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BOUNDEDNESS AND CONTINUITY OF MAXIMAL OPERATORS ASSOCIATED TO POLYNOMIAL COMPOUND CURVES ON TRIEBEL–LIZORKIN SPACES

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Abstract. In this paper we study the Triebel-Lizorkin space boundedness and continuity of maximal operators related to rough singular integrals associated to polynomial compound curves. We prove that the above operators are bounded and continuous on the inhomogeneous Triebel-Lizorkin space $F^{\alpha}_{p,q}(\mathbb{R}^n)$ for all $\alpha\in(0,1)$, $p\in[2,\infty)$ and $q\in(2p/(p+2),\infty)$ under the conditions that the integral kernels are given by $\Omega\in L(\log^+L)^{1/2}(\mathrm{S}^{n-1})\cup(\cup_{1<r<\infty}B_r^{0,-1/2}(\mathrm{S}^{n-1}))$. We also establish the boundedness and continuity of the above operators on the inhomogeneous Besov space $B^{\alpha}_{p,q}(\mathbb{R}^n)$ for $\alpha\in(0,1)$, $p\in[2,\infty)$ and $q\in(1,\infty)$. In addition, the corresponding results for maximal operators related to parametric Marcinkiewicz integrals are also considered.

1. Introduction

Let $\mathbb{R}_+ := (0,\infty)$ and $\mathscr{K}_2$ be the set of all measurable functions $h:\mathbb{R}_+\to\mathbb{R}$ with $\|h\|_{L^2(\mathbb{R}_+,dr/r)}\leqslant 1$, where $L^2(\mathbb{R}_+,dr/r)$ is the set of all measurable functions $h:\mathbb{R}_+\to\mathbb{R}$ that satisfy

$$\|h\|_{L^2(\mathbb{R}_+,dr/r)} := \Big(\int_0^\infty |h(r)|^2 r^{-1}dr\Big)^{1/2}<\infty.$$

Let $\mathbb{R}^n$ $(n\geqslant 2)$ be the n-dimensional Euclidean space and S^{n-1} denote the unit sphere in $\mathbb{R}^n$ equipped with the induced Lebesgue measure $d\sigma$. Assume that Ω is homogeneous of degree zero and integrable over S^{n-1} and satisfies the cancellation condition

$$\int_{\mathrm{S}^{n-1}}\Omega(u)d\sigma(u)=0. \tag{1}$$

Suppose that P is a real polynomial on $\mathbb{R}$ of degree N and satisfies $P(0)=0$. For a suitable function $\varphi:\mathbb{R}_+\to\mathbb{R}$, the corresponding maximal operator $\mathscr{S}_{\Omega,P,\varphi}$ along the “polynomial compound curve” $P(\varphi(|y|))y'$ on $\mathbb{R}^n$ is defined by

$$\mathscr{S}_{\Omega,P,\varphi}f(x)=\sup_{h\in\mathscr{K}_2}|T_{h,\Omega,P,\varphi}f(x)|, \tag{2}$$

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where $x \in \mathbb{R}^n$ and $T_{h,\Omega,P,\varphi}$ is the singular integral operator given by

$$T_{h,\Omega,P,\varphi}f(x) := \text{p.v.} \int_{\mathbb{R}^n} f(x - P(\varphi(|y|))y') \frac{h(|y|)\Omega(y)}{|y|^n} dy,$$

where $y' = y/|y|$ for any nonzero point $y \in \mathbb{R}^n$.

It is a long time interesting topic to study the rough singular integral operators. A celebrated work in this direction is due to Fefferman [12] who first introduced and studied the singular integral operator with rough radial kernels h, which has been investigated extensively by many authors. Due to the presence of h, a class of maximal operators related to the above rough singular integrals was first introduced by Chen and Lin [11], which is denoted by $\mathscr{S}_\Omega$ and corresponds to the special case of $\mathscr{S}_{\Omega,P,\varphi}$ with $P(t) = \varphi(t) = t$. Chen and Lin proved that if $\Omega \in \mathscr{C}(\mathrm{S}^{n-1})$, then $\mathscr{S}_\Omega$ is bounded on $L^p(\mathbb{R}^n)$ for any $p > 2n/(2n-1)$ and the range of p is best possible. Subsequently, the L^p mapping properties of $\mathscr{S}_\Omega$ have been discussed extensively by many authors. For example, see [26] for the case $\Omega \in H^1(\mathrm{S}^{n-1})$ (the Hardy space on S^{n-1}), [6, 7] for the case $\Omega \in L(\log^+ L)^{1/2}(\mathrm{S}^{n-1})$, [3, 5] for the case $\Omega \in B_r^{(0,-1/2)}(\mathrm{S}^{n-1})$ with some $r > 1$ (the block space generated by r-blocks). See Appendix for these definitions and relationships of the above rough kernels.

It is well known that the Triebel-Lizorkin spaces and Besov spaces contain many important function spaces, such as Lebesgue spaces, Hardy spaces, Sobolev spaces and Lipschitz spaces. During the last several years, a considerable amount of attention has been given to investigate the boundedness for various kinds of integral operators on Triebel-Lizorkin spaces and Besov spaces. For examples, see [1, 9, 10, 24] for singular integrals, [17, 18, 19, 27, 28] for Marcinkiewicz integrals, [15, 20, 22] for maximal operators. In this paper we shall establish the boundedness and continuity of maximal operators associated to polynomial compound curves with rough integral kernels $\Omega \in L(\log^+ L)^{1/2}(\mathrm{S}^{n-1}) \cup (\cup_{1<r<\infty} B_r^{0,-1/2}(\mathrm{S}^{n-1}))$ on the above function spaces.

We now recall the definitions of Triebel-Lizorkin spaces and Besov spaces. Let $\mathscr{S}'(\mathbb{R}^n)$ denote the tempered distribution class on $\mathbb{R}^n$. For $\alpha \in \mathbb{R}$ and $0 < p, q \leqslant \infty\,(p \neq \infty)$, we define the homogeneous Triebel-Lizorkin spaces $\dot{F}^\alpha_{p,q}(\mathbb{R}^n)$ and homogeneous Besov spaces $\dot{B}^\alpha_{p,q}(\mathbb{R}^n)$ by

$$\dot{F}^\alpha_{p,q}(\mathbb{R}^n) := \Big\{ f \in \mathscr{S}'(\mathbb{R}^n) : \|f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)} = \Big\| \Big(\sum_{i\in\mathbb{Z}} 2^{-i\alpha q} |\Psi_i * f|^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)} < \infty \Big\}, \quad (3)$$

$$\dot{B}^\alpha_{p,q}(\mathbb{R}^n) := \Big\{ f \in \mathscr{S}'(\mathbb{R}^n) : \|f\|_{\dot{B}^\alpha_{p,q}(\mathbb{R}^n)} = \Big(\sum_{i\in\mathbb{Z}} 2^{-i\alpha q} \|\Psi_i * f\|^q_{L^p(\mathbb{R}^n)} \Big)^{1/q} < \infty \Big\}, \quad (4)$$

where $\widehat{\Psi_i}(\xi) = \phi(2^i\xi)$ for $i \in \mathbb{Z}$ and $\phi \in \mathscr{C}_c^\infty(\mathbb{R}^n)$ satisfies the conditions: $0 \leqslant \phi(x) \leqslant 1$; $\mathrm{supp}(\phi) \subset \{x : 1/2 \leqslant |x| \leqslant 2\}$; $\phi(x) > c > 0$ if $3/5 \leqslant |x| \leqslant 5/3$. The inhomogeneous versions of Triebel-Lizorkin spaces and Besov spaces, which are denoted by $F^\alpha_{p,q}(\mathbb{R}^n)$ and $B^\alpha_{p,q}(\mathbb{R}^n)$, respectively, are obtained by adding the term $\|\Theta * f\|_{L^p(\mathbb{R}^n)}$ to the right hand side of (3) or (4) with $\sum_{i\in\mathbb{Z}}$ replaced by $\sum_{i\geqslant 1}$, where $\Theta \in \mathscr{S}(\mathbb{R}^n)$ (the Schwartz class), $\mathrm{supp}(\hat{\Theta}) \subset \{\xi : |\xi| \leqslant 2\}$, $\hat{\Theta}(x) > c > 0$ if $|x| \leqslant 5/3$.

The following properties are well known (see [13, 25] for example):

$$\dot{F}^{0}_{p,2}(\mathbb{R}^n) = L^p(\mathbb{R}^n) \ \text{ for } \ 1 < p < \infty; \tag{5}$$

$$\dot{F}^{\alpha}_{p,p}(\mathbb{R}^n) = \dot{B}^{\alpha}_{p,p}(\mathbb{R}^n) \ \text{ for } \ \alpha \in \mathbb{R} \ \text{ and } \ 1 < p < \infty; \tag{6}$$

and for any $1 < p, q < \infty$ and $\alpha > 0$,

$$F^{\alpha}_{p,q}(\mathbb{R}^n) \sim \dot{F}^{\alpha}_{p,q}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) \ \text{ and } \ \|f\|_{F^{\alpha}_{p,q}(\mathbb{R}^n)} \sim \|f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)} + \|f\|_{L^p(\mathbb{R}^n)}; \tag{7}$$

$$B^{\alpha}_{p,q}(\mathbb{R}^n) \sim \dot{B}^{\alpha}_{p,q}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) \ \text{ and } \ \|f\|_{B^{\alpha}_{p,q}(\mathbb{R}^n)} \sim \|f\|_{\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)} + \|f\|_{L^p(\mathbb{R}^n)}. \tag{8}$$

Recently, Liu [20] first studied the boundedness of maximal operators related to singular integrals associated to polynomial compound mappings on Triebel-Lizorkin spaces and Besov spaces. More precisely, let $d \geqslant 1$ and $\mathscr{P} = (P_1, \ldots, P_d)$ with each P_j being a real-valued polynomial on $\mathbb{R}^n$ and $\varphi \in \mathfrak{F}_1$ or $\mathfrak{F}_2$, where $\mathfrak{F}_1$ (resp., $\mathfrak{F}_2$) is the set of all functions $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following condition (a) (resp., (b)), where

(a) ϕ is an increasing $\mathscr{C}^1$ function such that $t\phi'(t) \geqslant C_\phi \phi(t)$ and $\phi(2t) \leqslant c_\phi \phi(t)$ for all $t > 0$, where C_ϕ and c_ϕ are independent of t.

