is the classical gamma function [74, 78, 79, 82, 84, 85]. In facts, $\mathscr{K}(r)$ and $\mathscr{E}(r)$ can be expressed by

$$\mathscr{K}(r)=\frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2};1;r^2\right)=\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2},n\right)^2}{(n!)^2}r^{2n}, \tag{1.2}$$

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$$\mathscr{E}(r)=\frac{\pi}{2}F\left(-\frac{1}{2},\frac{1}{2};1;r^2\right)=\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2},n\right)\left(\frac{1}{2},n\right)}{(n!)^2}r^{2n}.$$

There are close connections between the complete elliptic integrals and bivariate means. For example, the Toader mean $T(a,b)$ [22, 24, 25, 27] and the arithmetic-geometric mean $AG(a,b)$ [13, 14, 17, 44, 51–56] of two positive real numbers $a$ and $b$ with $a>b$ can be expressed as

$$T(a,b)=\frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^2\cos^2 t+b^2\sin^2 t}dt=\frac{2a}{\pi}\mathscr{E}\left(\sqrt{1-\left(\frac{b}{a}\right)^2}\right), \tag{1.3}$$

$$AG(a,b)=\frac{\pi a}{2\mathscr{K}\left(\sqrt{1-(\frac{b}{a})^2}\right)}. \tag{1.4}$$

The identity (1.4) is called Gaussian identity [11] and the arithmetic-geometric mean $AG(a,b)$ is defined as the common limit of the sequences $\{a_n\}$ and $\{b_n\}$ as follows:

$$a_0=a,\quad b_0=b,\quad a_{n+1}=\frac{a_n+b_n}{2},\quad b_{n+1}=\sqrt{a_nb_n}.$$

Recently, the complete elliptic integrals $\mathscr{K}(r)$ and $\mathscr{E}(r)$ have been the subject of intensive research [19, 20, 23, 26, 48, 60–62, 65, 66, 76, 77]. In particular, many remarkable inequalities and applications for $\mathscr{K}(r)$, $\mathscr{E}(r)$ and other related special functions can be found in the literature [1–3, 6–9, 12, 18, 21, 28–38, 42, 50, 67–73].

Carlson and Gustafson [16] proved that the double inequality

$$\log\frac{4}{r'}<\mathscr{K}(r)<\frac{4}{3+r^2}\log\frac{4}{r'} \tag{1.5}$$

holds for all $r\in(0,1)$. Here and in what follows we denote $r'=\sqrt{1-r^2}$.

The lower bound given in (1.5) was improved by Kühnau [41] as follows

$$\mathscr{K}(r)>\frac{9}{8+r^2}\log\frac{4}{r'}$$

for all $r\in(0,1)$.

In [4, 10, 17, 49, 56], the authors proved that the two-sided inequalities

$$\frac{\log r'}{r'-1}<\mathscr{K}(r)<\frac{\pi\log r'}{2(r'-1)},$$

$$\left[1+\left(\frac{\pi}{4\log 2}-1\right)r'^2\right]\log\frac{4}{r'}<\mathscr{K}(r)<\left(1+\frac{1}{4}r'^2\right)\log\frac{4}{r'},$$

$$\frac{\pi}{2}\left[\frac{\tanh^{-1}(r)}{r}\right]^{1/2}<\mathscr{K}(r)<\frac{\pi}{2}\frac{\tanh^{-1}(r)}{r} \tag{1.6}$$

are valid for all $r > 0$, where $\tanh^{-1}(r) = \log[(1+r)/(1-r)]/2$ is the inverse hyperbolic tangent function.

Alzer and Qiu [5], and Yang et al. [81] improved the lower bound given in (1.6) independently as follows:

$$\mathscr{K}(r) > \frac{\pi}{2}\left[\frac{\tanh^{-1}(r)}{r}\right]^{3/4}$$

for all $r \in (0,1)$.

