Mathematical
Inequalities
& Applications
Volume 22, Number 1 (2019), 1–23 doi:10.7153/mia-2019-22-01

SOME NEW INTEGRAL INEQUALITIES ON TIME SCALES

FEIFEI DU, WEI HU, LYNN ERBE AND ALLAN PETERSON

(*Communicated by M. Bohner*)

Abstract. In this paper, we establish some generalizations of inequalities on time scales, which have appeared in different articles. The inequalities that we will derive from our results when $g(t) = t$ are essentially new.

1. Introduction

Stefan Hilger introduced the theory of time scales in his PhD thesis [8] in 1988 (supervised by Bernd Aulbach) in order to unify continuous and discrete analysis. Since then, many authors have studied various inequalities and dynamic equations on time scales in detail [4, 5, 3, 2, 16, 20, 14, 7].

In [15], the following open problem was posed by Feng Qi: Under what conditions does the inequality

$$\int_a^b [f(x)]^t dx \geqslant \left[\int_a^b f(x)dx\right]^{t-1}$$

hold for $t > 1$? Various results have been studied by authors in [6, 7, 14, 20].

Kamel Brahim et al.[6] and Yu et al.[12] obtained some Feng-Qi type q-integral inequalities. Mohamad Rafi Segi Rahmat [16] pointed out some (q,h) analogues of integral inequalities on discrete time scales. L. Yin et al. [20] presented some Feng-Qi type inequalities on time scales.

This work is motivated by Waadallah T. Sulaiman [17, 18, 19] and Fayyaz et al. [7] who obtained integral inequalities on discrete time scales. We generalize the Feng-Qi type integral inequalities which appeared in these articles. To the best of the authors' knowledge, the inequalities that we will derive from our results when $g(t) = t$ are essentially new. In addition, we show that a recent result (Theorem 3.3 in [7]) is incorrect as stated without an additional assumption.

Mathematics subject classification (2010): 39A12, 39A70.
Keywords and phrases: Integral inequalities, time scales, Feng-Qi inequality.

Paper MIA-22-01

2. Preliminaries

For the convenience of the readers, we extracted some definitions and results that can be found in the monograph [4] as follows.

DEFINITION 1. A time scale $\mathbb{T}$ is a non-empty, closed subset of the real numbers $\mathbb{R}$. We define the forward and backward jump operators σ, $\rho : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$$

and

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\},$$

respectively.

DEFINITION 2. The forward and backward graininess functions are defined as follows:

$$\mu(t) := \sigma(t) - t$$

and

$$\nu(t) := t - \rho(t),$$

respectively.

DEFINITION 3. If $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$ we say that t is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated. Also, if $t < \sup\mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense, while if $t > \inf\mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. Points that are right-dense and left-dense at the same time are called dense.

DEFINITION 4. $\mathbb{T}^\kappa$ and $\mathbb{T}_\kappa$ are defined as follows:

$$\mathbb{T}^\kappa := \begin{cases} \mathbb{T}\backslash(\rho(\sup\mathbb{T}), \sup\mathbb{T}] & \text{if } \sup\mathbb{T} < \infty, \\ \mathbb{T} & \text{if } \sup\mathbb{T} = \infty, \end{cases}$$

and

$$\mathbb{T}_\kappa := \begin{cases} \mathbb{T}\backslash[\inf\mathbb{T}, \sigma(\inf\mathbb{T})) & \text{if } \inf\mathbb{T} > -\infty, \\ \mathbb{T} & \text{if } \inf\mathbb{T} = -\infty, \end{cases}$$

respectively.

DEFINITION 5. Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t (i.e., $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$\big|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]\big| \leqslant \varepsilon|\sigma(t) - s|$$

for all $s \in U$. We call $f^\Delta(t)$ the delta derivative of f at t.

DEFINITION 6. Assume $f:\mathbb{T}\to\mathbb{R}$ is a function and let $t\in\mathbb{T}_\kappa$. Then we define $f^\nabla(t)$ to be the number (provided it exists) with the property that given any $\varepsilon>0$, there is a neighborhood U of t (i.e., $U=(t-\delta,t+\delta)\cap\mathbb{T}$ for some $\delta>0$) such that

$$\left|[f(\rho(t))-f(s)]-f^\nabla(t)[\rho(t)-s]\right|\leqslant\varepsilon|\rho(t)-s|$$

for all $s\in U$. We call $f^\nabla(t)$ the nabla derivative of f at t.

LEMMA 1. *Assume $f:\mathbb{T}\to\mathbb{R}$ is continuous at t.*
(i) *If $\sigma(t)>t$, then f is delta differentiable at $t\in\mathbb{T}^\kappa$ with*

$$f^\Delta(t)=\frac{f(\sigma(t))-f(t)}{\mu(t)}.$$

(ii) *If $\sigma(t)=t$, then f is delta differentiable at $t\in\mathbb{T}^\kappa$ iff the limit $\lim\limits_{s\to t}\frac{f(t)-f(s)}{t-s}$ exits as a finite number. In this case*

$$f^\Delta(t)=\lim_{s\to t}\frac{f(t)-f(s)}{t-s}.$$

(iii) *If $\rho(t)<t$, then f is nabla differentiable at $t\in\mathbb{T}_\kappa$ with*

$$f^\nabla(t)=\frac{f(t)-f(\rho(t))}{\nu(t)}.$$

(iv) *If $\rho(t)=t$, then f is nabla differentiable at $t\in\mathbb{T}_\kappa$ iff the limit $\lim\limits_{s\to t}\frac{f(t)-f(s)}{t-s}$ exits as a finite number. In this case*

$$f^\nabla(t)=\lim_{s\to t}\frac{f(t)-f(s)}{t-s}.$$

LEMMA 2. *The delta-integral of f and the nabla-integral of g over the time scale interval $[a,b]_\mathbb{T}:=\{t\in\mathbb{T}:a\leqslant t\leqslant b \text{ and } a,b\in\mathrm{T}\}$ are defined by*

$$\int_a^b f(t)\Delta t=F(b)-F(a)$$

and

$$\int_a^b g(t)\nabla t=G(b)-G(a),$$

where $F^\Delta=f$ on $\mathbb{T}^\kappa$ and $G^\nabla=g$ on $\mathbb{T}_\kappa$, respectively.

DEFINITION 7. A function $f:\mathbb{T}\to\mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f:\mathbb{T}\to\mathbb{R}$ will be denoted in this paper by C_{rd}. The set of functions $f:\mathbb{T}\to\mathbb{R}$ that are differentiable and whose derivatives are rd-continuous is denoted by C^1_{rd}.

DEFINITION 8. A function $f:\mathbb{T}\to\mathbb{R}$ is called ld-continuous provided it is continuous at left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at right-dense points in $\mathbb{T}$. The set of ld-continuous functions $f:\mathbb{T}\to\mathbb{R}$ will be denoted in this paper by C_{ld}. The set of functions $f:\mathbb{T}\to\mathbb{R}$ that are differentiable and whose derivatives are ld-continuous is denoted by C^1_{ld}.

DEFINITION 9. ([1]) Let f be a real-valued function on $\mathbb{T}\times\mathbb{T}$.

(1) The function f is called rd-continuous in t if for every $\beta\in\mathbb{T}$, the function $f(t,\beta)$ is rd-continuous on $\mathbb{T}$.

(2) The function f is called rd-continuous in s if for every $\alpha\in\mathbb{T}$, the function $f(\alpha,s)$ is rd-continuous on $\mathbb{T}$.

Similarly, we have the following definitions:

DEFINITION 10. Let f be a real-valued function on $\mathbb{T}\times\mathbb{T}$.

(1) The function f is called ld-continuous in t if for every $\beta\in\mathbb{T}$, the function $f(t,\beta)$ is ld-continuous on $\mathbb{T}$.

(2) The function f is called ld-continuous in s if for every $\alpha\in\mathbb{T}$, the function $f(\alpha,s)$ is ld-continuous on $\mathbb{T}$.

DEFINITION 11. ([1]) $C_{rd}(\mathbb{T}\times\mathbb{T},\mathbb{R})$ denotes the set of functions $f:\mathbb{T}\times\mathbb{T}\to\mathbb{R}$ with the following properties:

(R1) f is rd-continuous in t.

(R2) f is rd-continuous in s.

(R3) if $(t_1,s_1)\in\mathbb{T}\times\mathbb{T}$ with t_1 right-dense or maximal and s_1 right dense or maximal, then f is continuous at (t_1,s_1).

(R4) if t_1 and s_1 are both left-dense, then the limit of $f(t,s)$ exists as (t,s) approaches (t_1,s_1) along any path in the region $R_{\mathrm{LL}}(t_1,s_1):=\{(t,s):t\in[a,t_1]\cap\mathbb{T},\ y\in[c,s_1]\cap\mathbb{T}\}$.

Similarly, we can give the definition of $C_{ld}(\mathbb{T}\times\mathbb{T},\mathbb{R})$ as follows:

DEFINITION 12. $C_{ld}(\mathbb{T}\times\mathbb{T},\mathbb{R})$ denotes the set of functions $f(t,s)$ on $\mathbb{T}\times\mathbb{T}$ with the following properties:

(L1) f is ld-continuous in t.

(L2) f is ld-continuous in s.

(L3) if $(t_1,s_1)\in\mathbb{T}\times\mathbb{T}$ with t_1 left-dense or minimal and s_1 left dense or minimal, then f is continuous at (t_1,s_1).