(b) ϕ is a decreasing $\mathscr{C}^1$ function such that $t\phi'(t) \leqslant -C_\phi \phi(t)$ and $\phi(t) \leqslant c_\phi \phi(2t)$ for all $t > 0$, where C_ϕ and c_ϕ are independent of t.

Define the maximal operators related to singular integrals associated to polynomial compound mappings $\mathscr{S}_{\Omega,\mathscr{P},\varphi}$ by

$$\mathscr{S}_{\Omega,\mathscr{P},\varphi} f(x) = \sup_{h \in \mathscr{K}_2} \left| \text{p.v.} \int_{\mathbb{R}^n} f(x - \mathscr{P}(\varphi(|y|)y')) \frac{h(|y|)\Omega(y)}{|y|^n} dy \right|.$$

We now introduce the main result of [20] as follows.

THEOREM A. ([20]) *Let* $\Omega \in H^1(\mathrm{S}^{n-1}) \cup L(\log^+ L)^{1/2}(\mathrm{S}^{n-1})$ *and satisfy* (1). *Then*

(i) $\mathscr{S}_{\Omega,\mathscr{P},\varphi}$ *is bounded on* $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ *for* $\alpha \in (0,1)$ *and* $(1/p, 1/q) \in \mathscr{R}$. *Here* $\mathscr{R}$ *denotes the set of all interiors of the convex hull of two squares* $(0,1/2)^2$ *and* $(1/2,1)^2$;

(ii) $\mathscr{S}_{\Omega,\mathscr{P},\varphi}$ *is bounded on* $\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)$ *for* $\alpha \in (0,1)$, $p \in [2,\infty)$ *and* $q \in (1,\infty)$.

The bounds of $\mathscr{S}_{\Omega,\mathscr{P},\varphi}$ *given above re independent of the coefficients of* P_j *for* $1 \leqslant j \leqslant d$.

REMARK 1. We remark that the class $\mathfrak{F}_1$ was first introduced by Al-Salman [4] who investigated the L^p bounds for the parabolic Marcinkiewicz integrals along surfaces on product domains. There are some model examples for the class $\mathfrak{F}_1$, such as $t^\alpha \ln^\beta(1+t)$ $(\alpha > 0,\ \beta \geqslant 0)$, $t\ln\ln(\mathbf{e}+t)$, real-valued polynomials P on $\mathbb{R}$ with positive coefficients and $P(0) = 0$ and so on. The model examples for function $\varphi \in \mathfrak{F}_2$ are $t^\delta (\delta < 0)$, $t^{-1}\ln(1+1/t)$. It should be pointed out that if $\varphi \in \mathfrak{F}_1$ (or $\varphi \in \mathfrak{F}_2$), there exist a constant $B_\varphi > 1$ such that $\varphi(2t) \geqslant B_\varphi \varphi(t)$ (or $\varphi(t) \geqslant B_\varphi \varphi(2t)$) (see [4]).

In light of the aforementioned facts concerning the above maximal operators, a question that arises naturally is the following

QUESTION. *Is $\mathscr{S}_{\Omega,P,\varphi}$ bounded and continuous on the Triebel-Lizorkin spaces and Besov spaces under the condition that $\Omega \in L(\log^+ L)^{1/2}(\mathrm{S}^{n-1})$ and $\varphi \in \mathfrak{F}_1$ or $\mathfrak{F}_2$?*

In this paper we will give an affirmative answer to the above question by proving the following results.

THEOREM 1. *Let P be a real polynomial on $\mathbb{R}$ of degree N and satisfy $P(0) = 0$ and $\varphi \in \mathfrak{F}_1$ or $\mathfrak{F}_2$. Let $\Omega \in L^s(\mathrm{S}^{n-1})$ for some $s \in (1,2]$ and satisfy (1) and $\mathscr{R}$ be given as in Theorem A. Then*

(i) *For $\alpha \in (0,1)$ and $(1/p, 1/q) \in \mathscr{R}$, there exists $C > 0$ such that*

$$\|\mathscr{S}_{\Omega,P,\varphi} f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)} \leqslant C(s-1)^{-1/2}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)},$$

where C is independent of s, Ω and the coefficients of P, but may depend on n, α, p, q, φ and N;

(ii) *$\mathscr{S}_{\Omega,P,\varphi}$ is continuous from $F^{\alpha}_{p,q}(\mathbb{R}^n)$ to $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ for all $\alpha \in (0,1)$, $p \in [2,\infty)$ and $q \in (2p/(p+2),\infty)$.*

THEOREM 2. *Let P be a real polynomial on $\mathbb{R}$ of degree N and satisfy $P(0) = 0$ and $\varphi \in \mathfrak{F}_1$ or $\mathfrak{F}_2$. Let $\Omega \in L(\log^+ L)^{1/2}(\mathrm{S}^{n-1}) \cup (\cup_{1<r<\infty} B_r^{(0,-1/2)}(\mathrm{S}^{n-1}))$ and satisfy (1). Then*

(i) *$\mathscr{S}_{\Omega,P,\varphi}$ is bounded on $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ for $\alpha \in (0,1)$ and $(1/p,1/q) \in \mathscr{R}$. Here $\mathscr{R}$ is given as in Theorem A;*

(ii) *$\mathscr{S}_{\Omega,P,\varphi}$ is continuous from $F^{\alpha}_{p,q}(\mathbb{R}^n)$ to $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ for all $\alpha \in (0,1)$, $p \in [2,\infty)$ and $q \in (2p/(p+2),\infty)$.*

REMARK 2. We point out that the introduce of the polynomial compound curves $P(\varphi(|y|))y'$ is greatly motivated by the Al-Salman's work [8], Liu and Zhang's work [23] and Liu et al.'s work [21]. In [8], Al-Salman established the L^p boundedness for the parabolic Marcinkiewicz integrals along surfaces $P(|y|)y'$ and $\varphi(|y|)y'$ provided that Ω belongs to the Grafakos-Stefanov class. In [23], Liu and Zhang proved certain L^p bounds for the parabolic Marcinkiewicz integrals associated to polynomials compound curves $P(\varphi(|y|))y'$ under the condition that $\Omega \in L(\log^+ L)^{\alpha}(\mathrm{S}^{n-1})$ with $\alpha = 1$ or $\alpha = 1/2$. In [21], Liu et al. established certain L^p estimates for the parametric Marcinkiewicz integrals along compound curves $\Phi(\varphi(|y|))y'$ with Φ satisfying certain growth conditions and $\varphi \in \mathfrak{F}_1$ or $\varphi \in \mathfrak{F}_2$ provided that $\Omega \in H^1(\mathrm{S}^{n-1})$ or $\Omega \in L(\log^+ L)^{1/2}(\mathrm{S}^{n-1})$.

REMARK 3. When $\varphi(t) = t$, Al-Salman [7] proved that $\mathscr{S}_{\Omega,P,\varphi}$ is of type (p,p) for $2 \leqslant p < \infty$ if $\Omega \in L(\log^+ L)^{1/2}(\mathrm{S}^{n-1})$. By (20), (24)–(25) in Section 3 and applying the arguments similar to those used in deriving Theorem 2.3 in [7], we can obtain

$$\|\mathscr{S}_{\Omega,P,\varphi} f\|_{L^p(\mathbb{R}^n)} \leqslant C(s-1)^{-1/2}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|f\|_{L^p(\mathbb{R}^n)} \tag{9}$$

for $2 \leqslant p < \infty$ if $\Omega \in L^s(S^{n-1})$ for some $s \in (1,2]$ and satisfies (1). Here $C > 0$ is independent of s, Ω and the coefficients of P. Applying (9) and an extrapolation argument (see [2]), we can get the following

$$\|\mathscr{S}_{\Omega,P,\varphi} f\|_{L^p(\mathbb{R}^n)} \leqslant C\|f\|_{L^p(\mathbb{R}^n)} \tag{10}$$

for $2 \leqslant p < \infty$ if $\Omega \in L(\log^+ L)^{1/2}(S^{n-1}) \cup (\cup_{1<r<\infty} B_r^{(0,-1/2)}(S^{n-1}))$.

In what follows, we set $\Delta_\zeta f(x) = f(x+\zeta) - f(x)$ for all $x, \zeta \in \mathbb{R}^n$. On can easily check that

$$\Delta_\zeta(\mathscr{S}_{\Omega,P,\varphi} f)(x) \leqslant \mathscr{S}_{\Omega,P,\varphi}(\Delta_\zeta f)(x) \ \ \forall x, \zeta \in \mathbb{R}^n; \tag{11}$$

$$|\mathscr{S}_{\Omega,P,\varphi} f - \mathscr{S}_{\Omega,P,\varphi} g| \leqslant |\mathscr{S}_{\Omega,P,\varphi}(f-g)| \tag{12}$$

for arbitrary function f, g defined on $\mathbb{R}^n$.

Applying (11)–(12), Remark 3 and Lemma 5, we can obtain the following theorem.

THEOREM 3. *Let P, φ, Ω be given as in Theorem 2. Then*

(i) $\mathscr{S}_{\Omega,P,\varphi}$ *is bounded on $\dot{B}^\alpha_{p,q}(\mathbb{R}^n)$ for $\alpha \in (0,1)$, $p \in [2,\infty)$ and $q \in (1,\infty)$;*

(ii) $\mathscr{S}_{\Omega,P,\varphi}$ *is continuous from $B^\alpha_{p,q}(\mathbb{R}^n)$ to $\dot{B}^\alpha_{p,q}(\mathbb{R}^n)$ for $\alpha \in (0,1)$, $p \in [2,\infty)$ and $q \in (1,\infty)$.*

It follows from (10) and (12) that $\mathscr{S}_{\Omega,P,\varphi}$ is bounded and continuous on $L^p(\mathbb{R}^n)$ for all $2 \leqslant p < \infty$ if $\Omega \in L(\log^+ L)^{1/2}(S^{n-1}) \cup (\cup_{1<r<\infty} B_r^{(0,-1/2)}(S^{n-1}))$. This together with (7)–(8) and Theorems 2-3 yields the following result.