The main purpose of the article is to establish the monotonicity properties of the functions involving the complete elliptic integral $\mathscr{K}(r)$ and provide the sharp bounds for $\mathscr{K}(r)$ and $AG(1,r)$ in terms of elementary functions.

## 2. Lemmas

In order to prove our main results we need several formulas and lemmas, which we present in this section.

The hypergeometric function $F(a,b,c;x)$ and the complete elliptic integrals $\mathscr{K}(r)$ and $\mathscr{E}(r)$ have the following formulas (See [11, (1.16), 1.19(4), 1.20(10), 1.48, (3.6)]):

$$\frac{d^n}{dx^n}F(a,b,c;x) = \frac{(a,n)(b,n)}{(c,n)}F(a+n,b+n;c+n;x), \tag{2.1}$$

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{2.2}$$

if $c > a+b$.

$$F(a,b;a+b+1;x) = (1-x)F(a+1,b+1;a+b+1;x), \tag{2.3}$$

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}F(a,b;a+b;x) + \log(1-x) + \psi(a) + \psi(b) + 2\gamma = O((1-x)\log(1-x)) \tag{2.4}$$

as $x \to 1$, where

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

is the psi function and

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right) = 0.57721566\cdots$$

is the Euler-Mascheroni constant.

$$\frac{d\mathscr{K}(r)}{dr} = \frac{\mathscr{E}(r) - r'^2\mathscr{K}(r)}{rr'^2}. \tag{2.5}$$

LEMMA 2.1. (See [80, Lemma 2.1]) *Let $r > 0$, $\{a_k\}_{k=0}^{\infty}$ be a nonnegative real sequence with $a_m > 0$ and $\sum_{k=m+1}^{\infty} a_k > 0$, and*

$$S(t) = \sum_{k=0}^{m} a_k t^k - \sum_{k=m+1}^{\infty} a_k t^k$$

*be a convergent power series on the interval $(0,r)$. Then $S(t) > 0$ for all $t \in (0,r)$ if $S(r^-) \geqslant 0$.*

LEMMA 2.2. (See [47, (2.13)]) *The inequality*

$$\frac{\Gamma^2(n+1/2)}{\Gamma^2(n+1)} < \frac{1}{n+1/4}$$

*is valid for all $n \in \mathbb{N}$.*

LEMMA 2.3. *Let $n \in \mathbb{N}$ and*

$$\lambda_n = \psi(n) - \frac{4n^2 - 17n + 1}{2(4n+1)} + \gamma. \tag{2.6}$$

*Then $\lambda_n < 0$ for $n \geqslant 11$.*

*Proof.* Elaborated computations lead to

$$\lambda_{11} = -\frac{107}{280} < 0, \tag{2.7}$$

$$\lambda_{n+1} - \lambda_n = -\frac{16n^3 - 8n^2 - 65n - 10}{2n(4n+1)(4n+5)} < 0 \tag{2.8}$$

for $n \geqslant 3$.

Therefore, Lemma 2.3 follows easily from (2.7) and (2.8). □

LEMMA 2.4. *Let $n, k \in \mathbb{N}$ with $k \leqslant n$ and*

$$v_k = \frac{1}{(k+1)(n-k+1)(k+1/4)(n-k+1/4)}, \tag{2.9}$$

$$\omega_n = \left(n + \frac{1}{4}\right) \sum_{k=0}^{n} v_k - \frac{6(2n+1)}{(n+1)(n+2)}. \tag{2.10}$$