(L4) if t_1 and s_1 are both right-dense, then the limit of $f(t,s)$ exists as (t,s) approaches (t_1,s_1) along any path in the region $R_{LL}(t_1,s_1) := \{(t,s) : t \in [t_1,b] \cap \mathbb{T},\ y \in [s_1,d] \cap \mathbb{T}\}$.

LEMMA 3. *Let $a,b \in \mathbb{T}, a < b$ and $f \in C_{rd}$.*
(i) If $\mathbb{T} = \mathbb{R}$, then

$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt,$$

where the integral on the right is the usual Riemann integral from calculus.
(ii) If $[a,b]_{\mathbb{T}}$ consists of only isolated points, then

$$\int_a^b f(t)\Delta t = \sum_{t \in [a,b)_{\mathbb{T}}} \mu(t) f(t).$$

LEMMA 4. *Let $a,b \in \mathbb{T}, a < b$ and $f \in C_{ld}$.*
(i) If $\mathbb{T} = \mathbb{R}$, then

$$\int_a^b f(t)\nabla t = \int_a^b f(t)dt,$$

where the integral on the right is the usual Riemann integral from calculus.
(ii) If $[a,b]_{\mathbb{T}}$ consists of only isolated points, then

$$\int_a^b g(t)\nabla t = \sum_{t \in (a,b]_{\mathbb{T}}} \nu(t) g(t).$$

LEMMA 5. *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^{\kappa}$.*
(i) If f is delta differentiable at t, then f is continuous at t.
(ii) If f is continuous, then f is rd-continuous.

LEMMA 6. *If $f \in C_{rd}$ and $t \in \mathbb{T}^{\kappa}$, then*

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t) f(t).$$

The following chain rule is due to Christian Pötzsche, who derived it first in 1998 (see also Stefan Keller's PhD thesis [13] and [11])

LEMMA 7. ([4], Theorem 1.90) *Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable. Then $f \circ g : \mathbb{T} \to \mathbb{R}$ is delta differentiable and the formula*

$$(f \circ g)^{\Delta}(t) = \left\{ \int_0^1 f'[g(t) + h\mu(t) g^{\Delta}(t)]dh \right\} g^{\Delta}(t)$$

holds.

LEMMA 8. *Assume $f:\mathbb{T}\to\mathbb{R}$ is delta differentiable at $t\in\mathbb{T}^\kappa$. Then*

$$f(\sigma(t)) = f(t)+\mu(t)f^\Delta(t).$$

LEMMA 9. *Assume $f:\mathbb{T}\to\mathbb{R}$ is delta differentiable on $\mathbb{T}^\kappa$. Then f is nabla differentiable at t and*

$$f^\nabla(t) = f^\Delta(\rho(t))$$

for $t\in\mathbb{T}_\kappa$ such that $\sigma(\rho(t))=t$.

LEMMA 10. (Remark 3.2, [10]) *If $g,h\in C_{rd}([a,b]_\mathbb{T},\mathbb{C})$, then $f\in C_{rd}(\mathbb{T}\times\mathbb{T},\mathbb{C})$, where f is defined by $f(t,s)=g(t)h(s)$ for $(t,s)\in\mathbb{T}\times\mathbb{T}$.*

Similarly, we have the following Lemma:

LEMMA 11. *If $g,h\in C_{ld}([a,b]_\mathbb{T},\mathbb{C})$, then $f\in C_{ld}(\mathbb{T}\times\mathbb{T},\mathbb{C})$, where f is defined by $f(t,s)=g(t)h(s)$ for $(t,s)\in\mathbb{T}\times\mathbb{T}$.*

3. Delta integral inequalities

In this section, we give some Feng-Qi type delta-integral inequalities on time scales. We begin with the following useful lemma.

LEMMA 12. *Let $p\geqslant 1$. Suppose $f:\mathbb{R}\to\mathbb{R}$ is continuously differentiable and $g:\mathbb{T}\to\mathbb{R}$ is delta differentiable for $t\in\mathbb{T}^\kappa$, and assume f, f', and g are nonnegative and nondecreasing functions. Then*

$$pf^{p-1}(g(t))f'(g(t))g^\Delta(t)\leqslant (f^p\circ g)^\Delta(t)\leqslant pf^{p-1}(g(\sigma(t)))f'(g(\sigma(t)))g^\Delta(t).$$

Proof. Let $u(x)=x^p$. Using Lemma 7 twice, we have

$$\begin{aligned}
(f^p\circ g)^\Delta(t) &= \big(u\circ(f\circ g)\big)^\Delta(t)\\
&= \left\{\int_0^1 u'[(f\circ g)(t)+h\mu(t)(f\circ g)^\Delta(t)]dh\right\}(f\circ g)^\Delta(t)\\
&= \left\{p\int_0^1 [(f\circ g)(t)+h\mu(t)(f\circ g)^\Delta(t)]^{p-1}dh\right\}(f\circ g)^\Delta(t)\\
&\geqslant \left\{p\int_0^1 [f(g(t))]^{p-1}dh\right\}\left\{\int_0^1 f'[g(t)+h\mu(t)g^\Delta(t)]dh\right\}g^\Delta(t)\\
&\geqslant pf^{p-1}(g(t))f'(g(t))g^\Delta(t).
\end{aligned}$$

By virtue of Lemma 7 and Lemma 8, we obtain

$$\begin{aligned}
(f^p\circ g)^\Delta(t) &= \left\{p\int_0^1 [(f\circ g)(t)+h\mu(t)(f\circ g)^\Delta(t)]^{p-1}dh\right\}(f\circ g)^\Delta(t)\\
&\leqslant \left\{p\int_0^1 [f(g(\sigma(t)))]^{p-1}dh\right\}\left\{\int_0^1 f'[g(t)+h\mu(t)g^\Delta(t)]dh\right\}g^\Delta(t)\\
&\leqslant pf^{p-1}(g(\sigma(t)))f'(g(\sigma(t)))g^\Delta(t). \quad\square
\end{aligned}$$

Let $g(t) = t$ in Lemma 12. Then we have the following result:

COROLLARY 1. *Let $p \geqslant 1$. Suppose $f : \mathbb{T} \to \mathbb{R}$ is delta differentiable for $t \in \mathbb{T}^\kappa$ and f and f' are nonnegative and nondecreasing functions. Then*

$$pf^{p-1}(t)f'(t) \leqslant (f^p)^\Delta(t) \leqslant pf^{p-1}(\sigma(t))f'(\sigma(t)).$$

Let $f(t) = t$ in Lemma 12. Then we have the following result:

COROLLARY 2. *Let $p \geqslant 1$. Suppose $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable for $t \in \mathbb{T}^\kappa$, and assume g is a nonnegative and nondecreasing function. Then*

$$pg^{p-1}(t)g^\Delta(t) \leqslant (g^p)^\Delta(t) \leqslant pg^{p-1}(\sigma(t))g^\Delta(t).$$

REMARK 1. Lemma 3.1 ([7]) is similar to Corollary 2, but only holds for discrete time scales. When $p \geqslant 1$ is an integer, the proof method of Lemma 3.1 ([20]) is valid for any time scale.

For $p = 1$ in Lemma 12, we have the following result:

COROLLARY 3. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable for $t \in \mathbb{T}^\kappa$, and assume g is nonnegative and f', g are nondecreasing functions. Then*

$$f'(g(t))g^\Delta(t) \leqslant (f \circ g)^\Delta(t) \leqslant f'(g(\sigma(t)))g^\Delta(t).$$

THEOREM 1. *Let $a, b \in \mathbb{T}$, $\alpha \geqslant \gamma + 1$, $\beta \geqslant 2$, and $\gamma > 0$. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and $g, \sigma : \mathbb{T} \to \mathbb{R}$ are delta differentiable for $t \in \mathbb{T}^\kappa$. Assume further that f and g are nonnegative and increasing functions such that $f^{\alpha-\gamma}(g(a)) \geqslant \beta\left(f^\gamma(g(a))\mu(a)\right)^{\beta-1}$ and*

$$(\alpha-\gamma)f^{\alpha-\gamma-1}(g(t))f'(g(t))g^\Delta(t) \geqslant \beta(\beta-1)f^{\gamma(\beta-1)}(g(\sigma^2(t)))(\sigma^2(t)-a)^{\beta-2}\sigma^\Delta(t),$$

where $\sigma^2(t) := \sigma(\sigma(t))$. Then

$$\int_a^b f^\alpha(g(t))\Delta t \geqslant \left(\int_a^b f^\gamma(g(t))\Delta t\right)^\beta.$$

Proof. For each $t \in [a,b]_\mathbb{T}$ let

$$F(t) := \int_a^t f^\alpha(g(\tau))\Delta\tau - \left(\int_a^t f^\gamma(g(\tau))\Delta\tau\right)^\beta.$$

Using Corollary 2, we have

$$\begin{aligned}F^{\Delta}(t) \geqslant & f^{\alpha}(g(t)) - \beta\left(\int_a^{\sigma(t)} f^{\gamma}(g(\tau))\Delta\tau\right)^{\beta-1} f^{\gamma}(g(t)) \\ = & f^{\gamma}(g(t))\left(f^{\alpha-\gamma}(g(t)) - \beta\left(\int_a^{\sigma(t)} f^{\gamma}(g(\tau))\Delta\tau\right)^{\beta-1}\right) \\ = & f^{\gamma}(g(t))h(t),\end{aligned}$$

where

$$h(t) := f^{\alpha-\gamma}(g(t)) - \beta\left(\int_a^{\sigma(t)} f^{\gamma}(g(\tau))\Delta\tau\right)^{\beta-1}.$$