COROLLARY 1. *Let P, φ, Ω be given as in Theorem 2. Then*

(i) $\mathscr{S}_{\Omega,P,\varphi}$ *is bounded and continuous on $F^\alpha_{p,q}(\mathbb{R}^n)$ for all $\alpha \in (0,1)$, $p \in [2,\infty)$ and $q \in (2p/(p+2),\infty)$.*

(ii) $\mathscr{S}_{\Omega,P,\varphi}$ *is bounded and continuous on $B^\alpha_{p,q}(\mathbb{R}^n)$ for all $\alpha \in (0,1)$, $p \in [2,\infty)$ and $q \in (1,\infty)$.*

REMARK 4. We remark that all of our continuity results are new, even in the special case $P(t) = \varphi(t) = t$. Also, all of our boundedness results are new, even in the special case $\varphi(t) = t$.

The paper is organized as follows. Section 2 contains some auxiliary lemmas. The proofs of Theorems 1 and 2 will be given in Section 3. Finally, we present the corresponding results for the maximal operators related to the parametric Marcinkiewicz integrals in Section 4. We would like to remark that the main methods and ideas employed in this paper is a combination of ideas and arguments from [1, 2, 16, 17, 22, 27], among others. Due to the application of some useful characterizations of Triebel-Lizorkin spaces (see Lemma 2), it makes that the proof of Triebel-Lizorkin space boundedness for maximal operators can be reduced to prove certain vector-valued inequalities, which can be deduced by certain Fourier transform estimates, maximal inequalities and interpolation arguments. The continuity part in Theorem 1 are motivated by the idea in [22].

It should be also pointed out that the proof of Theorem 2 is based on Theorem 1 and an extrapolation method (see [2]).

Throughout the paper, we denote p' by the conjugate index of p, which satisfies $1/p+1/p'=1$. The letter C or c, sometimes with certain parameters, will stand for positive constants not necessarily the same one at each occurrence, but are independent of the essential variables. In what follows, we set $\mathfrak{R}_n=\{\zeta\in\mathbb{R}^n;\,1/2<|\zeta|\leqslant 1\}$.

2. Preliminary Lemmas

Let us recall the following estimate of oscillatory integrals, which will play a key role in the estimates about Fourier transforms of some measures on $\mathbb{R}^n$.

LEMMA 1. ([16]) *Let $P_N(t)=\sum_{i=1}^N a_i t^i$ with $a_i\neq 0$ for all $1\leqslant i\leqslant N$. Suppose $\Omega\in L^s(\mathrm{S}^{n-1})$ for some $s>1$. Then, for any $r>0$ and $0<\varepsilon<\min\{1/s',1/N\}$, there exists a constant $C>0$ such that*

$$\int_{r/2}^r\Big|\int_{\mathrm{S}^{n-1}}\Omega(u')\mathbf{e}^{-iP_N(\varphi(t))\xi\cdot u'}d\sigma(u')\Big|^2\frac{dt}{t}\leqslant C\|\Omega\|_{L^s(\mathrm{S}^{n-1})}^2|\varphi(r)^N a_N\xi|^{-\varepsilon}\ \text{ if } \varphi\in\mathfrak{F}_1;$$

$$\int_{r/2}^r\Big|\int_{\mathrm{S}^{n-1}}\Omega(u')\mathbf{e}^{-iP_N(\varphi(t))\xi\cdot u'}d\sigma(u')\Big|^2\frac{dt}{t}\leqslant C\|\Omega\|_{L^s(\mathrm{S}^{n-1})}^2|\varphi(r/2)^N a_N\xi|^{-\varepsilon}\ \text{ if } \varphi\in\mathfrak{F}_2.$$

Here $C>0$ is independent of s, Ω and the coefficients of P_N, but depends on φ.

The following result is some useful characterizations of Triebel-Lizorkin spaces and Besov spaces, which are followed from [27].

LEMMA 2. ([27]) (i) *Let $\alpha\in(0,1)$, $p\in(1,\infty)$, $q\in(1,\infty]$ and $r\in[1,\min\{p,q\})$. Then*

$$\|f\|_{\dot F^{\alpha}_{p,q}(\mathbb{R}^n)}\sim\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\mathfrak{R}_n}|\Delta_{2^{-l}\zeta}f|^r d\zeta\Big)^{q/r}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}.$$

(ii) *Let $p\in[1,\infty)$, $q\in[1,\infty]$ and $r\in[1,p]$. Then*

$$\|f\|_{\dot B^{\alpha}_{p,q}(\mathbb{R}^n)}\sim\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big\|\Big(\int_{\mathfrak{R}_n}|\Delta_{2^{-l}\zeta}f|^r d\zeta\Big)^{1/r}\Big\|_{L^p(\mathbb{R}^n)}^q\Big)^{1/q}.$$

The results for the following vector-valued inequalities of the Hardy-Littlewood maximal functions will also be needed in our proofs.

LEMMA 3. ([17]) *Let $\mathrm{M}_{(n)}$ be the Hardy-Littlewood maximal operator on $\mathbb{R}^n$ and $\mathrm{M}_{\mathscr{P}}$ denote the Hardy-Littlewood maximal operator supported by polynomial mappings $\mathscr{P}$ defined by $\mathrm{M}_{\mathscr{P}}f(x)=\sup_{r>0}\frac{1}{r^n}\int_{|y|\leqslant r}|f(x-\mathscr{P}(y))|dy$, where $\mathscr{P}=(P_1,\dots,P_n)$ with each P_j being a real-valued polynomial in $\mathbb{R}^n$. Then the following results are valid:*

(i) *For any pair* $(p,q,r)\in(1,\infty)^3$*, it holds that*

$$\begin{aligned}&\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|\mathrm{M}_{(n)}g_{j,\zeta,k}|^2\Big)^{1/2}\Big\|^q_{L^r(\mathfrak{R}_n)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\&\leqslant C\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_{j,\zeta,k}|^2\Big)^{1/2}\Big\|^q_{L^r(\mathfrak{R}_n)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)};\end{aligned}\tag{13}$$

(ii) *For any pair* $(p,q,r)\in(1,\infty)^3$*, it holds that*

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}\|\mathrm{M}_{\mathscr{P}}f_{j,\zeta}\|^q_{L^r(\mathfrak{R}_n)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\leqslant C\Big\|\Big(\sum_{j\in\mathbb{Z}}\|f_{j,\zeta}\|^q_{L^r(\mathfrak{R}_n)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)},$$

where $C>0$ *is independent of the coefficients of* P_j *for all* $1\leqslant j\leqslant n$.

Let Ω be given as in (2) and $\Gamma:\mathbb{R}^n\to\mathbb{R}^n$ be a suitable mapping. Define the family of measures $\{\sigma_{t,\Gamma}\}_{t>0}$ and $\{|\sigma_{t,\Gamma}|\}_{t>0}$ on $\mathbb{R}^n$ by

$$\widehat{\sigma_{t,\Gamma}}(x)=\int_{\mathrm{S}^{n-1}}\mathbf{e}^{-2\pi i\Gamma(ty')\cdot x}\Omega(y')d\sigma(y'),\tag{14}$$

$$\widehat{|\sigma_{t,\Gamma}|}(x)=\int_{\mathrm{S}^{n-1}}\mathbf{e}^{-2\pi i\Gamma(ty')\cdot x}|\Omega(y')|d\sigma(y').\tag{15}$$

LEMMA 4. *Let* $v\geqslant1$, $\Omega\in L^1(\mathrm{S}^{n-1})$ *and* $\Gamma(y)=P(\varphi(|y|))y'$*, where* $\varphi\in\mathfrak{F}_1$ *or* $\mathfrak{F}_2$ *and* $P(\cdot)$ *is a real-valued polynomial on* $\mathbb{R}_+$*. If* $(1/p,1/q,1/r)$ *belongs to the interior of the convex hull of two cubes* $(0,1/2)^3$ *and* $(1/2,1)^3$*, then*

$$\begin{aligned}&\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\int_{\mathfrak{R}_n}\Big(\sum_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}}\big||\sigma_{t,\Gamma}|*g_{j,\zeta,k}\big|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\&\leqslant Cv^{1/2}\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_{j,\zeta,k}|^2\Big)^{1/2}\Big\|^q_{L^r(\mathfrak{R}_n)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)},\end{aligned}\tag{16}$$

where $C>0$ *is independent of* v, Ω *and the coefficients of* P.

Proof. We only consider the case $\varphi\in\mathfrak{F}_1$ and another case is analogous. By Hölder's inequality, (16) reduces to the following

$$\begin{aligned}&\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}}\big||\sigma_{t,\Gamma}|*g_{j,\zeta,k}\big|^2\frac{dt}{t}\Big)^{1/2}\Big\|^q_{L^r(\mathfrak{R}_n)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\&\leqslant Cv^{1/2}\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_{j,\zeta,k}|^2\Big)^{1/2}\Big\|^q_{L^r(\mathfrak{R}_n)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\end{aligned}\tag{17}$$

for $(1/p,1/q,1/r)$ belonging to the interior of the convex hull of two cubes $(0,1/2)^3$ and $(1/2,1)^3$. We first prove that for any pair $(p,q,r)\in(1,\infty)^3$,

$$\begin{aligned}&\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\sup_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}}\big||\sigma_{t,\Gamma}|*f_{j,\zeta}\big|\frac{dt}{t}\Big\|^q_{L^r(\mathfrak{R}_n)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\&\leqslant C(\varphi)v\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\Big\|\Big(\sum_{j\in\mathbb{Z}}\|f_{j,\zeta}\|^q_{L^r(\mathfrak{R}_n)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}.\end{aligned}\tag{18}$$

Let $[v]=\max\{k\in\mathbb{Z}:k\leqslant v\}$. By a change of variable and Fubini's theorem,

$$\begin{aligned}
&\sup_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}}\big||\sigma_{t,\Gamma}|*f_{j,\zeta}(x)\big|\frac{dt}{t}\\
&\leqslant \sup_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}}\int_{S^{n-1}}|f_{j,\zeta}(x-\Gamma(ty'))||\Omega(y')|d\sigma(y')\frac{dt}{t}\\
&\leqslant \int_{S^{n-1}}|\Omega(y')|\sup_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}}|f_{j,\zeta}(x-\Gamma(ty'))|\frac{dt}{t}d\sigma(y')\\
&\leqslant \int_{S^{n-1}}|\Omega(y')|\sum_{i=0}^{[v]}\sup_{k\in\mathbb{Z}}\int_{2^{kv+i}}^{2^{kv+i+1}}|f_{j,\zeta}(x-\Gamma(ty'))|\frac{dt}{t}d\sigma(y')\\
&\leqslant \int_{S^{n-1}}|\Omega(y')|\sum_{i=0}^{[v]}\sup_{k\in\mathbb{Z}}\int_{\varphi(2^{kv+i})}^{\varphi(2^{kv+i+1})}|f_{j,\zeta}(x-\Gamma(\varphi^{-1}(t)y'))|\frac{dt}{\varphi^{-1}(t)\varphi'(\varphi^{-1}(t))}d\sigma(y')\\
&\leqslant C(\varphi)v\int_{S^{n-1}}|\Omega(y')|\sup_{r>0}\frac{1}{r}\int_{|t|\leqslant r}|f_{j,\zeta}(x-\Gamma(\varphi^{-1}(t)y'))|dt\,d\sigma(y'),
\end{aligned}$$

which combining (ii) of Lemma 3 with Minkowski's inequality yields (18).