*Then $\omega_n < 0$ for $n \geqslant 8$.*

*Proof.* Let $n \geqslant 8$, $1 \leqslant k \leqslant n-1$ and

$$\xi_k = \frac{1}{k(k+1)(n-k)(n-k+1)}. \tag{2.11}$$

Then we clearly see that

$$\xi_k = \frac{1}{n(n+1)}\left(\frac{1}{k}+\frac{1}{n-k}\right) - \frac{1}{(n+1)(n+2)}\left(\frac{1}{k+1}+\frac{1}{n-k+1}\right),$$

$$\sum_{k=1}^{n-1}\xi_k = \frac{2(\psi(n)+\gamma)}{n(n+1)} - \frac{2(\psi(n)+1/n+\gamma-1)}{(n+1)(n+2)} = \frac{2(2\psi(n)+n+2\gamma-1)}{n(n+1)(n+2)}. \tag{2.12}$$

Note that

$$\left(k+\frac{1}{4}\right)\left(n-k+\frac{1}{4}\right) > k(n-k) \tag{2.13}$$

for all $k,n \in \mathbb{N}$.

It follows (2.9), (2.11) and (2.13) that

$$v_0 + v_n = \frac{32}{(n+1)(4n+1)},$$

$$\sum_{k=0}^{n} v_k = \sum_{k=1}^{n-1} v_k + \frac{32}{(n+1)(4n+1)} < \sum_{k=1}^{n-1}\xi_k + \frac{32}{(n+1)(4n+1)}. \tag{2.14}$$

From (2.10), (2.12) and (2.14) we have

$$\omega_n < \frac{2(2\psi(n)+n+2\gamma-1)(n+1/4)}{n(n+1)(n+2)} + \frac{32(n+1/4)}{(n+1)(4n+1)} - \frac{6(2n+1)}{(n+1)(n+2)} \tag{2.15}$$
$$= \frac{4n+1}{n(n+1)(n+2)}\lambda_n,$$

where $\lambda_n$ is given by (2.6).

Elaborated computations lead to

$$\omega_8 = -\frac{1855051}{114400650} < 0, \quad \omega_9 = -\frac{3251242}{111035925} < 0, \quad \omega_{10} = -\frac{1777462049611}{46588453411500} < 0. \tag{2.16}$$

Therefore, Lemma 2.4 follows easily from Lemma 2.3, (2.15) and (2.16). □

LEMMA 2.5. *Let $k,n \in \mathbb{N}$ with $k \leqslant n$, and*

$$W_n = \frac{\Gamma(n+1/2)}{\Gamma(1/2)\Gamma(n+1)}, \tag{2.17}$$

$$u_n = \pi\sum_{k=0}^{n}\frac{W_k^2 W_{n-k}^2}{(k+1)(n-k+1)} - \frac{6(2n+1)W_n^2}{(n+1)(n+2)}. \tag{2.18}$$

*Then $u_n < 0$ for all $n \geqslant 8$.*

*Proof.* From (2.17) we clearly see that the sequence $\{(k+1/4)W_k^2\}_{k=0}^{\infty}$ is strictly increasing, which leads to the conclusion that

$$\left(k+\frac{1}{4}\right)W_k^2 \leqslant \left(n+\frac{1}{4}\right)W_n^2,$$

$$W_k^2 W_{n-k}^2 \leqslant \frac{(n+1/4)^2 W_n^4}{(k+1/4)(n-k+1/4)} \tag{2.19}$$

for $0 \leqslant k \leqslant n$.

Let $\omega_n$ be defined by (2.10), then it follows from (2.9), (2.17), (2.18) and (2.19) together with Lemma 2.2 that

$$\begin{aligned}\frac{u_n}{W_n^2} &\leqslant \pi \sum_{k=0}^{n} \frac{(n+1/4)^2 W_n^2}{(k+1)(n-k+1)(k+1/4)(n-k+1/4)} - \frac{6(2n+1)}{(n+1)(n+2)} \\ &< \sum_{k=0}^{n} \frac{n+1/4}{(k+1)(n-k+1)(k+1/4)(n-k+1/4)} - \frac{6(2n+1)}{(n+1)(n+2)} = \omega_n.\end{aligned} \tag{2.20}$$

Therefore, Lemma 2.5 follows from Lemma 2.4 and (2.20). □

## 3. Main results

THEOREM 3.1. *The function* $r \mapsto r'^p e^{\mathscr{K}(r)}$ *is strictly increasing on* $(0,1)$ *if and only if* $p \leqslant \pi/4$ *and strictly decreasing on* $(0,1)$ *if and only if* $p \geqslant 1$.