Now, using Lemma 12 and Corollary 2, we have

$$\begin{aligned}h^{\Delta}(t) = & (f^{\alpha-\gamma}(g(t)))^{\Delta} - \beta\left(\left(\int_a^{\sigma(t)} f^{\gamma}(g(\tau))\Delta\tau\right)^{\beta-1}\right)^{\Delta} \\ \geqslant & (\alpha-\gamma)f^{\alpha-\gamma-1}(g(t))f'(g(t))g^{\Delta}(t) \\ & -\beta(\beta-1)\left(\int_a^{\sigma^2(t)} f^{\gamma}(g(\tau))\Delta\tau\right)^{\beta-2}\left(\int_a^{\sigma(t)} f^{\gamma}(g(\tau))\Delta\tau\right)^{\Delta} \\ = & (\alpha-\gamma)f^{\alpha-\gamma-1}(g(t))f'(g(t))g^{\Delta}(t) \\ & -\beta(\beta-1)\left(\int_a^{\sigma^2(t)} f^{\gamma}(g(\tau))\Delta\tau\right)^{\beta-2} f^{\gamma}(g(\sigma(t)))\sigma^{\Delta}(t),\end{aligned}$$

where

$$\begin{aligned}\left(\int_a^{\sigma(t)} f^{\gamma}(g(\tau))\Delta\tau\right)^{\Delta} &= \left(\int_a^{t} f^{\gamma}(g(\tau))\Delta\tau + \int_t^{\sigma(t)} f^{\gamma}(g(\tau))\Delta\tau\right)^{\Delta} \\ &= \left(\int_a^{t} f^{\gamma}(g(\tau))\Delta\tau\right)^{\Delta} + \left[f^{\gamma}(g(t))(\sigma(t)-t)\right]^{\Delta} \\ &= f^{\gamma}(g(t)) + (f^{\gamma}\circ g)^{\Delta}(t)\mu(t) + f^{\gamma}(g(\sigma(t)))(\sigma^{\Delta}(t)-1) \\ &= f^{\gamma}(g(\sigma(t))) + f^{\gamma}(g(\sigma(t)))(\sigma^{\Delta}(t)-1) \\ &= f^{\gamma}(g(\sigma(t)))\sigma^{\Delta}(t).\end{aligned}$$

Since $\gamma > 0$ and f, g are increasing, we have that $f^{\gamma}\circ g$ is increasing. Then

$$\int_a^{\sigma^2(t)} f^{\gamma}(g(\tau))\Delta\tau \leqslant f^{\gamma}(g(\sigma^2(t)))(\sigma^2(t)-a).$$

Hence we obtain

$$\begin{aligned}h^{\Delta}(t) \geqslant & (\alpha-\gamma)f^{\alpha-\gamma-1}(g(t))f'(g(t))g^{\Delta}(t) \\ & -\beta(\beta-1)f^{\gamma(\beta-2)}(g(\sigma^2(t)))(\sigma^2(t)-a)^{\beta-2}f^{\gamma}(g(\sigma(t)))\sigma^{\Delta}(t)\end{aligned}$$

$$
\begin{aligned}
\geqslant&(\alpha-\gamma)f^{\alpha-\gamma-1}(g(t))f'(g(t))g^{\Delta}(t)\\
&-\beta(\beta-1)f^{\gamma(\beta-1)}(g(\sigma^2(t)))(\sigma^2(t)-a)^{\beta-2}\sigma^{\Delta}(t)\\
\geqslant&0.
\end{aligned}
$$

So h is nondecreasing. But

$$
\begin{aligned}
h(a)&=f^{\alpha-\gamma}(g(a))-\beta\left(\int_a^{\sigma(a)}f^{\gamma}(g(\tau))\Delta\tau\right)^{\beta-1}\\
&=f^{\alpha-\gamma}(g(a))-\beta\left(f^{\gamma}(g(a))\mu(a)\right)^{\beta-1}\\
&\geqslant 0.
\end{aligned}
$$

Therefore $h(t)\geqslant h(a)\geqslant 0$ and it follows that $F^{\Delta}(t)\geqslant 0$. So $F(t)\geqslant F(a)=0$, which completes the proof. $\square$

If $f(t)=t$ in Theorem 1, we have the following result:

COROLLARY 4. *Let $a,b\in\mathbb{T}$, $\alpha\geqslant\gamma+1$, $\beta\geqslant 2$, and $\gamma>0$. Suppose $g,\sigma:\mathbb{T}\to\mathbb{R}$ are delta differentiable for $t\in\mathbb{T}^{\kappa}$, and assume g is a nonnegative and increasing function such that $f^{\alpha-\gamma}(g(a))\geqslant\beta\left(f^{\gamma}(g(a))\mu(a)\right)^{\beta-1}$ and*

$$
(\alpha-\gamma)g^{\alpha-\gamma-1}(t)g^{\Delta}(t)\geqslant\beta(\beta-1)g^{\gamma(\beta-1)}(\sigma^2(t))(\sigma^2(t)-a)^{\beta-2}\sigma^{\Delta}(t),
$$

where $\sigma^2(t):=\sigma(\sigma(t))$. Then

$$
\int_a^b g^{\alpha}(t)\Delta t\geqslant\left(\int_a^b g^{\gamma}(t)\Delta t\right)^{\beta}.
$$

If $g(t)=t$ in Theorem 1, we have the following result:

COROLLARY 5. *Let $a,b\in\mathbb{T}$, $\alpha\geqslant\gamma+1$, $\beta\geqslant 2$, and $\gamma>0$. Suppose f is differentiable for $t\in\mathbb{R}$ and $\sigma:\mathbb{T}\to\mathbb{R}$ is delta differentiable for $t\in\mathbb{T}^{\kappa}$, and assume f is a nonnegative and increasing function such that $f^{\alpha-\gamma}(a)\geqslant\beta\left(f^{\gamma}(a)\mu(a)\right)^{\beta-1}$ and*

$$
(\alpha-\gamma)f^{\alpha-\gamma-1}(t)f'(t)\geqslant\beta(\beta-1)f^{\gamma(\beta-1)}(\sigma^2(t))(\sigma^2(t)-a)^{\beta-2}\sigma^{\Delta}(t),
$$

where $\sigma^2(t):=\sigma(\sigma(t))$. Then

$$
\int_a^b f^{\alpha}(t)\Delta t\geqslant\left(\int_a^b f^{\gamma}(t)\Delta t\right)^{\beta}.
$$

If $\gamma=1$, $\beta=\alpha-1$ in Corollary 5, we obtain the following result:

COROLLARY 6. *Let $a,b\in\mathbb{T}$, $\alpha\geqslant 3$. Suppose $f,\sigma:\mathbb{T}\to\mathbb{R}$ are delta differentiable for $t\in\mathbb{T}^{\kappa}$, and assume f is a nonnegative and increasing function such that*

$$f^{\alpha-1}(a)\geqslant(\alpha-1)(f(a)\mu(a))^{\alpha-2} \tag{1}$$

and

$$f^{\alpha-2}(t)f^{\Delta}(t)\geqslant(\alpha-2)f^{\alpha-2}(\sigma^2(t))(\sigma^2(t)-a)^{\alpha-3}\sigma^{\Delta}(t),$$

where $\sigma^2(t):=\sigma(\sigma(t))$. Then

$$\int_a^b f^{\alpha}(t)\Delta t\geqslant\left(\int_a^b f(t)\Delta t\right)^{\alpha-1}.$$

REMARK 2. Nonnegativity of f does not guarantee (1) holds. So it seems that Theorem 3.3 in [7] is incorrect since $F_1(a)\geqslant 0$ does not hold without (1). A similar comment applies to Theorem 3.2 in [16].

Furthermore, if $\mathbb{T}=\mathbb{R}$ in Theorem 1, we have the following result, providing another sufficient condition for Feng-Qi inequality which is different from Theorem 1.1 in [14].