By duality, we have that, for any $1<p,q,r<\infty$, there exist functions $\{f_{j,\zeta}\}_{j,\zeta}$ with $\|\{f_{j,\zeta}\}\|_{L^{p'}(\mathbb{R}^n,\ell^{q'}(L^{r'}(\mathfrak{R}_n)))}=1$ such that

$$\begin{aligned}
&\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\sum_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}}\big||\sigma_{t,\Gamma}|*g_{j,\zeta,k}\big|\frac{dt}{t}\Big\|_{L^r(\mathfrak{R}_n)}^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&=\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^n}\int_{\mathfrak{R}_n}\sum_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}}\big||\sigma_{t,\Gamma}|*g_{j,\zeta,k}(x)\big|\frac{dt}{t}|f_{j,\zeta}(x)|d\zeta dx\\
&\leqslant \sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^n}\int_{\mathfrak{R}_n}\sum_{k\in\mathbb{Z}}|g_{j,\zeta,k}(x)|\int_{2^{kv}}^{2^{(k+1)v}}|\sigma_{t,\Gamma}|*\widetilde{|f_{j,\zeta}|}(-x)\frac{dt}{t}d\zeta dx,
\end{aligned}$$

where $\widetilde{f_{j,\zeta}}(x)=f_{j,\zeta}(-x)$. This together with Hölder's inequality and (18) implies that

$$\begin{aligned}
&\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\sum_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}}\big||\sigma_{t,\Gamma}|*g_{j,\zeta,k}\big|\frac{dt}{t}\Big\|_{L^r(\mathfrak{R}_n)}^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\leqslant \Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\sum_{k\in\mathbb{Z}}|g_{j,\zeta,k}|\Big\|_{L^r(\mathfrak{R}_n)}^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\quad\times\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\sup_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}}\big||\sigma_{t,\Gamma}|*\widetilde{|f_{j,\zeta}|}\big|\frac{dt}{t}\Big\|_{L^{r'}(\mathfrak{R}_n)}^{q'}\Big)^{1/q'}\Big\|_{L^{p'}(\mathbb{R}^n)}\\
&\leqslant C(\varphi)v\|\Omega\|_{L^1(S^{n-1})}\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\sum_{k\in\mathbb{Z}}|g_{j,\zeta,k}|\Big\|_{L^r(\mathfrak{R}_n)}^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)},
\end{aligned}\tag{19}$$

On the other hand, we get by Hölder's inequality that

$$\begin{aligned}\big||\sigma_{t,\Gamma}|*g_{j,\zeta,k}(x)\big| &\leqslant \int_{\mathrm{S}^{n-1}}|g_{j,\zeta,k}(x-\Gamma(ty'))||\Omega(y')|d\sigma(y')\\ &=\|\Omega\|_{L^1(\mathrm{S}^{n-1})}^{1/2}(|\sigma_{t,\Gamma}|*|g_{j,\zeta,k}|^2(x))^{1/2}.\end{aligned}$$

Combining this inequality with (19) yields that

$$\begin{aligned}&\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}}\big||\sigma_{t,\Gamma}|*g_{j,\zeta,k}\big|^2\frac{dt}{t}\Big)^{1/2}\Big\|_{L^r(\mathfrak{R}_n)}^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\ &\leqslant C\|\Omega\|_{L^1(\mathrm{S}^{n-1})}^{1/2}\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}}\big||\sigma_{t,\Gamma}|*|g_{j,\zeta,k}|^2\big|\frac{dt}{t}\Big\|_{L^{r/2}(\mathfrak{R}_n)}^{q/2}\Big)^{2/q}\Big\|_{L^{p/2}(\mathbb{R}^n)}^{1/2}\\ &\leqslant C(\varphi)\|\Omega\|_{L^1(\mathrm{S}^{n-1})}^{1/2}v^{1/2}\|\Omega\|_{L^1(\mathrm{S}^{n-1})}^{1/2}\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\sum_{k\in\mathbb{Z}}|g_{j,\zeta,k}|^2\Big\|_{L^{r/2}(\mathfrak{R}_n)}^{q/2}\Big)^{2/q}\Big\|_{L^{p/2}(\mathbb{R}^n)}^{1/2}\\ &\leqslant C(\varphi)v^{1/2}\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_{j,\zeta,k}|^2\Big)^{1/2}\Big\|_{L^r(\mathfrak{R}_n)}^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\end{aligned}$$

for $(p,q,r)\in(2,\infty)^3$. This gives (17) for $(p,q,r)\in(2,\infty)^3$. By duality we have (17) for $(p,q,r)\in(1,2)^3$. Interpolating these two cases, we see that (17) holds for $(1/p,1/q,1/r)$ belonging to the interior of the convex hull of two cubes $(0,1/2)^3$ and $(1/2,1)^3$. This finishes the proof of Lemma 4. □

We end this section by the following criterion on the boundedness and continuity of a class of sublinear operators on Besov spaces.

LEMMA 5. ([22]) *Let T be a sublinear operator. Assume that $T: L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)$ for some $p\in(1,\infty)$. If*

$$|\Delta_\zeta(Tf)(x)|\leqslant|T(\Delta_\zeta(f))(x)|$$

for any x, $\zeta\in\mathbb{R}^n$. Then T is bounded on $\dot{B}^s_{p,q}(\mathbb{R}^n)$ for any $s\in(0,1)$ and $q\in(1,\infty)$. Specially, if T also satisfies the following

$$|Tf-Tg|\leqslant|T(f-g)|$$

for arbitrary function f,g defined on $\mathbb{R}^n$. Then T is continuous from $B^s_{p,q}(\mathbb{R}^n)$ to $\dot{B}^s_{p,q}(\mathbb{R}^n)$ for any $s\in(0,1)$ and $q\in(1,\infty)$.

3. Proofs of Theorems 1–2

Proof of Theorem 1. Let $\Omega\in L^s(\mathrm{S}^{n-1})$ for some $1<s\leqslant 2$ and $P(t)=\sum_{i=1}^N b_it^i$. Without loss of generality we may assume that $b_i\neq 0$ for all $1\leqslant i\leqslant N$. Let $\sigma_{t,\Gamma}$ and $|\sigma_{t,\Gamma}|$ be defined as in (14) and (15), respectively. We only prove Theorem 1 for the case $\varphi\in\mathfrak{F}_1$ by the following two steps and the other case is analogous.

Step 1. Proof of (i) of Theorem 1. Define $\Gamma_0,\Gamma_1,\dots,\Gamma_N$ by $\Gamma_0(y)=(0,\dots,0)$ and $\Gamma_\lambda(y)=P_\lambda(\varphi(|y|))y'$ for $1\leqslant\lambda\leqslant N$, where $P_\lambda(t)=\sum_{i=1}^{\lambda}b_it^i$. For $t>0$ and $0\leqslant\lambda\leqslant N$, we denote $\sigma_{t,\lambda}=\sigma_{t,\Gamma_\lambda}$ and $|\sigma_{t,\lambda}|=|\sigma_{t,\Gamma_\lambda}|$. By a change of variables and Hölder's inequality, it is no difficult to see that

$$\mathscr{S}_{\Omega,P,\varphi}f(x)\leqslant\Big(\int_0^\infty|\sigma_{t,N}*f(x)|^2\frac{dt}{t}\Big)^{1/2}. \tag{20}$$

By (20), (11) and (i) of Lemma 2, we obtain

$$\begin{aligned}&\|\mathscr{S}_{\Omega,P,\varphi}f\|_{\dot F^\alpha_{p,q}(\mathbb{R}^n)}\\ &\leqslant C\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\mathfrak{R}_n}|\Delta_{2^{-l}\zeta}(\mathscr{S}_{\Omega,P,\varphi}f)|d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\ &\leqslant C\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\mathfrak{R}_n}|\mathscr{S}_{\Omega,P,\varphi}(\Delta_{2^{-l}\zeta}f)|d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\ &\leqslant C\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\mathfrak{R}_n}\Big(\int_0^\infty|\sigma_{t,N}*\Delta_{2^{-l}\zeta}f|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\end{aligned} \tag{21}$$

for $\alpha\in(0,1)$ and $(p,q)\in(1,\infty)^2$. Hence, to prove (i) of Theorem 1, it suffices to show that

$$\begin{aligned}&\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\mathfrak{R}_n}\Big(\int_0^\infty|\sigma_{t,N}*\Delta_{2^{-l}\zeta}f|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\ &\leqslant C(s-1)^{-1/2}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|f\|_{\dot F^\alpha_{p,q}(\mathbb{R}^n)}\end{aligned} \tag{22}$$

for $\alpha\in(0,1)$ and $(1/p,1/q)$ belonging to the set of all interiors of the convex hull of two squares $(0,1/2)^2$ and $(1/2,1)^2$. Here $C>0$ is independent of s,Ω and the coefficients of P.