*Proof.* Let $x = r^2 \in (0,1)$ and

$$G_1(x) = (1-x)^{p/2} e^{\mathscr{K}(\sqrt{x})} = (1-x)^{p/2} e^{\frac{\pi}{2}F(1/2,1/2;1;x)} = r'^p e^{\mathscr{K}(r)}. \tag{3.1}$$

Then (2.1) and (2.3) lead to

$$\begin{aligned}G_1'(x) &= -\frac{p}{2}(1-x)^{p/2-1} e^{\mathscr{K}(\sqrt{x})} + \frac{\pi}{8}(1-x)^{p/2} F(3/2,3/2;2;x) e^{\mathscr{K}(\sqrt{x})} \\ &= -\frac{p}{2}(1-x)^{p/2-1} e^{\mathscr{K}(\sqrt{x})} + \frac{\pi}{8}(1-x)^{p/2-1} F(1/2,1/2;2;x) e^{\mathscr{K}(\sqrt{x})} \\ &= -\frac{1}{2}(1-x)^{p/2-1} e^{\mathscr{K}(\sqrt{x})}\left(p - \frac{\pi}{4}F(1/2,1/2;2;x)\right).\end{aligned} \tag{3.2}$$

From (3.1) and (3.2) we clearly see that the function $r \to r'^p e^{\mathscr{K}(r)}$ is strictly increasing on $(0,1)$ if and only

$$p \leqslant \frac{\pi}{4} \inf_{x\in(0,1)} F\left(\frac{1}{2},\frac{1}{2};2;x\right)$$

and strictly decreasing on $(0,1)$ if and only if

$$p \geqslant \frac{\pi}{4} \sup_{x\in(0,1)} F\left(\frac{1}{2},\frac{1}{2};2;x\right).$$

It follows from (1.1) and (2.2) that

$$\inf_{x\in(0,1)} F\left(\frac{1}{2},\frac{1}{2};2;x\right) = F\left(\frac{1}{2},\frac{1}{2};2;0^+\right) = 1,$$

$$\sup_{x\in(0,1)} F\left(\frac{1}{2},\frac{1}{2};2;x\right) = F\left(\frac{1}{2},\frac{1}{2};2;1^-\right) = \frac{\Gamma(2)\Gamma(1)}{\Gamma^2(3/2)} = \frac{4}{\pi}. \quad \square$$

THEOREM 3.2. *The function*

$$r \mapsto \frac{r'}{r^2}\left[\mathscr{E}(r) - r'^2\mathscr{K}(r)\right]e^{\mathscr{K}(r)}$$

*is strictly increasing from* $(0,1)$ *onto* $(\pi e^{\pi/2}/4, 4)$.

*Proof.* It follows from (1.2), (2.1), (2.3) and (2.5) that

$$\frac{\mathscr{E}(r) - r'^2\mathscr{K}(r)}{rr'^2} = \frac{d\mathscr{K}(r)}{dr} = \frac{\pi r}{4}F(3/2,3/2;2;r^2) = \frac{\pi r}{4r'^2}F(1/2,1/2;2;r^2),$$

$$\begin{aligned} \frac{r'}{r^2}\left[\mathscr{E}(r) - r'^2\mathscr{K}(r)\right]e^{\mathscr{K}(r)} &= \frac{\pi}{4}\sqrt{1-r^2}F(1/2,1/2;2;r^2)e^{\mathscr{K}(r)} \\ &= \frac{\pi}{4}\sqrt{1-r^2}F(1/2,1/2;2;r^2)e^{\frac{\pi}{2}F(1/2,1/2;1;r^2)}. \end{aligned} \tag{3.3}$$