COROLLARY 7. *Let $a,b\in\mathbb{T}$, $\alpha\geqslant 3$. Suppose $f:\mathbb{R}\to\mathbb{R}$ is differentiable at $t\in\mathbb{R}$, and assume f is a nonnegative and increasing function. If*

$$f^{\alpha-2}(t)f'(t)\geqslant(\alpha-2)f^{\alpha-2}(t)(t-a)^{\alpha-3}$$

is satisfied, then

$$\int_a^b f^{\alpha}(t)dt\geqslant\left(\int_a^b f(t)dt\right)^{\alpha-1}.$$

THEOREM 2. *Let $\rho^m(a)$, $b\in\mathbb{T}$, $m\in\mathbb{N}_2$, $\alpha\geqslant\gamma+1$, $\beta\geqslant 2$, and $\gamma>0$. Suppose $f:\mathbb{R}\to\mathbb{R}$ is continuously differentiable and $g,\sigma:\mathbb{T}\to\mathbb{R}$ are delta differentiable for $t\in\mathbb{T}^{\kappa}$, and assume f and g are nonnegative and increasing functions such that $f^{\alpha-\gamma}(g(a))\geqslant\beta\left(f^{\gamma}(g(\rho^m(a)))\mu(a)\right)^{\beta-1}$ and*

$$(\alpha-\gamma)f'(g(t))g^{\Delta}(t)\geqslant\beta(\beta-1)f^{\gamma\beta-\alpha+1}(g(\rho^{m-2}(t)))(\sigma^2(t)-a)^{\beta-2}\sigma^{\Delta}(t),$$

where $\sigma^2(t):=\sigma(\sigma(t))$. Then

$$\int_a^b f^{\alpha}(g(t))\Delta t\geqslant\left(\int_a^b f^{\gamma}(g(\rho^m(t)))\Delta t\right)^{\beta}.$$

Proof. For each $t\in[a,b]_{\mathbb{T}}$ let

$$F(t):=\int_a^t f^{\alpha}(g(\tau))\Delta\tau-\left(\int_a^t f^{\gamma}(g(\rho^m(\tau)))\Delta\tau\right)^{\beta}.$$

Using Corollary 2, we have

$$\begin{aligned}F^{\Delta}(t) \geqslant & f^{\alpha}(g(t)) - \beta\left(\int_a^{\sigma(t)} f^{\gamma}(g(\rho^m(\tau)))\Delta\tau\right)^{\beta-1} f^{\gamma}(g(\rho^m(t)))\\ \geqslant & f^{\alpha}(g(t)) - \beta\left(\int_a^{\sigma(t)} f^{\gamma}(g(\rho^m(\tau)))\Delta\tau\right)^{\beta-1} f^{\gamma}(g(t))\\ = & f^{\gamma}(g(t))\left(f^{\alpha-\gamma}(g(t)) - \beta\left(\int_a^{\sigma(t)} f^{\gamma}(g(\rho^m(\tau)))\Delta\tau\right)^{\beta-1}\right)\\ = & f^{\gamma}(g(t))h(t),\end{aligned}$$

where

$$h(t) := f^{\alpha-\gamma}(g(t)) - \beta\left(\int_a^{\sigma(t)} f^{\gamma}(g(\rho^m(\tau)))\Delta\tau\right)^{\beta-1}.$$

Now, using Lemma 12 and Corollary 2,

$$\begin{aligned}h^{\Delta}(t) = & (f^{\alpha-\gamma}(g(t)))^{\Delta} - \beta\left(\left(\int_a^{\sigma(t)} f^{\gamma}(g(\rho^m(\tau)))\Delta\tau\right)^{\beta-1}\right)^{\Delta}\\ \geqslant & (\alpha-\gamma)f^{\alpha-\gamma-1}(g(t))f'(g(t))g^{\Delta}(t)\\ & -\beta(\beta-1)\left(\int_a^{\sigma^2(t)} f^{\gamma}(g(\rho^m(\tau)))\Delta\tau\right)^{\beta-2}\left(\int_a^{\sigma(t)} f^{\gamma}(g(\rho^m(\tau)))\Delta\tau\right)^{\Delta}\\ = & (\alpha-\gamma)f^{\alpha-\gamma-1}(g(t))f'(g(t))g^{\Delta}(t)\\ & -\beta(\beta-1)\left(\int_a^{\sigma^2(t)} f^{\gamma}(g(\rho^m(\tau)))\Delta\tau\right)^{\beta-2} f^{\gamma}(g(\rho^{m-1}(t)))\sigma^{\Delta}(t),\end{aligned}$$

where

$$\begin{aligned}\left(\int_a^{\sigma(t)} f^{\gamma}(g(\rho^m(\tau)))\Delta\tau\right)^{\Delta} &= \left(\int_a^{t} f^{\gamma}(g(\rho^m(\tau)))\Delta\tau + \int_t^{\sigma(t)} f^{\gamma}(g(\rho^m(\tau)))\Delta\tau\right)^{\Delta}\\ &= \left(\int_a^{t} f^{\gamma}(g(\rho^m(\tau)))\Delta\tau\right)^{\Delta} + \left[f^{\gamma}(g(\rho^m(t)))\mu(t)\right]^{\Delta}\\ &= f^{\gamma}(g(\rho^{m-1}(t)))\sigma^{\Delta}(t).\end{aligned}$$

Since $\gamma > 0$ and $f,\ g$ are increasing, we have that $f^{\gamma}\circ g$ is increasing. It follows that

$$\int_a^{\sigma^2(t)} f^{\gamma}(g(\rho^m(\tau)))\Delta\tau \leqslant f^{\gamma}(g(\rho^{m-2}(t)))(\sigma^2(t)-a).$$

Hence we obtain

$$\begin{aligned}h^{\Delta}(t) \geqslant & (\alpha-\gamma)f^{\alpha-\gamma-1}(g(t))f'(g(t))g^{\Delta}(t)\\ & -\beta(\beta-1)f^{\gamma(\beta-2)}(g(\rho^{m-2}(t)))(\sigma^2(t)-a)^{\beta-2}f^{\gamma}(g(\rho^{m-1}(t)))\sigma^{\Delta}(t)\end{aligned}$$

$$
\begin{aligned}
&\geqslant(\alpha-\gamma)f^{\alpha-\gamma-1}(g(\rho^{m-2}(t)))f'(g(t))g^{\Delta}(t)\\
&\quad-\beta(\beta-1)f^{\gamma(\beta-1)}(g(\rho^{m-2}(t)))(\sigma^2(t)-a)^{\beta-2}\sigma^{\Delta}(t)\\
&=f^{\alpha-\gamma-1}(g(\rho^{m-2}(t)))\Big((\alpha-\gamma)f'(g(t))g^{\Delta}(t)\\
&\quad-\beta(\beta-1)f^{\gamma\beta-\alpha+1}(g(\rho^{m-2}(t)))(\sigma^2(t)-a)^{\beta-2}\sigma^{\Delta}(t)\Big)\\
&\geqslant 0.
\end{aligned}
$$

Therefore, h is nondecreasing. But

$$
\begin{aligned}
h(a)&=f^{\alpha-\gamma}(g(a))-\beta\left(\int_a^{\sigma(a)}f^{\gamma}(g(\rho^m(\tau)))\Delta\tau\right)^{\beta-1}\\
&=f^{\alpha-\gamma}(g(a))-\beta\left(f^{\gamma}(g(\rho^m(a)))\mu(a)\right)^{\beta-1}\\
&\geqslant 0.
\end{aligned}
$$

Then $h(t)\geqslant h(a)\geqslant 0$ it follows that $F^{\Delta}(t)\geqslant 0$. So $F(t)\geqslant F(a)=0$, which completes the proof. □

If $f(t)=t$ in Theorem 2, we have the following result:

COROLLARY 8. *Let* $\rho^m(a),\ b\in\mathbb{T},\ m\in\mathbb{N}_2,\ \alpha\geqslant\gamma+1,\ \beta\geqslant 2,$ *and* $\gamma>0$. *Suppose* $g,\sigma:\mathbb{T}\to\mathbb{R}$ *are delta differentiable for* $t\in\mathbb{T}^{\kappa}$, *and assume* g *is a nonnegative and increasing function such that* $g^{\alpha-\gamma}(a)\geqslant\beta\left(g^{\gamma}(\rho^m(a))\mu(a)\right)^{\beta-1}$ *and*

$$(\alpha-\gamma)g^{\Delta}(t)\geqslant\beta(\beta-1)g^{\gamma\beta-\alpha+1}(\rho^{m-2}(t))(\sigma^2(t)-a)^{\beta-2}\sigma^{\Delta}(t),$$

where $\sigma^2(t):=\sigma(\sigma(t))$. *Then*

$$\int_a^b g^{\alpha}(t)\Delta t\geqslant\left(\int_a^b g^{\gamma}(\rho^m(t))\Delta t\right)^{\beta}.$$

If $g(t)=t$ in Theorem 2, we have the following result:

COROLLARY 9. *Let* $\rho^m(a),\ b\in\mathbb{T},\ m\in\mathbb{N}_2,\ \alpha\geqslant\gamma+1,\ \beta\geqslant 2,$ *and* $\gamma>0$. *Suppose* f *is differentiable for* $t\in\mathbb{R}$ *and* $\sigma:\mathbb{T}\to\mathbb{R}$ *is delta differentiable for* $t\in\mathbb{T}^{\kappa}$, *and assume* f *and* g *are nonnegative and increasing functions such that* $f^{\alpha-\gamma}(a)\geqslant\beta\left(f^{\gamma}(\rho^m(a))\mu(a)\right)^{\beta-1}$ *and*

$$(\alpha-\gamma)f'(t)\geqslant\beta(\beta-1)f^{\gamma\beta-\alpha+1}(\rho^{m-2}(t))(\sigma^2(t)-a)^{\beta-2}\sigma^{\Delta}(t),$$

where $\sigma^2(t):=\sigma(\sigma(t))$. *Then*

$$\int_a^b f^{\alpha}(t)\Delta t\geqslant\left(\int_a^b f^{\gamma}(\rho^m(t))\Delta t\right)^{\beta}.$$

LEMMA 13. *Let $\varphi \geqslant 0$ be non-increasing on $[a,b]_{\mathbb{T}}$, and assume $\varphi : \mathbb{T} \to \mathbb{R}$ is delta differentiable for $t \in \mathbb{T}^{\kappa}$. If*

$$\int_a^{\sigma(t)} \psi(\tau)\Delta\tau \geqslant 0, \ \ \forall t \in [a,b]_{\mathbb{T}}, \tag{2}$$

then

$$\int_a^b \varphi(t)\psi(t)\Delta t \geqslant 0. \tag{3}$$

If inequality (2) is reversed, then inequality (3) is also reversed.