Below we shall prove (22). By a change of variable and Hölder's inequality,

$$\begin{aligned}&\Big(\int_{2^{ks'}}^{2^{(k+1)s'}}|\widehat{\sigma_{t,\lambda}}(\xi)-\widehat{\sigma_{t,\lambda-1}}(\xi)|^2\frac{dt}{t}\Big)^{1/2}\\ &\leqslant\Big(\int_{2^{ks'}}^{2^{(k+1)s'}}\Big(\int_{\mathrm{S}^{n-1}}|\Omega(y')||\mathbf{e}^{-2\pi iP_\lambda(\varphi(t))\xi\cdot y'}-\mathbf{e}^{-2\pi iP_{\lambda-1}(\varphi(t))\xi\cdot y'}|d\sigma(y')\Big)^2\frac{dt}{t}\Big)^{1/2}\\ &\leqslant C\Big(\int_{2^{ks'}}^{2^{(k+1)s'}}(\min\{1,|\varphi(t)^\lambda b_\lambda\xi|\})^2\frac{dt}{t}\|\Omega\|^2_{L^1(\mathrm{S}^{n-1})}\Big)^{1/2}\\ &\leqslant C(s-1)^{-1/2}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}|\varphi(2^{(k+1)s'})^\lambda b_\lambda\xi|^{1/(2\lambda s')}.\end{aligned} \tag{23}$$

Since $1<s\leqslant2$, then $s'\leqslant2(s-1)^{-1}$. Choose $v\in\mathbb{Z}$ such that $v<s'\leqslant v+1$. By a

change of variables again and Lemma 1,

$$\begin{aligned}&\Big(\int_{2^{ks'}}^{2^{(k+1)s'}}|\widehat{\sigma_{t,\lambda}}(\xi)|^2\frac{dt}{t}\Big)^{1/2}\\&\leqslant\Big(\sum_{j=0}^{v}\int_{2^{ks'+j}}^{2^{ks'+j+1}}\Big|\int_{S^{n-1}}\Omega(y')\mathrm{e}^{-2\pi iP_\lambda(\varphi(t))\xi\cdot y'}d\sigma(y')\Big|^2\frac{dt}{t}\Big)^{1/2}\\&\leqslant\Big(\sum_{j=0}^{v}C(\varphi)\|\Omega\|^2_{L^s(S^{n-1})}\min\{1,|\varphi(2^{ks'+j+1})^\lambda b_\lambda\xi|^{-1/(\lambda s')}\}\Big)^{1/2}\\&\leqslant C(\varphi)(s-1)^{-1/2}\|\Omega\|_{L^s(S^{n-1})}\min\{1,|\varphi(2^{ks'})^\lambda b_\lambda\xi|^{-1/(2\lambda s')}\}.\end{aligned}\tag{24}$$

Let $\psi\in\mathscr{C}_0^\infty(\mathbb{R})$ be supported in $\{|t|\leqslant 1\}$ and $\psi(t)\equiv 1$ for $|t|<1/2$. For $1\leqslant\lambda\leqslant N$, define the family of measures $\{\omega_{t,\lambda}\}_{t>0}$ by

$$\widehat{\omega_{t,\lambda}}(\xi)=\widehat{\sigma_{t,\lambda}}(\xi)\prod_{j=\lambda+1}^{N}\psi(|\varphi(t)^j b_j\xi|)-\widehat{\sigma_{t,\lambda-1}}(\xi)\prod_{j=\lambda}^{N}\psi(|\varphi(t)^j b_j\xi|).\tag{25}$$

Note that $\sigma_{t,0}=0$ by (1). One can easily get from (23)–(25) that

$$\sigma_{t,N}=\sum_{\lambda=1}^{N}\omega_{t,\lambda};\tag{26}$$

$$\begin{aligned}&\Big(\int_{2^{ks'}}^{2^{(k+1)s'}}|\widehat{\omega_{t,\lambda}}(\xi)|^2\frac{dt}{t}\Big)^{1/2}\\&\leqslant C(\varphi)(s-1)^{-1/2}\|\Omega\|_{L^s(S^{n-1})}\min\{1,|\varphi(2^{(k+1)s'})^\lambda b_\lambda\xi|,|\varphi(2^{ks'})^\lambda b_\lambda\xi|^{-1}\}^{1/(\lambda s')}.\end{aligned}\tag{27}$$

By (26) and Minkowski's inequality,

$$\begin{aligned}&\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\mathfrak{R}_n}\Big(\int_0^\infty|\sigma_{t,N}*\Delta_{2^{-l}\zeta}f|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\&=\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\mathfrak{R}_n}\Big(\int_0^\infty\Big|\sum_{\lambda=1}^{N}\omega_{t,\lambda}*\Delta_{2^{-l}\zeta}f\Big|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\&\leqslant\sum_{\lambda=1}^{N}\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\mathfrak{R}_n}\Big(\int_0^\infty|\omega_{t,\lambda}*\Delta_{2^{-l}\zeta}f|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\&=:\sum_{\lambda=1}^{N}A_\lambda.\end{aligned}\tag{28}$$

Therefore, to prove (22), it suffices to show that

$$A_\lambda\leqslant C(s-1)^{-1/2}\|\Omega\|_{L^s(S^{n-1})}\|f\|_{\dot F^\alpha_{p,q}(\mathbb{R}^n)}\tag{29}$$

for any $1\leqslant\lambda\leqslant N$. Here $C>0$ is independent of s,Ω and the coefficients of P.

Let $\eta_0\in\mathscr{C}^\infty(\mathbb{R})$ be an even function satisfying $0\leqslant\eta_0(t)\leqslant 1$, $\eta_0(0)=1$ and $\eta_0(t)=0$ for $|t|\geqslant 1$. Set $\eta(\xi)=1$ for $|\xi|\leqslant 1$, $\eta(\xi)=\eta_0(\frac{|\xi|-1}{a-1})$, where $a=$

$B_\varphi^{\lambda(s'-1)} > 1$ and B_φ is given as in Remark 1. Then, η satisfies $\chi_{|\xi|\leqslant 1}(\xi) \leqslant \eta(\xi) \leqslant \chi_{|\xi|\leqslant a}(\xi)$ and $|\partial^\alpha \eta(\xi)| \leqslant c_\alpha (a-1)^{-|\alpha|}$ for $\xi \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}^n$, where c_α is independent of a. Let $a_k = \varphi(2^{-ks'})^{-\lambda}|b_\lambda|^{-1}$. Note that $\inf_{k\in\mathbb{Z}} \frac{a_{k+1}}{a_k} \geqslant a$. We define the sequence of functions $\{\psi_k\}_{k\in\mathbb{Z}}$ on $\mathbb{R}^n$ by

$$\psi_k(\xi) = \eta(a_{k+1}^{-1}\xi) - \eta(a_k^{-1}\xi), \;\; \xi \in \mathbb{R}^n.$$

Observing that $\mathrm{supp}(\psi_k) \subset \{a_k \leqslant |\xi| \leqslant aa_{k+1}\}$, $\mathrm{supp}(\psi_k) \cap \mathrm{supp}(\psi_j) = \emptyset$ for $|j-k| \geqslant 2$ and $\sum_{k\in\mathbb{Z}} \psi_k(\xi) = 1$ for $\xi \in \mathbb{R}^n \setminus \{0\}$.

Define the multiplier operator $S_{k,\lambda}$ on $\mathbb{R}^n$ by

$$\widehat{S_{k,\lambda} f}(\xi) = \psi_k(|\xi|)\hat{f}(\xi).$$

We get by Minkowski's inequality that

$$\begin{aligned} &A_\lambda \\ &= \Big\| \Big(\sum_{l\in\mathbb{Z}} 2^{lq\alpha} \Big(\int_{\mathfrak{R}_n} \Big(\sum_{k\in\mathbb{Z}} \int_{2^{ks'}}^{2^{(k+1)s'}} \Big| \omega_{t,\lambda} * \sum_{j\in\mathbb{Z}} S_{j-k,\lambda} \Delta_{2^{-l}\zeta} f \Big|^2 \frac{dt}{t} \Big)^{1/2} d\zeta \Big)^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)} \\ &\leqslant \sum_{j\in\mathbb{Z}} \Big\| \Big(\sum_{l\in\mathbb{Z}} 2^{lq\alpha} \Big(\int_{\mathfrak{R}_n} \Big(\sum_{k\in\mathbb{Z}} \int_{2^{ks'}}^{2^{(k+1)s'}} |\omega_{t,\lambda} * S_{j-k,\lambda} \Delta_{2^{-l}\zeta} f|^2 \frac{dt}{t} \Big)^{1/2} d\zeta \Big)^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)}. \end{aligned} \tag{30}$$

Define the mixed norm $\|\cdot\|_{E^\alpha_{p,q}}$ for measurable functions on $\mathbb{R}^n \times \mathfrak{R}_n \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{R}_+$ by

$$\|g\|_{E^\alpha_{p,q}} := \Big\| \Big(\sum_{l\in\mathbb{Z}} 2^{lq\alpha} \Big(\int_{\mathfrak{R}_n} \Big(\sum_{k\in\mathbb{Z}} \int_0^\infty |g(t,x,\zeta,l,k)|^2 \frac{dt}{t} \Big)^{1/2} d\zeta \Big)^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)}.$$

For any $j \in \mathbb{Z}$, let

$$V_{j,\lambda}(f)(t,x,\zeta,l,k) := \omega_{t,\lambda} * S_{j-k,\lambda} \Delta_{2^{-l}\zeta} f(x) \chi_{[2^{ks'},2^{(k+1)s'}]}(t).$$

Then we have

$$A_\lambda \leqslant \sum_{j\in\mathbb{Z}} \|V_{j,\lambda}(f)\|_{E^\alpha_{p,q}}. \tag{31}$$

By (27), (6), (ii) of Lemma 2, Hölder's inequality, Minkowski's inequality, Fubini's theorem and Plancherel's theorem, we have that

$$\begin{aligned}
&\|V_{j,\lambda}(f)\|_{E^{\alpha}_{2,2}}\\
&=\Big(\int_{\mathbb{R}^n}\sum_{l\in\mathbb{Z}}2^{2l\alpha}\Big(\int_{\mathfrak{R}_n}\Big(\sum_{k\in\mathbb{Z}}\int_{2^{ks'}}^{2^{(k+1)s'}}|\omega_{t,\lambda}*S_{j-k,\lambda}\Delta_{2^{-l}\zeta}f(x)|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^2dx\Big)^{1/2}\\
&\leqslant C\Big(\sum_{l\in\mathbb{Z}}2^{2l\alpha}\int_{\mathfrak{R}_n}\sum_{k\in\mathbb{Z}}\int_{2^{ks'}}^{2^{(k+1)s'}}\int_{\mathbb{R}^n}|\omega_{t,\lambda}*S_{j-k,\lambda}\Delta_{2^{-l}\zeta}f(x)|^2dx\frac{dt}{t}d\zeta\Big)^{1/2}\\
&\leqslant C\Big(\sum_{l\in\mathbb{Z}}2^{2l\alpha}\int_{\mathfrak{R}_n}\sum_{k\in\mathbb{Z}}\int_{E_{j-k,s}}\int_{2^{ks'}}^{2^{(k+1)s'}}|\widehat{\omega_{t,\lambda}}(x)|^2\frac{dt}{t}|\widehat{\Delta_{2^{-l}\zeta}f}(x)|^2dxd\zeta\Big)^{1/2}\\
&\leqslant C(\varphi)(s-1)^{-1/2}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}B_{\varphi}^{-|j|}\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big\|\Big(\int_{\mathfrak{R}_n}|\Delta_{2^{-l}\zeta}f|^2d\zeta\Big)^{1/2}\Big\|^2_{L^2(\mathbb{R}^n)}\Big)^{1/2}\\
&\leqslant C(\varphi)(s-1)^{-1/2}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}B_{\varphi}^{-|j|}\|f\|_{\dot{F}^{\alpha}_{2,2}(\mathbb{R}^n)}.
\end{aligned}\tag{32}$$

Here $E_{j-k,\lambda}=\{x\in\mathbb{R}^n:\varphi(2^{(k-j)s'})^{-\lambda}\leqslant|b_\lambda\xi|\leqslant B_{\varphi}^{\lambda s'}\varphi(2^{(k-j-1)s'})^{-\lambda}\}$ and $C>0$ is independent of s, Ω and the coefficients of P.