Let $x = r^2$, $W_n$ and $u_n$ be respectively defined by (2.17) and (2.18), and

$$\begin{aligned} G_2(x) &= \frac{\pi}{4}\sqrt{1-x}F(1/2,1/2;2;x)e^{\mathscr{K}(\sqrt{x})} \\ &= \frac{\pi}{4}\sqrt{1-x}F(1/2,1/2;2;x)e^{\frac{\pi}{2}F(1/2,1/2;1;x)}, \\ G_3(x) &= \frac{32\sqrt{1-x}}{\pi}e^{-\mathscr{K}(\sqrt{x})}G_2'(x). \end{aligned} \tag{3.4}$$

Then it follows from (1.1), (2.1), (2.3), (2.17) and (2.18) that

$$\begin{aligned} G_3(x) &= \frac{32\sqrt{1-x}}{\pi}e^{-\mathscr{K}(\sqrt{x})}G_2'(x) \\ &= \pi(1-x)F\left(\frac{1}{2},\frac{1}{2};2;x\right)F\left(\frac{3}{2},\frac{3}{2};2;x\right) - 4F\left(\frac{1}{2},\frac{1}{2};2;x\right) \\ &\quad +(1-x)F\left(\frac{3}{2},\frac{3}{2};3;x\right) \\ &= \pi F^2\left(\frac{1}{2},\frac{1}{2};2;x\right) - 4F\left(\frac{1}{2},\frac{1}{2};2;x\right) + (1-x)F\left(\frac{3}{2},\frac{3}{2};3;x\right) \\ &= \pi\left(\sum_{n=0}^{\infty}\frac{W_n^2}{n+1}x^n\right)^2 - 4\sum_{n=0}^{\infty}\frac{W_n^2}{n+1}x^n - \sum_{n=0}^{\infty}\frac{2(4n-1)W_n^2}{(n+1)(n+2)}x^n \\ &= \pi\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{W_k^2W_{n-k}^2}{(k+1)(n-k+1)}\right)x^n - \sum_{n=0}^{\infty}\frac{6(2n+1)W_n^2}{(n+1)(n+2)}x^n = \sum_{n=0}^{\infty}u_nx^n. \end{aligned} \tag{3.5}$$

Elaborated computations show that

$$u_0=\pi-3>0,\quad u_1=\frac{\pi-3}{4}>0,\quad u_2=\frac{14\pi-45}{128}<0, \tag{3.6}$$

$$u_3=\frac{31\pi-105}{512}<0,\quad u_4=\frac{626\pi-2205}{16384}<0,\quad u_5=\frac{1718\pi-6237}{65536}<0, \tag{3.7}$$

$$u_6=\frac{79948\pi-297297}{4194304}<0,\quad u_7=\frac{242659\pi-920205}{16777216}<0. \tag{3.8}$$

From (1.1), (2.2), (2.4), (3.4) and (3.5) we get

$$G_2(0^+)=\frac{\pi}{4}F\left(\frac{1}{2},\frac{1}{2};2;0^+\right)e^{\mathscr{K}(0^+)}=\frac{\pi}{4}e^{\pi/2}, \tag{3.9}$$

$$F\left(\frac{1}{2},\frac{1}{2};2;1^-\right)=\frac{\Gamma(2)\Gamma(1)}{\Gamma^2(3/2)}=\frac{4}{\pi},$$

$$\begin{aligned}G_2(1^-)&=\lim_{x\to1^-}\sqrt{1-x}e^{\frac{\pi}{2}F(1/2,1/2;1;x)} \\ &=\lim_{x\to1^-}\sqrt{1-x}e^{2\log2-\log\sqrt{1-x}+O((1-x)\log(1-x))}=4,\end{aligned} \tag{3.10}$$

$$\begin{aligned}G_3(1^-)&=\pi\left(\frac{4}{\pi}\right)^2-\frac{16}{\pi}+\lim_{x\to1^-}(1-x)F\left(\frac{3}{2},\frac{3}{2};3;x\right) \\ &=\frac{8}{\pi}\lim_{x\to1^-}(1-x)\left[4(\log2-1)-\log(1-x)+O((1-x)\log(1-x))\right]=0.\end{aligned} \tag{3.11}$$

From Lemmas 2.1 and 2.5 together with (3.5)–(3.8) and (3.11) we have

$$G_3(x)>0 \tag{3.12}$$

for $x\in(0,1)$.