Proof. By the product rule,

$$\left[\varphi(t)\int_a^t \psi(\tau)\Delta\tau\right]^{\Delta} = \psi(t)\varphi(t) + \left(\int_a^{\sigma(t)} \psi(\tau)\Delta\tau\right)\varphi^{\Delta}(t), \forall t \in [a,b]_{\mathbb{T}}.$$

Therefore,

$$\int_a^b \psi(t)\varphi(t)\Delta t = -\int_a^b \left(\int_a^{\sigma(t)} \psi(\tau)\Delta\tau\right)\varphi^{\Delta}(t)\Delta t + \varphi(b)\int_a^b \psi(\tau)\Delta\tau \geqslant 0$$

being the sum of two nonnegative terms. □

THEOREM 3. *Suppose f, g, and h are nonnegative functions, where h is defined on $[a,b]_{\mathbb{T}}$, and f, g are defined on the range of h; Assume further that f is non-increasing, h is nondecreasing, and $f \circ h$, $g \circ h \in \mathrm{C_{rd}}$. If*

$$\int_a^{\sigma(t)} f^{\beta}(h(\tau))\Delta\tau \geqslant \int_a^{\sigma(t)} g^{\beta}(h(\tau))\Delta\tau, \quad \forall t \in [a,b]_{\mathbb{T}} \ \ \textit{and} \ \ \beta > 0,$$

then

$$\int_a^b f^{\alpha+\beta}(h(t))\Delta t \geqslant \int_a^b f^{\alpha}(h(t))g^{\beta}(h(t))\Delta t, \ \ \alpha \geqslant 0.$$

Proof. The proof follows from Lemma 13 by putting

$$\varphi(t) := f^{\alpha}(h(t)), \quad \text{and} \quad \psi(t) := f^{\beta}(h(t)) - g^{\beta}(h(t)). \quad \square$$

LEMMA 14. (Change of integration order [9], Lemma 1) *Assume $a,b \in \mathbb{T}$ and $f \in \mathrm{C}_{rd}(\mathbb{T}\times\mathbb{T},\mathbb{R})$, then*

$$\int_a^b \int_a^{\eta} f(\eta,\xi)\Delta\xi\Delta\eta = \int_a^b \int_{\sigma(\xi)}^b f(\eta,\xi)\Delta\eta\Delta\xi.$$

The proofs of the following two theorems are similar to the proofs in the nabla cases, which will be mentioned in the next section, and therefore are omitted.

THEOREM 4. *Suppose f, g, and h are nonnegative functions, where h is defined on $[a,b]_{\mathbb{T}}$, and f, g are defined on the range of h. Assume further that either $f\circ h$ or $g\circ h$ is nondecreasing, $f\circ h\in C^1_{rd}$, $g\circ h\in C^1_{rd}$, and $(f^{\alpha}\circ g)^{\Delta}(t)$ and $(g^{\beta}\circ h)^{\Delta}(t)$ exist for $t\in[a,b]_{\mathbb{T}^{\kappa}}$. If*

$$\int_{\sigma(t)}^{b} f^{\beta}(h(\tau))\Delta\tau \geqslant \int_{\sigma(t)}^{b} g^{\beta}(h(\tau))\Delta\tau,\quad \forall t\in[a,b]_{\mathbb{T}}\quad \text{and}\quad \beta>0,$$

then

$$\int_{a}^{b} f^{\alpha+\beta}(h(\tau))\Delta\tau \geqslant \int_{a}^{b} f^{\alpha}(h(\tau))g^{\beta}(h(\tau))\Delta\tau$$

holds for all positive numbers α and β.

THEOREM 5. *Suppose f, g, and h are nonnegative functions, where h is defined on $[a,b]_{\mathbb{T}}$, and f, g are defined on the range of h. Assume further that $g\circ h$ is nondecreasing, $g\circ h\in C^1_{rd}$, $f\circ h\in C_{rd}$ and $(g^{-\alpha}\circ h)^{\Delta}(t)$ exists for $t\in[a,b]_{\mathbb{T}^{\kappa}}$. If*

$$\int_{\sigma(t)}^{b} f^{\beta}(h(\tau))\Delta\tau \geqslant \int_{\sigma(t)}^{b} g^{\beta}(h(\tau))\Delta\tau,\quad \forall t\in[a,b]_{\mathbb{T}}\quad \text{and}\quad \beta>0,$$

then

$$\int_{a}^{b} f^{\beta-\alpha}(h(\tau))\Delta\tau \leqslant \int_{a}^{b} f^{\beta}(h(\tau))g^{-\alpha}(h(\tau))\Delta\tau$$

holds for all $\beta>\alpha>0$.

4. Nabla integral inequalities

In this section, we give some Feng-Qi type nabla-integral inequalities on time scales. We begin with the following useful lemma.

LEMMA 15. *Let $p\geqslant 1$. Suppose $f:\mathbb{R}\to\mathbb{R}$ is continuously differentiable and $g:\mathbb{T}\to\mathbb{R}$ is nabla differentiable for $t\in\mathbb{T}_{\kappa}$, and assume f, f', and g are nonnegative and nondecreasing functions. Then*

$$pf^{p-1}(g(\rho(t)))f'(g(\rho(t)))g^{\nabla}(t) \leqslant (f^{p}\circ g)^{\nabla}(t) \leqslant pf^{p-1}(g(t))f'(g(t))g^{\nabla}(t).$$

Proof. Using Lemma 9, we obtain $(f^{p}\circ g)^{\nabla}(t)=(f^{p}\circ g)^{\Delta}(\rho(t))$. The rest of the proof is similar to the proof of Lemma 12 and therefore is omitted. □

If $g(t)=t$ in Lemma 15, then we have the following result:

COROLLARY 10. *Let $p\geqslant 1$. Suppose $f:\mathbb{T}\to\mathbb{R}$ is nabla differentiable for $t\in\mathbb{T}_{\kappa}$, and assume f, f' are nonnegative and nondecreasing functions. Then*

$$pf^{p-1}(\rho(t))f'(\rho(t)) \leqslant (f^{p})^{\nabla}(t) \leqslant pf^{p-1}(t)f'(t).$$

If $f(t)=t$ in Lemma 15, then we have the following result ([7], Lemma 4.1):

COROLLARY 11. *Let $p \geqslant 1$. Suppose $g:\mathbb{T}\to\mathbb{R}$ is nabla differentiable for $t\in\mathbb{T}_\kappa$, and assume g is nonnegative and nondecreasing function. Then*

$$pg^{p-1}(\rho(t))g^{\nabla}(t)\leqslant (g^p)^{\nabla}(t)\leqslant pg^{p-1}(t)g^{\nabla}(t).$$

For $p=1$ in Lemma 15, we have the following result:

COROLLARY 12. *Suppose $f:\mathbb{R}\to\mathbb{R}$ is continuously differentiable and $g:\mathbb{T}\to\mathbb{R}$ is nabla differentiable for $t\in\mathbb{T}_\kappa$, and assume g is nonnegative and f',g are nondecreasing functions. Then*

$$f'(g(\rho(t)))g^{\nabla}(t)\leqslant (f\circ g)^{\nabla}(t)\leqslant f'(g(t))g^{\nabla}(t).$$

THEOREM 6. *Let $a,b\in\mathbb{T}$, $\alpha\geqslant\gamma+1$, $\beta\geqslant 2$, and $\gamma>0$. Suppose $f:\mathbb{R}\to\mathbb{R}$ is continuously differentiable and $g:\mathbb{T}\to\mathbb{R}$ is nabla differentiable for $t\in\mathbb{T}_\kappa$, and assume f, f', and g are nonnegative and increasing functions. If*

$$(\alpha-\gamma)f^{\alpha-\gamma-1}(g(\rho(t)))f'(g(\rho(t)))g^{\nabla}(t)\geqslant\beta(\beta-1)f^{\gamma(\beta-1)}(g(t))(t-a)^{\beta-2}$$

is satisfied, then

$$\int_a^b f^{\alpha}(g(t))\nabla t\geqslant\left(\int_a^b f^{\gamma}(g(t))\nabla t\right)^{\beta}.$$

Proof. For each $t\in[a,b]_{\mathbb{T}}$ let

$$F(t):=\int_a^t f^{\alpha}(g(\tau))\nabla\tau-\left(\int_a^t f^{\gamma}(g(\tau))\nabla\tau\right)^{\beta}.$$

Using Corollary 11, we have

$$\begin{aligned}F^{\nabla}(t)\geqslant& f^{\alpha}(g(t))-\beta\left(\int_a^t f^{\gamma}(g(\tau))\nabla\tau\right)^{\beta-1}f^{\gamma}(g(t))\\ =&f^{\gamma}(g(t))\left(f^{\alpha-\gamma}(g(t))-\beta\left(\int_a^t f^{\gamma}(g(\tau))\nabla\tau\right)^{\beta-1}\right)\\ =&f^{\gamma}(g(t))h(t),\end{aligned}$$

where

$$h(t):=f^{\alpha-\gamma}(g(t))-\beta\left(\int_a^t f^{\gamma}(g(\tau))\nabla\tau\right)^{\beta-1}.$$