Below we shall prove that

$$\|V_{j,\lambda}(f)\|_{E^{\alpha}_{p,q}}\leqslant C(s-1)^{-1/2}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)}\tag{33}$$

for $(1/p,1/q)$ belonging to the set of all interiors of the convex hull of two squares $(0,1/2)^2$ and $(1/2,1)^2$. Here $C>0$ is independent of s, Ω and the coefficients of P. In fact, interpolating between (32) and (33) implies that for any $\alpha\in(0,1)$ and $(1/p,1/q)$ belonging to the set of all interiors of the convex hull of two squares $(0,1/2)^2$ and $(1/2,1)^2$, there exists $\theta\in(0,1)$ such that

$$\|V_{j,\lambda}(f)\|_{E^{\alpha}_{p,q}}\leqslant C(s-1)^{-1/2}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}B_{\varphi}^{-\theta|j|}\|f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)},\tag{34}$$

where C is independent of s, Ω and the coefficients of P. Then (29) follows from (31) and (34).

We now prove (33). For $1\leqslant\lambda\leqslant N$, let Φ^λ be a radial function in $\mathscr{S}(\mathbb{R}^n)$ defined by $\widehat{\Phi^\lambda}(x)=\psi(|x|)$, where $x\in\mathbb{R}^n$ and ψ is given as in (25). Define X_λ by

$$X_\lambda f(x)=\sup_{k\in\mathbb{Z}}\sup_{t\in[2^{ks'},2^{(k+1)s'}]}|X_{k,t;\lambda}f(x)|,$$

where $X_{k,t;\lambda}f(x)=(\varphi(t)^\lambda b_\lambda)^{-n}\Phi^\lambda((\varphi(t)^\lambda b_\lambda)^{-1}x)$. One can easily check that

$$|X_\lambda f(x)|\leqslant C\mathrm{M}_{(n)}f(x),$$

which together with (i) of Lemma 3 yields

$$\begin{aligned}
&\Big\|\Big(\sum_{l\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|X_\lambda g_{l,\zeta,k}|^2\Big)^{1/2}\Big\|^q_{L^r(\mathfrak{R}_n)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\leqslant C\Big\|\Big(\sum_{l\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_{l,\zeta,k}|^2\Big)^{1/2}\Big\|^q_{L^r(\mathfrak{R}_n)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}
\end{aligned}\tag{35}$$

for $1 \leqslant \lambda \leqslant N$ and $(p,q,r) \in (1,\infty)^3$. For convenience, define $X^\lambda f = X_\lambda \circ X_{\lambda+1} \circ \cdots \circ X_N f$ for $1 \leqslant \lambda \leqslant N$. Then (35) yields that

$$\begin{aligned}
&\Big\|\Big(\sum_{l\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|X^\lambda g_{l,\zeta,k}|^2\Big)^{1/2}\Big\|_{L^r(\mathfrak{R}_n)}^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)} \\
&\leqslant C\Big\|\Big(\sum_{l\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_{l,\zeta,k}|^2\Big)^{1/2}\Big\|_{L^r(\mathfrak{R}_n)}^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}
\end{aligned} \tag{36}$$

for $1 \leqslant \lambda \leqslant N$ and $(p,q,r) \in (1,\infty)^3$. On the other hand, by the definition of $X_{k,t;\lambda}$,

$$\begin{aligned}
\omega_{t,\lambda} * f = &\ \sigma_{t,\lambda} * (X_{k,t;\lambda+1} \circ X_{k,t;\lambda+2} \circ \cdots \circ X_{k,t;N} f) \\
&- \sigma_{t,\lambda-1} * (X_{k,t;\lambda} \circ X_{k,t;\lambda+1} \circ \cdots \circ X_{k,t;N} f).
\end{aligned}$$

It follows that

$$\int_{2^{ks'}}^{2^{(k+1)s'}} |\omega_{t,1} * f|^2 \frac{dt}{t} \leqslant \int_{2^{ks'}}^{2^{(k+1)s'}} \big||\sigma_{t,1}| * X^2 f\big|^2 \frac{dt}{t}; \tag{37}$$

$$\int_{2^{ks'}}^{2^{(k+1)s'}} |\omega_{t,\lambda} * f|^2 \frac{dt}{t} \leqslant 2\int_{2^{ks'}}^{2^{(k+1)s'}} \big||\sigma_{t,\lambda}| * X^{\lambda+1} f\big|^2 \frac{dt}{t} + 2\int_{2^{ks'}}^{2^{(k+1)s'}} \big||\sigma_{t,\lambda-1}| * X^{\lambda} f\big|^2 \frac{dt}{t} \tag{38}$$

for $2 \leqslant \lambda \leqslant N$. We get by Lemma 4 that for any $1 \leqslant \lambda \leqslant N$ and $(1/p, 1/q, 1/r)$ belonging to the interior of the convex hull of two cubes $(0,1/2)^3$ and $(1/2,1)^3$, there exists $C > 0$ independent of s, Ω and the coefficients of P such that

$$\begin{aligned}
&\Big\|\Big(\sum_{l\in\mathbb{Z}}\Big(\int_{\mathfrak{R}_n}\Big(\sum_{k\in\mathbb{Z}}\int_{2^{ks'}}^{2^{(k+1)s'}}\big||\sigma_{t,\lambda}| * g_{l,\zeta,k}\big|^2\frac{dt}{t}\Big)^{1/2} d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)} \\
&\leqslant C(s-1)^{-1/2}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\Big\|\Big(\sum_{l\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_{l,\zeta,k}|^2\Big)^{1/2}\Big\|_{L^r(\mathfrak{R}_n)}^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}.
\end{aligned} \tag{39}$$

By (36)–(39), there exists $C > 0$ independent of s, Ω and the coefficients of P such that

$$\begin{aligned}
&\Big\|\Big(\sum_{l\in\mathbb{Z}}\Big(\int_{\mathfrak{R}_n}\Big(\sum_{k\in\mathbb{Z}}\int_{2^{ks'}}^{2^{(k+1)s'}}|\omega_{t,\lambda} * g_{l,\zeta,k}|^2\frac{dt}{t}\Big)^{1/2} d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)} \\
&\leqslant C(s-1)^{-1/2}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\Big\|\Big(\sum_{l\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_{l,\zeta,k}|^2\Big)^{1/2}\Big\|_{L^r(\mathfrak{R}_n)}^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}.
\end{aligned} \tag{40}$$

for $1 \leqslant \lambda \leqslant N$ and $(1/p, 1/q, 1/r)$ belonging to the interior of the convex hull of two cubes $(0,1/2)^3$ and $(1/2,1)^3$. Let $\alpha \in (0,1)$ and $(1/p, 1/q)$ belong to the set of all interiors of the convex hull of two squares $(0,1/2)^2$ and $(1/2,1)^2$. We can choose a positive integer r such that $1 \leqslant r < \min\{p,q\}$ and $(1/p, 1/q, 1/r)$ belongs to the interior of the convex hull of two cubes $(0,1/2)^3$ and $(1/2,1)^3$. Then (40) together with (i) of Lemma 2 and Lemma 2.5 in [27] implies that

$$\begin{aligned}
&\|V_{j,\lambda}(f)\|_{E^{\alpha}_{p,q}}\\
&=\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\mathfrak{R}_n}\Big(\sum_{k\in\mathbb{Z}}\int_{2^{ks'}}^{2^{(k+1)s'}}|\omega_{t,\lambda}*S_{j-k,\lambda}\Delta_{2^{-l}\zeta}f|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\leqslant C(s-1)^{-1/2}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big\|\Big(\sum_{k\in\mathbb{Z}}|S_{j-k,\lambda}\Delta_{2^{-l}\zeta}f|^2\Big)^{1/2}\Big\|^q_{L^r(\mathfrak{R}_n)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\leqslant C(s-1)^{-1/2}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\Big(\frac{B_\varphi^{\lambda s'}}{B_\varphi^{\lambda s'}-1}\Big)^{n+2}\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\|\Delta_{2^{-l}\zeta}f\|^q_{L^r(\mathfrak{R}_n)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\leqslant C\Big(\frac{B_\varphi^{\lambda}}{B_\varphi^{\lambda}-1}\Big)^{n+2}(s-1)^{-1/2}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)}.
\end{aligned}$$

This proves (33) and completes the proof of (i) of Theorem 1.

Step 2. Proof of (ii) of Theorem 1. Let $\alpha\in(0,1)$, $p\in[2,\infty)$ and $q\in(2p/(p+2),\infty)$. Let $f_j\to f$ in $F^{\alpha}_{p,q}(\mathbb{R}^n)$ as $j\to\infty$. By (7), we see that $f_j\to f$ in $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ and in $L^p(\mathbb{R}^n)$ as $j\to\infty$. We notice that $\mathscr{S}_{\Omega,P,\varphi}f_j\to\mathscr{S}_{\Omega,P,\varphi}f$ in $L^p(\mathbb{R}^n)$ as $j\to\infty$. Below we want to show that $\mathscr{S}_{\Omega,P,\varphi}f_j\to\mathscr{S}_{\Omega,P,\varphi}f$ in $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ as $j\to\infty$. We shall prove this claim by contradiction. Without loss of generality we may assume that there exists $c>0$ such that

$$\|\mathscr{S}_{\Omega,P,\varphi}f_j-\mathscr{S}_{\Omega,P,\varphi}f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)}>c$$

for every j.