Therefore, Theorem 3.2 follows from (3.3)–(3.5), (3.9), (3.10) and (3.12). □

THEOREM 3.3. *The function*

$$r\mapsto e^{\mathscr{K}(r)}-\frac{p}{r'}$$

*is strictly decreasing on* $(0,1)$ *if and only if* $p\geqslant4$ *and strictly increasing on* $(0,1)$ *if and only if* $p\leqslant\pi e^{\pi/2}/4=3.7781401\cdots$.

*Proof.* Let $x=r^2$, $G_2(x)$ be defined by (3.4), and

$$G_4(x)=e^{\mathscr{K}(\sqrt{x})}-\frac{p}{\sqrt{1-x}}=e^{\mathscr{K}(r)}-\frac{p}{r'}=e^{\frac{\pi}{2}F(1/2,1/2;1;x)}-\frac{p}{\sqrt{1-x}}. \tag{3.13}$$

Then (2.1) and (2.3) lead to

$$G_4'(x) = \frac{\pi}{8}F\left(\frac{3}{2},\frac{3}{2};2;x\right)e^{\mathscr{K}(\sqrt{x})} - \frac{p}{2(1-x)^{3/2}} \tag{3.14}$$
$$= \frac{\frac{\pi}{4}\sqrt{1-x}F\left(\frac{1}{2},\frac{1}{2};2;x\right)e^{\mathscr{K}(\sqrt{x})} - p}{2(1-x)^{3/2}} = \frac{G_2(x)-p}{2(1-x)^{3/2}}.$$

It follows from Theorem 3.2 that $G_2(x)$ is strictly increasing from $(0,1)$ onto $(\pi e^{\pi/2}/4, 4)$. Therefore, Theorem 3.3 follows from (3.13) and (3.14) together with the monotonicity of $G_2(x)$ on the interval $(0,1)$. □

From (1.3), (1.4) and Theorem 3.2 we get Corollary 3.4 immediately.

COROLLARY 3.4. *The double inequalities*

$$r'^2\mathscr{K}(r) + p\frac{r^2}{r'}e^{-\mathscr{K}(r)} < \mathscr{E}(r) < r'^2\mathscr{K}(r) + q\frac{r^2}{r'}e^{-\mathscr{K}(r)},$$

$$\frac{r^2}{AG(1,r)} + \frac{2pr'^2}{\pi r}e^{-\frac{\pi}{2AG(1,r)}} < T(1,r) < \frac{r^2}{AG(1,r)} + \frac{2qr'^2}{\pi r}e^{-\frac{\pi}{2AG(1,r)}} \tag{3.15}$$

*hold for all* $r \in (0,1)$ *if and only if* $p \leqslant \pi e^{\pi/2}/4 = 3.7781401\cdots$ *and* $q \geqslant 4$.

REMARK 3.5. Let $x,y,z$ be nonnegative real numbers such that at most one of them is 0. Then the symmetric elliptic integral of the second kind $R_G(x,y,z)$ is defined by

$$R_G(x,y,z) = \frac{1}{4}\int_0^\infty [(t+x)(t+y)(t+z)]^{-1/2}\left(\frac{x}{t+x}+\frac{y}{t+y}+\frac{z}{t+z}\right),$$

and the Toader mean $T(a,b)$ can be written as

$$T(a,b) = \frac{4}{\pi}R_G\left(a^2,b^2,0\right) =: R_E\left(a^2,b^2\right).$$

Using the inequalities for $R_E$ presented in [45, 46] one can obtain bounds for $T$ which are sharper than those in (3.15).