Now, using Lemma 15 and Corollary 11,

$$\begin{aligned}h^{\nabla}(t)=&(f^{\alpha-\gamma}(g(t)))^{\nabla}-\beta\left(\left(\int_a^t f^{\gamma}(g(\tau))\nabla\tau\right)^{\beta-1}\right)^{\nabla}\\ \geqslant&(\alpha-\gamma)f^{\alpha-\gamma-1}(g(\rho(t)))f'(g(\rho(t)))g^{\nabla}(t)\\ &-\beta(\beta-1)\left(\int_a^t f^{\gamma}(g(\tau))\nabla\tau\right)^{\beta-2}f^{\gamma}(g(t)).\end{aligned}$$

Since $\gamma > 0$ and f, g are increasing, we have that $f^\gamma \circ g$ is increasing. It follows that

$$\int_a^t f^\gamma(g(\tau))\nabla\tau \leqslant f^\gamma(g(t))(t-a).$$

Hence we obtain

$$\begin{aligned} h^\nabla(t) \geqslant &(\alpha-\gamma) f^{\alpha-\gamma-1}(g(\rho(t)))f'(g(\rho(t)))g^\nabla(t) \\ &- \beta(\beta-1) f^{\gamma(\beta-1)}(g(t))(t-a)^{\beta-2} \\ \geqslant &0. \end{aligned}$$

Therefore, $h(t)$ is nondecreasing. But $h(a) = f^{\alpha-\gamma}(g(a)) \geqslant 0$. Then $h(t) \geqslant h(a) \geqslant 0$, and it follows that $F^\nabla(t) \geqslant 0$. So $F(t) \geqslant F(a) = 0$, which completes the proof. □

If we let $g(t) = t$ in Theorem 6, we get the following result:

COROLLARY 13. *Let $a,b \in \mathbb{T}$, $\alpha \geqslant \gamma+1$, $\beta \geqslant 2$, and $\gamma > 0$. Suppose f, f' are nonnegative and increasing functions. If*

$$(\alpha-\gamma) f^{\alpha-\gamma-1}(\rho(t))f'(\rho(t)) \geqslant \beta(\beta-1) f^{\gamma(\beta-1)}(t)(t-a)^{\beta-2}$$

is satisfied, then

$$\int_a^b f^\alpha(t)\nabla t \geqslant \left(\int_a^b f^\gamma(t)\nabla t\right)^\beta.$$

If we let $f(t) = t$ in Theorem 6, then we get the following result:

COROLLARY 14. *Let $\alpha \geqslant \gamma+1$, $\beta \geqslant 2$, and $\gamma > 0$. Suppose $g : \mathbb{T} \to \mathbb{R}$ is nabla differentiable for $t \in \mathbb{T}_\kappa$, and assume g is a nonnegative and increasing function. If*

$$(\alpha-\gamma) g^{\alpha-\gamma-1}(\rho(t))g^\nabla(t) \geqslant \beta(\beta-1) g^{\gamma(\beta-1)}(t)(t-a)^{\beta-2}$$

is satisfied, then

$$\int_a^b g^\alpha(t)\nabla t \geqslant \left(\int_a^b g^\gamma(t)\nabla t\right)^\beta.$$

If $\gamma = 1$, $\beta = \alpha - 1$ in Corollary 14, we obtain the following result:

COROLLARY 15. *Let $a,b \in \mathbb{T}$, $\alpha \geqslant 3$. Suppose f is a nonnegative and increasing function, and assume $f : \mathbb{T} \to \mathbb{R}$ is nabla differentiable for $t \in \mathbb{T}_\kappa$. If*

$$f^{\alpha-2}(\rho(t))f^\nabla(t) \geqslant (\alpha-2) f^{\alpha-2}(t)(t-a)^{\alpha-3}$$

is satisfied, then

$$\int_a^b f^\alpha(t)\nabla t \geqslant \left(\int_a^b f(t)\nabla t\right)^{\alpha-1}.$$

REMARK 3. The result in Theorem 4.3 of [7] is a special case of the above corollary but does not follow from its conditions.

THEOREM 7. *Let* $\rho^m(a),\ b\in\mathbb{T},\ m\in\mathbb{N}_0,\ \alpha\geqslant\gamma+1,\ \beta\geqslant 2,$ *and* $\gamma>0$. *Suppose* $f:\mathbb{R}\to\mathbb{R}$ *is continuously differentiable and* $g:\mathbb{T}\to\mathbb{R}$ *is nabla differentiable for* $t\in\mathbb{T}_\kappa$, *and assume* $f,\ f'$, *and* g *are nonnegative and increasing functions. If*

$$f'(g(\rho(t)))g^\nabla(t)>\frac{\beta(\beta-1)}{\alpha-\gamma}f^{\gamma\beta-\alpha+1}(g(\rho^m(t)))(t-a)^{\beta-2}$$

is satisfied, then

$$\int_a^b f^\alpha(g(t))\nabla t\geqslant\left(\int_a^b f^\gamma(g(\rho^m(t)))\nabla t\right)^\beta.$$

Proof. For each $t\in[a,b]_\mathbb{T}$ let

$$F(t):=\int_a^t f^\alpha(g(\tau))\nabla\tau-\left(\int_a^t f^\gamma(g(\rho^m(\tau)))\nabla\tau\right)^\beta.$$

Using Corollary 11, we have

$$\begin{aligned}F^\nabla(t)\geqslant&f^\alpha(g(t))-\beta\left(\int_a^t f^\gamma(g(\rho^m(\tau)))\nabla\tau\right)^{\beta-1}f^\gamma(g(\rho^m(t)))\\ \geqslant&f^\alpha(g(t))-\beta\left(\int_a^t f^\gamma(g(\rho^m(\tau)))\nabla\tau\right)^{\beta-1}f^\gamma(g(t))\\ =&f^\gamma(g(t))\left(f^{\alpha-\gamma}(g(t))-\beta\left(\int_a^t f^\gamma(g(\rho^m(\tau)))\nabla\tau\right)^{\beta-1}\right)\\ =&f^\gamma(g(t))h(t),\end{aligned}$$

where

$$h(t):=f^{\alpha-\gamma}(g(t))-\beta\left(\int_a^t f^\gamma(g(\rho^m(\tau)))\nabla\tau\right)^{\beta-1}.$$

Now, using Lemma 15 and Corollary 11,

$$\begin{aligned}h^\nabla(t)=&(f^{\alpha-\gamma}(g(t)))^\nabla-\beta\left(\left(\int_a^t f^\gamma(g(\rho^m(\tau)))\nabla\tau\right)^{\beta-1}\right)^\nabla\\ \geqslant&(\alpha-\gamma)f^{\alpha-\gamma-1}(g(\rho(t)))f'(g(\rho(t)))g^\nabla(t)\\ &-\beta(\beta-1)\left(\int_a^t f^\gamma(g(\rho^m(\tau)))\nabla\tau\right)^{\beta-2}f^\gamma(g(\rho^m(t))).\end{aligned}$$

Since $\gamma>0$ and $f,\ g$ are increasing, we have that $f^\gamma\circ g$ is increasing. It follows that

$$\int_a^t f^\gamma(g(\rho^m(\tau)))\nabla\tau\leqslant f^\gamma(g(\rho^m(t)))(t-a).$$

Hence we obtain

$$\begin{aligned}
h^{\nabla}(t) \geqslant &(\alpha-\gamma) f^{\alpha-\gamma-1}(g(\rho(t))) f'(g(\rho(t))) g^{\nabla}(t) \\
&-\beta(\beta-1) f^{\gamma(\beta-1)}(g(\rho^{m}(t)))(t-a)^{\beta-2} \\
\geqslant &(\alpha-\gamma) f^{\alpha-\gamma-1}(g(\rho^{m}(t))) f'(g(\rho(t))) g^{\nabla}(t) \\
&-\beta(\beta-1) f^{\gamma(\beta-1)}(g(\rho^{m}(t)))(t-a)^{\beta-2} \\
= &(\alpha-\gamma) f^{\alpha-\gamma-1}(g(\rho^{m}(t)))\Big(f'(g(\rho(t))) g^{\nabla}(t) \\
&-\frac{\beta(\beta-1)}{\alpha-\gamma} f^{\gamma\beta-\alpha+1}(g(\rho^{m}(t)))(t-a)^{\beta-2}\Big) \\
\geqslant &0.
\end{aligned}$$

Therefore, h is nondecreasing. But $h(a)=f^{\alpha-\gamma}(g(a)) \geqslant 0$, so $h(t) \geqslant h(a) \geqslant 0$ and it follows that $F^{\nabla}(t) \geqslant 0$. So $F(t) \geqslant F(a)=0$, which completes the proof. □

If $f(t)=t$ in Theorem 7, we have the following result:

COROLLARY 16. *Let $\rho^{m}(a),\ b \in \mathbb{T},\ m \in \mathbb{N}_0,\ \alpha \geqslant \gamma+1,\ \beta \geqslant 2,$ and $\gamma>0$. Suppose $g: \mathbb{T} \rightarrow \mathbb{R}$ is nabla differentiable for $t \in \mathbb{T}_\kappa$, and assume g is a nonnegative and increasing function. If*

$$g^{\nabla}(t)>\frac{\beta(\beta-1)}{\alpha-\gamma} g^{\gamma\beta-\alpha+1}(\rho^{m}(t))(t-a)^{\beta-2}$$

is satisfied, then

$$\int_a^b g^{\alpha}(t) \nabla t \geqslant\left(\int_a^b g^{\gamma}(\rho^{m}(t)) \nabla t\right)^{\beta}.$$