By Minkowski's inequality and the fact that $\|\Delta_{2^{-k}\zeta}g\|_{L^p(\mathbb{R}^n)}\leqslant 2\|g\|_{L^p(\mathbb{R}^n)}$, we have

$$\begin{aligned}
&\Big(\int_{\mathbb{R}^n}\Big(\int_{\mathfrak{R}_n}|\Delta_{2^{-k}\zeta}(\mathscr{S}_{\Omega,P,\varphi}f_j-\mathscr{S}_{\Omega,P,\varphi}f)(x)|d\zeta\Big)^p dx\Big)^{1/p}\\
&\leqslant\int_{\mathfrak{R}_n}\Big(\int_{\mathbb{R}^n}|\Delta_{2^{-k}\zeta}(\mathscr{S}_{\Omega,P,\varphi}f_j-\mathscr{S}_{\Omega,P,\varphi}f)(x)|^p dx\Big)^{1/p}d\zeta\\
&\leqslant 2\int_{\mathfrak{R}_n}\|\mathscr{S}_{\Omega,P,\varphi}f_j-\mathscr{S}_{\Omega,P,\varphi}f\|_{L^p(\mathbb{R}^n)}d\zeta\\
&=2|\mathfrak{R}_n|\|\mathscr{S}_{\Omega,P,\varphi}f_j-\mathscr{S}_{\Omega,P,\varphi}f\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

Since $\mathscr{S}_{\Omega,P,\varphi}f_j\to\mathscr{S}_{\Omega,P,\varphi}f$ in $L^p(\mathbb{R}^n)$ as $j\to\infty$, we see that by extracting a subsequence we may assume that

$$\int_{\mathfrak{R}_n}|\Delta_{2^{-l}\zeta}(\mathscr{S}_{\Omega,P,\varphi}f_j-\mathscr{S}_{\Omega,P,\varphi}f)(x)|d\zeta\to 0\ \text{ as } j\to\infty \tag{41}$$

for every $l\in\mathbb{Z}$ and almost every $x\in\mathbb{R}^n$. On the other hand, by (11) and (12), it holds that

$$|\Delta_{2^{-l}\zeta}(\mathscr{S}_{\Omega,P,\varphi}f_j-\mathscr{S}_{\Omega,P,\varphi}f)(x)|\leqslant 2\mathscr{S}_{\Omega,P,\varphi}(\Delta_{2^{-l}\zeta}f)(x)+\mathscr{S}_{\Omega,P,\varphi}(\Delta_{2^{-l}\zeta}(f_j-f))(x)$$

for $(x,l,\zeta)\in\mathbb{R}^n\times\mathbb{Z}\times\mathfrak{R}_n$. For convenience we set

$$\|g\|_{p,q,\alpha}:=\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\mathfrak{R}_n}|g(x,l,\zeta)|d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}$$

for $\alpha \in \mathbb{R}$ and $(p,q) \in (1,\infty)^2$. It follows from (i) of Lemma 2 that $\|f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)} \sim \|\Delta_{2^{-l}\zeta} f\|_{p,q,\alpha}$ for $\alpha \in (0,1)$ and $(p,q) \in (1,\infty)^2$. By (21) and (22), we obtain

$$\begin{aligned}\|\mathscr{S}_{\Omega,P,\varphi}(\Delta_{2^{-l}\zeta} f)\|_{p,q,\alpha} &\leqslant C\Big\|\Big(\sum_{l\in\mathbb{Z}} 2^{lq\alpha}\Big(\int_{\mathfrak{R}_d} |\mathscr{S}_{\Omega,P,\varphi}(\Delta_{2^{-l}\zeta} f)|d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)} \\ &\leqslant C(s-1)^{-1/2}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)}.\end{aligned}$$

It follows that $\|\mathscr{S}_{\Omega,P,\varphi}(\Delta_{2^{-l}\zeta}(f_j - f))\|_{p,q,\alpha} \leqslant C\|f_j - f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)} \to 0$ as $j \to \infty$. One can extract a subsequence such that $\sum_{j=1}^{\infty} \|\mathscr{S}_{\Omega,P,\varphi}(\Delta_{2^{-l}\zeta}(f_j - f))\|_{p,q,\alpha} < \infty$. Define a function $G : \mathbb{R}^n \times \mathbb{Z} \times \mathfrak{R}_n \to \mathbb{R}$ by

$$G(x,l,\zeta) = \sum_{j=1}^{\infty} \mathscr{S}_{\Omega,P,\varphi}(\Delta_{2^{-l}\zeta}(f_j - f))(x) + 2\mathscr{S}_{\Omega,P,\varphi}(\Delta_{2^{-l}\zeta} f)(x).$$

One can easily check that $\|G\|_{p,q,\alpha} < \infty$ and

$$|\Delta_{2^{-l}\zeta}(\mathscr{S}_{\Omega,P,\varphi} f_j - \mathscr{S}_{\Omega,P,\varphi} f)(x)| \leqslant G(x,l,\zeta) \text{ for almost every } (x,l,\zeta) \in \mathbb{R}^n \times \mathbb{Z} \times \mathfrak{R}_n. \tag{42}$$

It follows that

$$\int_{\mathfrak{R}_n} |\Delta_{2^{-l}\zeta}(\mathscr{S}_{\Omega,P,\varphi} f_j - \mathscr{S}_{\Omega,P,\varphi} f)(x)|d\zeta \leqslant \int_{\mathfrak{R}_n} G(x,l,\zeta)d\zeta \tag{43}$$

for almost every $x \in \mathbb{R}^n$ and $l \in \mathbb{Z}$. Since $\|G\|_{p,q,\alpha} < \infty$, then

$$\Big(\sum_{l\in\mathbb{Z}}\Big(\int_{\mathfrak{R}_n} G(x,l,\zeta)d\zeta\Big)^q\Big)^{1/q} < \infty \tag{44}$$

for almost every $x \in \mathbb{R}^n$. It follows from (41), (43)–(44) and the dominated convergence theorem that

$$\Big(\sum_{l\in\mathbb{Z}}\Big(\int_{\mathfrak{R}_n} |\Delta_{2^{-l}\zeta}(\mathscr{S}_{\Omega,P,\varphi} f_j - \mathscr{S}_{\Omega,P,\varphi} f)(x)|d\zeta\Big)^q\Big)^{1/q} \to 0 \text{ as } j \to \infty \tag{45}$$

for almost every $x \in \mathbb{R}^n$. By (42) again,

$$\begin{aligned}&\Big(\sum_{l\in\mathbb{Z}}\Big(\int_{\mathfrak{R}_n} |\Delta_{2^{-l}\zeta}(\mathscr{S}_{\Omega,P,\varphi} f_j - \mathscr{S}_{\Omega,P,\varphi} f)(x)|d\zeta\Big)^q\Big)^{1/q} \\ &\leqslant \Big(\sum_{l\in\mathbb{Z}}\Big(\int_{\mathfrak{R}_n} |G(x,l,\zeta)|d\zeta\Big)^q\Big)^{1/q}\end{aligned} \tag{46}$$

for almost every $x \in \mathbb{R}^n$. By (45)–(46), the fact $\|G\|_{p,q,\alpha} < \infty$ and the dominated convergence theorem,

$$\|\Delta_{2^{-l}\zeta}(\mathscr{S}_{\Omega,P,\varphi} f_j - \mathscr{S}_{\Omega,P,\varphi} f)\|_{p,q,\alpha} \to 0 \text{ as } j \to \infty.$$

This yields $\|\mathscr{S}_{\Omega,P,\varphi} f_j - \mathscr{S}_{\Omega,P,\varphi} f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)} \to 0$ as $j \to \infty$ and get a contradiction. □

Proof of Theorem 2. Let $\Omega \in L(\log^+ L)^{1/2}(\mathrm{S}^{n-1}) \cup (\cup_{1<r<\infty} B_r^{(0,-1/2)}(\mathrm{S}^{n-1}))$. (20) together with (21)–(22) and the extrapolation argument as in the proof of Theorem 2.3 in [2] yields

$$\begin{aligned}
&\|\mathscr{S}_{\Omega,P,\varphi} f\|_{\dot F^{\alpha}_{p,q}(\mathbb{R}^n)}\\
&\leqslant C\Big\|\Big(\sum_{l\in\mathbb{Z}} 2^{lq\alpha}\Big(\int_{\mathfrak{R}_n} |\Delta_{2^{-l}\zeta}(\mathscr{S}_{\Omega,P,\varphi} f)|d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\leqslant C\Big\|\Big(\sum_{l\in\mathbb{Z}} 2^{lq\alpha}\Big(\int_{\mathfrak{R}_n} |\mathscr{S}_{\Omega,P,\varphi}(\Delta_{2^{-l}\zeta} f)|d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\leqslant C\Big\|\Big(\sum_{l\in\mathbb{Z}} 2^{lq\alpha}\Big(\int_{\mathfrak{R}_n}\Big(\int_0^\infty |\sigma_{t,N} * \Delta_{2^{-l}\zeta} f|^2\frac{dt}{t}\Big)^{1/2} d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\leqslant C\|f\|_{\dot F^{\alpha}_{p,q}(\mathbb{R}^n)}
\end{aligned} \tag{47}$$

for $0<\alpha<1$ and $(1/p,1/q) \in \mathscr{R}$. This yields (i) of Theorem 2. (ii) of Theorem 2 follows from (47) and the similar arguments as in the proof of (ii) of Theorem 1. □

4. Additional results

Let $h \in \mathscr{K}_2$, $\varphi \in \mathfrak{F}_1$ or $\mathfrak{F}_2$ and Ω, P be given as in (2). For $\rho = \sigma + i\tau$ ($\sigma, \tau \in \mathbb{R}$ with $\sigma > 0$), we define the maximal operator $\mathscr{M}^{\rho}_{\Omega,P,\varphi}$ by

$$\mathscr{M}^{\rho}_{\Omega,P,\varphi} f(x) = \sup_{h\in\mathscr{K}_2} |\mathfrak{M}^{\rho}_{h,\Omega,P,\varphi} f(x)|,$$

where $\mathfrak{M}^{\rho}_{h,\Omega,P,\varphi}$ is the parametric Marcinkiewicz integral operator

$$\mathfrak{M}^{\rho}_{h,\Omega,P,\varphi} f(x) := \Big(\int_0^\infty \Big|\frac{1}{t^\rho}\int_{|y|\leqslant t} \frac{h(|y|)\Omega(y)}{|y|^{n-\rho}} f(x - P(\varphi(|y|))y')dy\Big|^2 \frac{dt}{t}\Big)^{1/2}.$$

For $P(t) = \varphi(t) = t$, we shall simply denote $\mathscr{M}^{\rho}_{\Omega,P,\varphi}$ by $\mathscr{M}^{\rho}_{\Omega}$. Al-Qassem et al. [1] showed that $\mathscr{M}_{\Omega,\rho}$ is bounded on $L^p(\mathbb{R}^n)$ for $2 \leqslant p < \infty$ if $\Omega \in L(\log^+ L)^{1/2}(\mathrm{S}^{n-1}) \cup (\cup_{1<r<\infty} B_r^{(0,-1/2)}(\mathrm{S}^{n-1}))$. Recently, the boundedness for Marcinkiewicz integral on Tribel-Lizorkin spaces and Besov spaces has been investigated by many authors (see [17, 18, 19, 27, 28] for example). In this paper we shall establish the boundedness and continuity for maximal operator $\mathscr{M}^{\rho}_{\Omega,P,\varphi}$ on the above function spaces.