COROLLARY 3.6. *The double inequality*

$$\frac{\pi}{2} + p\log\frac{1}{r'} < \mathscr{K}(r) < \frac{\pi}{2} + q\log\frac{1}{r'} \tag{3.16}$$

*is valid for all* $r \in (0,1)$ *if and only if* $p \leqslant \pi/4$ *and* $q \geqslant 1$. *Moreover, we have*

$$\log\frac{4}{r'} < \mathscr{K}(r) < \frac{\pi}{2} + \log\frac{1}{r'} \tag{3.17}$$

*for all* $r \in (0,1)$.

*Proof.* If $p \leqslant \pi/4$ and $q \geqslant 1$, then inequality (3.16) follows from Theorem 3.1 and the fact that

$$\lim_{r\to 0^+} r'^p e^{\mathscr{K}(r)} = e^{\pi/2}.$$

If the first inequality of (3.16) holds for all $r \in (0,1)$, then (1.2) leads to

$$\lim_{r\to 0^+} \frac{\mathscr{K}(r) - \left(\frac{\pi}{2} + p\log\frac{1}{r'}\right)}{r^2} = \frac{\pi}{8} - \frac{p}{2} \geqslant 0,$$

which leads to $p \leqslant \pi/4$.

If the second inequality of (3.16) holds for all $r \in (0,1)$, then it follows from (2.4) that

$$\lim_{x\to 1^-} \frac{\mathscr{K}(\sqrt{x}) - \left(\frac{\pi}{2} + q\log\frac{1}{\sqrt{1-x}}\right)}{\log(1-x)}$$
$$= \lim_{x\to 1^-} \frac{\log 4 - \frac{\pi}{2} + \frac{q-1}{2}\log(1-x) + O((1-x)\log(1-x))}{\log(1-x)} = \frac{q-1}{2} \geqslant 0,$$

which leads to $q \geqslant 1$.

Inequality (3.17) follows from the second inequality of (3.16) and Theorem 3.1 together with the fact that

$$\lim_{r\to 1^-} r' e^{\mathscr{K}(r)} = \lim_{x\to 1^-} \sqrt{1-x}\, e^{\frac{\pi}{2}F(1/2,1/2;1;x)} = 4. \quad \square$$

COROLLARY 3.7. *The double inequality*

$$\log\left(e^{\pi/2} - p + \frac{p}{r'}\right) < \mathscr{K}(r) < \log\left(e^{\pi/2} - q + \frac{q}{r'}\right) \tag{3.18}$$

*holds for all* $r \in (0,1)$ *if and only if* $p \leqslant \pi e^{\pi/2}/4 = 3.7781401\cdots$ *and* $q \geqslant 4$. *Moreover, we have*

$$\log\frac{4}{r'} < \mathscr{K}(r) < \log\left(e^{\pi/2} - 4 + \frac{4}{r'}\right) \tag{3.19}$$

*for all* $r \in (0,1)$.

*Proof.* If $p \leqslant \pi e^{\pi/2}/4$ and $q \geqslant 4$, then inequality (3.18) follows from Theorem 3.3 and the fact that

$$\lim_{r\to 0^+}\left(e^{\mathscr{K}(r)} - \frac{p}{r'}\right) = e^{\pi/2} - p.$$

If the first inequality of (3.18) holds for all $r \in (0,1)$, then (1.2) leads to

$$\lim_{x\to 0^+} \frac{\sqrt{1-x}\left[e^{\mathscr{K}(\sqrt{x})} + p - e^{\pi/2}\right] - p}{x}$$
$$= \lim_{x\to 0^+} \frac{\sqrt{1-x}\left[e^{\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2},n\right)^2}{(n!)^2}x^n} + p - e^{\pi/2}\right] - p}{x} = \frac{\pi e^{\pi/2}}{8} - \frac{p}{2} \geqslant 0,$$

which leads to $p \leqslant \pi e^{\pi/2}/4$.