If $g(t)=t$ in Theorem 7, we have the following result:

COROLLARY 17. *Let $\rho^{m}(a),\ b \in \mathbb{T},\ m \in \mathbb{N}_0,\ \alpha \geqslant \gamma+1,\ \beta \geqslant 2,$ and $\gamma>0$. Suppose $f,\ f'$ are nonnegative and increasing functions. If*

$$f'(\rho(t))>\frac{\beta(\beta-1)}{\alpha-\gamma} f^{\gamma\beta-\alpha+1}(\rho^{m}(t))(t-a)^{\beta-2}$$

is satisfied, then

$$\int_a^b f^{\alpha}(t) \nabla t \geqslant\left(\int_a^b f^{\gamma}(\rho^{m}(t)) \nabla t\right)^{\beta}.$$

LEMMA 16. *Suppose φ is nonnegative and nondecreasing on $[a,b]_{\mathbb{T}}$, and assume $\varphi: \mathbb{T} \rightarrow \mathbb{R}$ is nabla differentiable for $t \in \mathbb{T}_\kappa$. If*

$$\int_{\rho(t)}^b \psi(\tau) \nabla \tau \geqslant 0, \quad \forall t \in[a,b]_{\mathbb{T}}, \tag{4}$$

then

$$\int_a^b \varphi(t)\psi(t)\nabla t \geqslant 0. \tag{5}$$

If inequality (4) *is reversed, then inequality* (5) *is also reversed.*

Proof. By the product rule,

$$\left[\varphi(t)\int_t^b \psi(\tau)\nabla\tau\right]^{\nabla} = -\psi(t)\varphi(t) + \left(\int_{\rho(t)}^b \psi(\tau)\nabla\tau\right)\varphi^{\nabla}(t), \forall t \in [a,b]_{\mathbb{T}}.$$

Therefore,

$$\int_a^b \psi(t)\varphi(t)\nabla t = \int_a^b \left(\int_{\rho(t)}^b \psi(\tau)\nabla\tau\right)\varphi^{\nabla}(t)\nabla t + \varphi(a)\int_a^b \psi(\tau)\nabla\tau \geqslant 0$$

being the sum of two nonnegative terms. □

THEOREM 8. *Suppose f, g, and h are nonnegative functions, where h is defined on $[a,b]_{\mathbb{T}}$, and f, g are defined on the range of h. Assume further that f, h are nondecreasing. If*

$$\int_{\rho(t)}^b f^{\beta}(h(\tau))\nabla\tau \geqslant \int_{\rho(t)}^b g^{\beta}(h(\tau))\nabla\tau, \quad \forall t \in [a,b]_{\mathbb{T}} \quad \text{and} \quad \beta > 0,$$

then

$$\int_a^b f^{\alpha+\beta}(h(t))\nabla t \geqslant \int_a^b f^{\alpha}(h(t))g^{\beta}(h(t))\nabla t, \ \ \alpha \geqslant 0.$$

Proof. The proof follows from Lemma 16 by putting

$$\varphi(t) := f^{\alpha}(h(t)), \quad \text{and} \quad \psi(t) := f^{\beta}(h(t)) - g^{\beta}(h(t)). \quad \square$$

LEMMA 17. *Assume that $a,b \in \mathbb{T}$ and $f \in C_{ld}(\mathbb{T}\times\mathbb{T},\mathbb{R})$, then*

$$\int_a^b \int_a^{\eta} f(\eta,\xi)\nabla\xi\nabla\eta = \int_a^b \int_{\rho(\xi)}^b f(\eta,\xi)\nabla\eta\nabla\xi. \tag{6}$$

Proof. The proof of this lemma is similar to the proof of Lemma 14, and therefore is omitted. □

THEOREM 9. *Suppose f, g, and h are nonnegative functions, where h is defined on $[a,b]_{\mathbb{T}}$, and f, g are defined on the range of h. Assume further that either $f\circ h$ or $g\circ h$ is nondecreasing, $f\circ h \in C^1_{ld}$, $g\circ h \in C^1_{ld}$, and $(f^{\alpha}\circ h)^{\nabla}(t)$ and $(g^{\beta}\circ h)^{\nabla}(t)$ exist for $t \in [a,b]_{\mathbb{T}_{\kappa}}$. If*

$$\int_{\rho(t)}^b f^{\beta}(h(\tau))\nabla\tau \geqslant \int_{\rho(t)}^b g^{\beta}(h(\tau))\nabla\tau, \quad \forall t \in [a,b]_{\mathbb{T}} \quad \text{and} \quad \beta > 0, \tag{7}$$

then

$$\int_a^b f^{\alpha+\beta}(h(\tau))\nabla\tau \geqslant \int_a^b f^{\alpha}(h(\tau))g^{\beta}(h(\tau))\nabla\tau$$

holds for all positive numbers α *and* β.

Proof. Suppose that $f\circ h$ is nondecreasing. Using the Fundamental Theorem for nabla case and Lemma 11, we have

$$\begin{aligned}
&\int_a^b f^{\alpha+\beta}(h(\tau))\nabla\tau\\
&=\int_a^b f^{\beta}(h(\tau))f^{\alpha}(h(\tau))\nabla\tau\\
&=\int_a^b f^{\beta}(h(\tau))\Big(\int_a^{\tau}(f^{\alpha}\circ h)^{\nabla}(t)\nabla t+f^{\alpha}(h(a))\Big)\nabla\tau\\
&\overset{(6)}{=}\int_a^b \Big((f^{\alpha}\circ h)^{\nabla}(t)\int_{\rho(t)}^b f^{\beta}(h(\tau))\nabla\tau\Big)\nabla t+f^{\alpha}(h(a))\int_a^b f^{\beta}(h(\tau))\nabla\tau\\
&\overset{(7)}{\geqslant}\int_a^b \Big((f^{\alpha}\circ h)^{\nabla}(t)\int_{\rho(t)}^b g^{\beta}(h(\tau))\nabla\tau\Big)\nabla t+f^{\alpha}(h(a))\int_a^b g^{\beta}(h(\tau))\nabla\tau\\
&\overset{(6)}{=}\int_a^b g^{\beta}(h(\tau))\Big(\int_a^{\tau}(f^{\alpha}\circ h)^{\nabla}(t)\nabla t+f^{\alpha}(h(a))\Big)\nabla\tau\\
&=\int_a^b f^{\alpha}(h(\tau))g^{\beta}(h(\tau))\nabla\tau.
\end{aligned}$$

Now suppose $g\circ h$ is nondecreasing. Notice that $\alpha,\beta>0$, so from (7) we have

$$\int_{\rho(t)}^b f^{\alpha}(h(\tau))\nabla\tau \geqslant \int_{\rho(t)}^b g^{\alpha}(h(\tau))\nabla\tau,\quad \forall t\in[a,b]_{\mathbb{T}}\quad\text{and}\quad \alpha>0. \tag{8}$$

Using the Fundamental Theorem for nabla case, we have

$$\begin{aligned}
&\int_a^b f^{\alpha}(h(\tau))g^{\beta}(h(\tau))\nabla\tau \qquad (9)\\
&=\int_a^b f^{\alpha}(h(\tau))\Big(\int_a^{\tau}(g^{\beta}\circ h)^{\nabla}(t)\nabla t+g^{\beta}(h(a))\Big)\nabla\tau\\
&\overset{(6)}{=}\int_a^b \Big((g^{\beta}\circ h)^{\nabla}(t)\int_{\rho(t)}^b f^{\alpha}(h(\tau))\nabla\tau\Big)\nabla t+g^{\beta}(h(a))\int_a^b f^{\alpha}(h(\tau))\nabla\tau\\
&\overset{(8)}{\geqslant}\int_a^b \Big((g^{\beta}\circ h)^{\nabla}(t)\int_{\rho(t)}^b g^{\alpha}(h(\tau))\nabla\tau\Big)\nabla t+g^{\beta}(h(a))\int_a^b g^{\alpha}(h(\tau))\nabla\tau\\
&\overset{(6)}{=}\int_a^b g^{\alpha}(h(\tau))\Big(\int_a^{\tau}(g^{\beta}\circ h)^{\nabla}(t)\nabla t+g^{\beta}(h(a))\Big)\nabla\tau\\
&=\int_a^b g^{\beta+\alpha}(h(\tau))\nabla\tau.
\end{aligned}$$

Using the weighted AM-GM inequality, we have

$$\frac{\alpha}{\alpha+\beta}f^{\alpha+\beta}(h(\tau))+\frac{\beta}{\alpha+\beta}g^{\alpha+\beta}(h(\tau))\geqslant f^{\alpha}(h(\tau))g^{\beta}(h(\tau)).$$

Integrating the above inequality gives

$$\begin{aligned}&\int_a^b f^{\alpha}(h(\tau))g^{\beta}(h(\tau))\nabla\tau\\ \leqslant&\frac{\alpha}{\alpha+\beta}\int_a^b f^{\alpha+\beta}(h(\tau))\nabla\tau+\frac{\beta}{\alpha+\beta}\int_a^b g^{\alpha+\beta}(h(\tau))\nabla\tau\\ \overset{(9)}{\leqslant}&\frac{\alpha}{\alpha+\beta}\int_a^b f^{\alpha+\beta}(h(\tau))\nabla\tau+\frac{\beta}{\alpha+\beta}\int_a^b f^{\alpha}(h(\tau))g^{\beta}(h(\tau))\nabla\tau.\end{aligned}$$

It is easy to see that the result follows from the last inequality. □

Let $h(t)=t$, $g(t)=t$, $a=0$, and $[a,b]_{\mathbb{T}}=[a,b]_q=\{bq^k:0\leqslant k\leqslant n,0<q<1\}$. We get the following result.