THEOREM 4. *Let* P, φ, Ω *be given as in Theorem* 2. *Then*

(i) $\mathscr{M}^{\rho}_{\Omega,P,\varphi}$ *is bounded and continuous on* $F^{\alpha}_{p,q}(\mathbb{R}^n)$ *for all* $\alpha \in (0,1)$, $p \in [2,\infty)$ *and* $q \in (2p/(p+2),\infty)$;

(ii) $\mathscr{M}^{\rho}_{\Omega,P,\varphi}$ *is bounded and continuous on* $B^{\alpha}_{p,q}(\mathbb{R}^n)$ *for all* $\alpha \in (0,1)$, $p \in [2,\infty)$ *and* $q \in (1,\infty)$.

Proof. By arguments similar to those used in deriving (33) and (35) in [1], one can obtain

$$\mathscr{M}^{\rho}_{\Omega,P,\varphi} f(x) \leqslant C(\rho)\mathscr{S}_{\Omega,P,\varphi} f(x) \quad \forall x \in \mathbb{R}^n. \tag{48}$$

Observe that

$$|\Delta_\zeta(\mathscr{M}^\rho_{\Omega,P,\varphi}f)(x)| \leqslant |\mathscr{M}^\rho_{\Omega,P,\varphi}(\Delta_\zeta f)(x)| \quad \forall x,\, \zeta \in \mathbb{R}^n. \tag{49}$$

By (48), (49), (47) and (i) of Lemma 2,

$$\begin{aligned}
\|\mathscr{M}^\rho_{\Omega,P,\varphi}f\|_{\dot F^\alpha_{p,q}(\mathbb{R}^n)} &\leqslant C\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\mathfrak{R}_n}|\Delta_{2^{-l}\zeta}(\mathscr{M}^\rho_{\Omega,P,\varphi}f)|d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\leqslant C\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\mathfrak{R}_n}|\mathscr{M}^\rho_{\Omega,P,\varphi}(\Delta_{2^{-l}\zeta}f)|d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\leqslant C\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\mathfrak{R}_n}|\mathscr{S}_{\Omega,P,\varphi}(\Delta_{2^{-l}\zeta}f)|d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\leqslant C\|f\|_{\dot F^\alpha_{p,q}(\mathbb{R}^n)}
\end{aligned} \tag{50}$$

for $0<\alpha<1$ and $(1/p,1/q)\in\mathscr{R}$. Here $\mathscr{R}$ is given as in Theorem A. On the other hand, by (48) and (10), we see that

$$\|\mathscr{M}^\rho_{\Omega,P,\varphi}f\|_{L^p(\mathbb{R}^n)} \leqslant C\|f\|_{L^p(\mathbb{R}^n)} \tag{51}$$

for $2\leqslant p<\infty$. From (7), (50) and (51) we see that $\mathscr{M}^\rho_{\Omega,P,\varphi}$ is bounded on $F^\alpha_{p,q}(\mathbb{R}^n)$ for $\alpha\in(0,1)$, $p\in[2,\infty)$ and $q\in(2p/(p+2),\infty)$. One can easily check that

$$|\mathscr{M}^\rho_{\Omega,P,\varphi}f-\mathscr{M}^\rho_{\Omega,P,\varphi}g| \leqslant |\mathscr{M}^\rho_{\Omega,P,\varphi}(f-g)| \tag{52}$$

for arbitrary functions f,g defined on $\mathbb{R}^n$. Using (50)–(52) and the arguments similar to those used in deriving (ii) of Theorem 1, we can obtain that $\mathscr{M}^\rho_{\Omega,P,\varphi}$ is continuous from $F^\alpha_{p,q}(\mathbb{R}^n)$ to $\dot F^\alpha_{p,q}(\mathbb{R}^n)$ for all $\alpha\in(0,1)$, $p\in[2,\infty)$ and $q\in(2p/(p+2),\infty)$. Observe from (51)–(52) that $\mathscr{M}^\rho_{\Omega,P,\varphi}$ is bounded and continuous on $L^p(\mathbb{R}^n)$ for all $p\in[2,\infty)$. This together with (7) yields (i) of Theorem 4. (ii) of Theorem 4 follows from (8), (48)–(49), (51)–(52) and Lemma 5. □

5. Appendix

In this section we give the definitions of several rough kernels we used. Recall that the Hardy space $H^1(\mathrm{S}^{n-1})$ is the set of all $L^1(\mathrm{S}^{n-1})$ functions which satisfy $\|f\|_{H^1(\mathrm{S}^{n-1})}<\infty$, where

$$\|\Omega\|_{H^1(\mathrm{S}^{n-1})} := \int_{\mathrm{S}^{n-1}}\sup_{0\leqslant r<1}\Big|\int_{\mathrm{S}^{n-1}}\Omega(\theta)\frac{1-r^2}{|rw-\theta|^n}d\sigma(\theta)\Big|d\sigma(w).$$

The class $L(\log^+L)^\alpha(\mathrm{S}^{n-1})$ (for $\alpha>0$) denotes the class of all measurable functions Ω on S^{n-1} which satisfy

$$\|\Omega\|_{L(\log^+L)^\alpha(\mathrm{S}^{n-1})} := \int_{\mathrm{S}^{n-1}}|\Omega(\theta)|\log^\alpha(|\Omega(\theta)|+2)d\sigma(\theta)<\infty.$$

The block spaces in $\mathbb{R}^n$ originated from the work of Taibleson and Weiss on the convergence of the Fourier series in connection with developments of the real Hardy spaces. The block spaces on S^{n-1} was introduced by Jiang and Lu [14] in studying the homogeneous singular integral operators. A q-block on S^{n-1} is an $L^q(\mathrm{S}^{n-1})$ $(1<q\leqslant\infty)$ function b which satisfies $\mathrm{supp}(b)=I$ and $\|b\|_q\leqslant|I|^{1-1/q}$, where $|I|=\sigma(I)$, and $I=\{x\in\mathrm{S}^{n-1}:|x-x_0|<\alpha\}$ for some $\alpha\in(0,1]$ and $x_0\in\mathrm{S}^{n-1}$. The block $B_q^{(0,v)}(\mathrm{S}^{n-1})$ is defined by

$$B_q^{(0,v)}(\mathrm{S}^{n-1}):=\{\Omega\in L^1(\mathrm{S}^{n-1}):\Omega=\sum_{\mu=1}^{\infty}\lambda_\mu b_\mu,\ M_q^{(0,v)}(\{\lambda_\mu\})<\infty\},$$

where $v>-1$, $\lambda_\mu\in\mathbb{C}$, b_μ is a q-block supported on a cap I_μ on S^{n-1} and

$$M_q^{(0,v)}(\{\lambda_\mu\})=\sum_{\mu=1}^{\infty}|\lambda_\mu|\big(1+\log^{(v+1)}(|I_\mu|^{-1})\big).$$

The norm of $B_q^{(0,v)}(\mathrm{S}^{n-1})$ is given by

$$\|\Omega\|_{B_q^{(0,v)}(\mathrm{S}^{n-1})}:=N_q^{(0,v)}(\Omega)=\inf\{M_q^{(0,v)}(\{\lambda_\mu\})\},$$

where the infimum is taken over all q-block decompositions of Ω.

We notice that the following inclusion relations are valid:

$$L^r(\mathrm{S}^{n-1})\subsetneq L(\log^+L)^{\beta_1}(\mathrm{S}^{n-1})\subsetneq L(\log^+L)^{\beta_2}(\mathrm{S}^{n-1})\ \ \forall r>1 \text{ and } 0<\beta_2<\beta_1;$$

$$L(\log^+L)^{\beta}(\mathrm{S}^{n-1})\subsetneq H^1(\mathrm{S}^{n-1})\ \ \forall\beta\geqslant 1;$$

$$L(\log^+L)^{\beta}(\mathrm{S}^{n-1})\nsubseteq H^1(\mathrm{S}^{n-1})\nsubseteq L(\log^+L)^{\beta}(\mathrm{S}^{n-1})\ \ \forall 0<\beta<1.$$

$$\bigcup_{r>1}L^r(\mathrm{S}^{n-1})\subsetneq B_q^{(0,v)}(\mathrm{S}^{n-1})\ \ \forall q>1 \text{ and } v>-1;$$

$$B_q^{(0,v_2)}(\mathrm{S}^{n-1})\subsetneq B_q^{(0,v_1)}(\mathrm{S}^{n-1})\ \ \forall q>1 \text{ and } v_2>v_1>-1;$$

$$\bigcup_{q>1}B_q^{(0,v)}(\mathrm{S}^{n-1})\nsubseteq\bigcup_{r>1}L^r(\mathrm{S}^{n-1})\ \ \forall v>-1;$$

$$B_q^{(0,v)}(\mathrm{S}^{n-1})\subset H^1(\mathrm{S}^{n-1})+L(\log^+L)^{1+v}(\mathrm{S}^{n-1})\ \forall q>1 \text{ and } v>-1.$$

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