If the second inequality of (3.18) holds for all $r \in (0,1)$, then (1.2) and (2.4) lead to

$$\begin{aligned} q &\geqslant \lim_{x\to 1^-} \sqrt{1-x}\left[e^{\frac{\pi}{2}F(1/2,1/2;1;x)} - e^{\pi/2} + q\right] \\ &= \lim_{x\to 1^-} \sqrt{1-x}\left[e^{\log 4 - \frac{1}{2}\log(1-x) + O((1-x)\log(1-x))} - e^{\pi/2} + q\right] = 4. \end{aligned}$$

Inequality (3.19) follows from the first inequality of (3.17) and the second inequality of (3.18). □

REMARK 3.8. Let

$$R_F(x,y,z) = \frac{1}{2}\int_0^\infty [(t+x)(t+y)(t+z)]^{-1/2}\,dt$$

be the symmetric elliptic integral of the first kind. Then the complete elliptic integral of the first kind $\mathscr{K}(r)$ can be expressed by

$$\mathscr{K}(r) = R_F\left(r'^2, 1, 0\right) := \frac{\pi}{2} R_K\left(r'^2, 1\right).$$

Bounds for complete elliptic integral $\mathscr{K}$ included in Corollaries 3.6 and 3.7 are not necessarily simple and sharp. Using the known bounds for $R_K$ given in [39, 40, 43, 46] on can obtain simple and sharp bounds which are sharper than those given in Corollaries 3.6 and 3.7.

From (1.4), Corollary 3.6 and Corollary 3.7 we get Corollary 3.9 immediately.

COROLLARY 3.9. *The double inequalities*

$$\frac{1}{1-p\log r} < AG(1,r) < \frac{1}{1-q\log r},$$

$$\frac{\pi}{2\log\left(e^{\pi/2} - \lambda + \frac{\lambda}{r}\right)} < AG(1,r) < \frac{\pi}{2\log\left(e^{\pi/2} - \mu + \frac{\mu}{r}\right)}$$

*hold for all* $r \in (0,1)$ *if and only if* $p \geqslant 2/\pi$, $q \leqslant 1/2$, $\lambda \geqslant 4$ *and* $\mu \leqslant \pi e^{\pi/2}/4$. *Moreover, one has*

$$\frac{1}{1-\frac{2}{\pi}\log r} < AG(1,r) < \frac{\pi}{2\log\frac{4}{r}},$$

$$AG(1,r) > \frac{\pi}{2\log\left(e^{\pi/2} - 4 + \frac{4}{r}\right)} \tag{3.20}$$

*for all* $r \in (0,1)$.

REMARK 3.10. Neuman and Sándor [45, Theorem 3.2] proved that the inequality

$$AG(1,r) > \frac{\frac{1+r}{2} - \sqrt{r}}{\log \frac{1+r}{2\sqrt{r}}} \tag{3.21}$$

holds for all $r \in (0,1)$.

Let

$$I(r) = \frac{\frac{1+r}{2} - \sqrt{r}}{\log \frac{1+r}{2\sqrt{r}}}, \quad J(r) = \frac{\pi}{2\log\left(e^{\pi/2} - 4 + \frac{4}{r}\right)}. \tag{3.22}$$

Then numerical computations lead to

$$I(0.05) = 0.3531\cdots < J(0.05) = 0.3576\cdots, \tag{3.23}$$

$$I(0.1) = 0.4223\cdots < J(0.1) = 0.4235\cdots. \tag{3.24}$$

From (3.22)–(3.24) we know that the lower bound for $AG(1,r)$ given in (3.20) is better than that given in (3.21) for some $r \in (0,1)$.

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