COROLLARY 18. *If f is a nonnegative function on $[0,b]_q$ and satisfies*

$$\int_{qt}^b f^{\beta}(\tau)d_q\tau\geqslant\int_{qt}^b \tau^{\beta}d_q\tau$$

for all $t\in[0,b]_q$ and $\beta>0$, then the inequality

$$\int_0^b f^{\beta+\alpha}(\tau)d_q\tau\geqslant\int_0^b f^{\alpha}(\tau)\tau^{\beta}d_q\tau$$

holds for all positive numbers α and β.

REMARK 4. A similar result can be found in Theorem 3 in [12], where the sufficient condition seems to be incorrect due to the improper use of Lemma 17.

THEOREM 10. *Suppose $f,g,$ and h are nonnegative functions, where h is defined on $[a,b]_{\mathbb{T}}$, and f, g are defined on the range of h. Assume further that $g\circ h$ is decreasing, $g\circ h\in\mathrm{C}^1_{ld}$, $f\circ h\in\mathrm{C}_{ld}$ and $(g^{-\alpha}\circ h)^{\nabla}(t)$ exists for $t\in[a,b]_{\mathbb{T}_\kappa}$. If*

$$\int_{\rho(t)}^b f^{\beta}(h(\tau))\nabla\tau\geqslant\int_{\rho(t)}^b g^{\beta}(h(\tau))\nabla\tau,\quad\forall t\in[a,b]_{\mathbb{T}}\ \text{ and }\ \beta>0,\tag{10}$$

then

$$\int_a^b f^{\beta-\alpha}(h(\tau))\nabla\tau\leqslant\int_a^b f^{\beta}(h(\tau))g^{-\alpha}(h(\tau))\nabla\tau$$

holds for all $\beta>\alpha>0$.

Proof. Using the Fundamental Theorem for the nabla case and Lemma 11, we have

$$\begin{aligned}
&\int_a^b f^\beta(h(\tau))g^{-\alpha}(h(\tau))\nabla\tau \qquad (11)\\
&=\int_a^b f^\beta(h(\tau))\Big(\int_a^\tau (g^{-\alpha}\circ h)^\nabla(t)\nabla t+g^{-\alpha}(h(a))\Big)\nabla\tau\\
&\overset{(6)}{=}\int_a^b \Big((g^{-\alpha}\circ h)^\nabla(t)\int_{\rho(t)}^b f^\beta(h(\tau))\nabla\tau\Big)\nabla t+g^{-\alpha}(h(a))\int_a^b f^\beta(h(u))\nabla u\\
&\overset{(10)}{\geqslant}\int_a^b \Big((g^{-\alpha}\circ h)^\nabla(t)\int_{\rho(t)}^b g^\beta(h(\tau))\nabla\tau\Big)\nabla t+g^{-\alpha}(h(a))\int_a^b g^\beta(h(\tau))\nabla\tau\\
&\overset{(6)}{=}\int_a^b g^\beta(h(\tau))\Big(\int_a^\tau (g^{-\alpha}\circ h)^\nabla(t)\nabla t+g^{-\alpha}(h(a))\Big)\nabla\tau\\
&=\int_a^b g^{\beta-\alpha}(h(\tau))\nabla\tau.
\end{aligned}$$

Using the weighted AM-GM inequality, we get

$$f^{\alpha_1}(h(\tau))g^{\beta_1}(h(\tau))\leqslant\frac{\alpha_1}{\alpha_1+\beta_1}f^{\alpha_1+\beta_1}(h(\tau))+\frac{\beta_1}{\alpha_1+\beta_1}g^{\alpha_1+\beta_1}(h(\tau)),\ \alpha_1,\beta_1>0.$$

Let $\alpha_1+\beta_1=\beta,\ \ \beta_1=\alpha$. Then $\beta>\alpha>0$,

$$f^{\beta-\alpha}(h(\tau))\leqslant\frac{\beta-\alpha}{\beta}f^\beta(h(\tau))g^{-\alpha}(h(\tau))+\frac{\alpha}{\beta}g^{\beta-\alpha}(h(\tau)).$$

Integrating the above inequality yields

$$\begin{aligned}
&\int_a^b f^{\beta-\alpha}(h(\tau))\nabla\tau\\
&\leqslant\frac{\beta-\alpha}{\beta}\int_a^b f^\beta(h(\tau))g^{-\alpha}(h(\tau))\nabla\tau+\frac{\alpha}{\beta}\int_a^b g^{\beta-\alpha}(h(\tau))\nabla\tau\\
&\overset{(11)}{\leqslant}\frac{\beta-\alpha}{\beta}\int_a^b f^\beta(h(\tau))g^{-\alpha}(h(\tau))\nabla\tau+\frac{\alpha}{\beta}\int_a^b f^\beta(h(\tau))g^{-\alpha}(h(\tau))\nabla\tau\\
&=\int_a^b f^\beta(h(\tau))g^{-\alpha}(h(\tau))\nabla\tau. \quad \square
\end{aligned}$$

REFERENCES

[1] C. D. AHLBRANDT AND C. MORIAN, *Partial differential equations on time scales*, J. Comput. Appl. Math. **141**, (2002), 35–55.
[2] R. P. AGARWAL, D. O'REGAN AND S. H. SAKER, *Hardy Type Inequalities on Time Scales*, Springer, Switzerland, 2016.
[3] E. AKIN, *Cauchy functions for dynamic equations on a measure chain*, Math. Anal. Appl. **267**, (2002), 97–115.

[4] M. BOHNER AND A. PETERSON, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, 2001.

[5] M. BOHNER AND A. PETERSON, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.

[6] K. BRAHIM, N. BETTAIBI AND M. SELLAMI, *On some Feng-Qi type q-integral inequalities*, J. Inequal. Pure Appl. Math. **9**, 2(2008), 1–7.

[7] T. FAYYAZ, N. IRSHAD, A. KHAN, G. RAHMAN AND G. ROQIA, *Generalized integral inequalities on time scales*, J. Inequal. Appl. **235**, (2016), 1–12.

[8] S. HILGER, *Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannig-faltigkeiten*, PhD thesis, Universität Würzburg, 1988.

[9] B. KARPUZ, *Unbounded oscillation of higher-order nonlinear delay dynamic equations of neutral type with oscillating coefficients*, Electron. J. Qual. Theory Differ. Equ. **34**, (2009), 1–14.

[10] B. KARPUZ, *Volterra theory on time scales*, Results. Math. **65**, (2014), 263–292.

[11] S. KELLER, *Asymptotisches Verhalten invarianter Faserbündel bei Diskretisierung und Mittelwertbildung im Rahmen der Analysis auf Zeitskalen*, PhD thesis, Universität Augsburg, 1999.

[12] Y. MIAO AND F. QI, *Several q-integral inequalities*, J. Math. Inequal. **3**, 1(2009), 115–121.

[13] C. PÖTZSCHE, *Chain rule and invariance principle on measure chains*, J. Comput. Appl. Math. **141**, (2002), 249–254.

[14] F. QI, A. J. LI, W. Z. ZHAO, D. W. NIU AND J. CAO, *Extensions of several integral inequalities*, J. Inequal. Pure Appl. Math. **7**, 3(2006), 1–4.

[15] F. QI, *Several integral inequalities*, J. Inequal. Pure Appl. Math. **1**, 2(2000), 1–3.

[16] M. R. SEGI RAHMAT, *On some (q,h)-analogues of integral inequalities on discrete time scales*, Comput. Math. Appl. **62**, 2(2000), 1790–1797.

[17] W. SULAIMAN, *New Types of Q-Integral Inequalities*, Adv. Pure. Math. **1**, (2011), 77–80.

[18] W. SULAIMAN, *A Study on New q-Integral Inequalities*, Appl. Math. **1**, (2011), 465–469.

[19] W. SULAIMAN, *Several Ideas on Some Integral Inequalities*, Adv. Pure. Math. **1**, (2011), 63–66.

[20] L. YIN, Q. LUO AND F. QI, *Several integral inequalities on time scales*, J. Math. Inequal. **6**, (2012), 419–429.

(Received March 9, 2017)

Feifei Du
School of Mathematics
Sun Yat-Sen University
Guangzhou, China, 510275
e-mail: qinjin65@126.com

Wei Hu
Department of Mathematics
University of Nebraska-Lincoln
Lincoln, NE 68588-0130, USA
e-mail: wei.hu@huskers.unl.edu

Lynn Erbe
Department of Mathematics
University of Nebraska-Lincoln
Lincoln, NE 68588-0130, USA
e-mail: lerbe2@math.unl.edu

Allan Peterson
Department of Mathematics
University of Nebraska-Lincoln
Lincoln, NE 68588-0130, USA
e-mail: apeterson1@math.unl.edu