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ESSENTIAL NORM OF WEIGHTED COMPOSITION OPERATORS BETWEEN BLOCH–TYPE SPACES IN THE OPEN UNIT BALL

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Abstract. In this paper, we give an estimation for the essential norm of weighted composition operators between Bloch-type spaces in the open unit ball of $\mathbb{C}^n$.

1. Introduction

Let μ be a positive continuous function on $[0,1)$. We say that μ is normal, if there exist positive numbers a and b, $0<a<b$, and $\delta\in[0,1)$ such that (see [25]),

$$\frac{\mu(r)}{(1-r)^a} \text{ is decreasing on } [\delta,1) \text{ and } \lim_{r\to 1}\frac{\mu(r)}{(1-r)^a}=0;$$
$$\frac{\mu(r)}{(1-r)^b} \text{ is increasing on } [\delta,1) \text{ and } \lim_{r\to 1}\frac{\mu(r)}{(1-r)^b}=\infty.$$

Let $\mathbb{B}$ be the open unit ball of $\mathbb{C}^n$, $H(\mathbb{B})$ the space of all holomorphic functions on $\mathbb{B}$. When $n=1$, $\mathbb{B}$ is the open unit disk $\mathbb{D}$ of the complex plane and $H(\mathbb{D})$ is the holomorphic function space on $\mathbb{D}$. For $z\in\mathbb{B}$, let $z=(z_1,z_2,\cdots,z_n)$ and $\{z^j\}_{j=1}^{\infty}$ denote a sequence in $\mathbb{B}$. For convenience, all the vectors in the paper will be written as row vectors, A^T and A^H will be the transpose and conjugate transpose of a matrix or vector A respectively, both $\langle z,w\rangle$ and $\langle z,w^T\rangle$ mean the inner product of z and w, that is, $\langle z,w\rangle=\langle z,w^T\rangle=\sum_{j=1}^n z_j\overline{w_j}$.

Let ω be normal on $[0,1)$. An $f\in H(\mathbb{B})$ is said to belong to the Bloch-type space in the open unit ball, denoted by $\mathscr{B}_\omega(\mathbb{B})$, or $\mathscr{B}_\omega$ for simplicity, if

$$\|f\|_{\mathscr{B}_\omega}=|f(0)|+\sup_{z\in\mathbb{B},w\in\mathbb{C}^n\setminus\{0\}}\frac{\omega(|z|)|\langle\nabla f(z),\overline{w}\rangle|}{\sqrt{(1-|z|^2)|w|^2+|\langle z,w\rangle|^2}}<\infty.$$

Here $\nabla f(z)$ is the gradient of f, that is,

$$\nabla f(z)=\left(\frac{\partial f}{\partial z_1}(z),\frac{\partial f}{\partial z_2}(z),\cdots,\frac{\partial f}{\partial z_n}(z)\right).$$

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The Bloch-type spaces on $\mathbb{B}$ are the extensions of the Bloch-type spaces on $\mathbb{D}$ and have different forms of expression which are equivalent, see [33, 37, 38] for example. In this paper, we will also use the equivalent norm $\|\cdot\|_{\mathscr{B}_{\omega,1}}$ of $\mathscr{B}_\omega$, where

$$\|f\|_{\mathscr{B}_{\omega,1}} = |f(0)| + \sup_{z\in\mathbb{B}} \omega(|z|)|\nabla f(z)|.$$

The equivalence of $\|\cdot\|_{\mathscr{B}_\omega}$ and $\|\cdot\|_{\mathscr{B}_{\omega,1}}$ is induced by the following expression, which was proved in [34],

$$\sup_{w\in\mathbb{C}^n\setminus\{0\}} \frac{|\langle\nabla f(z),\overline{w}\rangle|}{\sqrt{(1-|z|^2)|w|^2+|\langle z,w\rangle|^2}} \approx |\nabla f(z)|, \qquad f\in H(\mathbb{B}) \text{ and } z\in\mathbb{B}. \tag{1}$$

It is well known that $\mathscr{B}_\omega$ is a Banach space with the norm $\|\cdot\|_{\mathscr{B}_\omega}$. When $0<\alpha<\infty$ and $\omega(t)=(1-t^2)^\alpha$, we get the α-Bloch space (often also called Bloch-type space), denoted by $\mathscr{B}^\alpha=\mathscr{B}^\alpha(\mathbb{B})$. In particular, when $\omega(t)=1-t^2$, we get the Bloch space, denoted by $\mathscr{B}=\mathscr{B}(\mathbb{B})$. See [33, 40] for more information on the Bloch space $\mathscr{B}$ and the Bloch-type space $\mathscr{B}_\omega$, for example.

Assume $u\in H(\mathbb{B})$ and $\varphi(z)=(\varphi_1(z),\varphi_2(z),\cdots,\varphi_n(z))$ is a holomorphic self-map of $\mathbb{B}$. The weighted composition operator uC_φ is defined by

$$uC_\varphi f = u(z)f(\varphi(z)),\ f\in H(\mathbb{B}).$$

When $u=1$, uC_φ is the composition operator.

It is of some interest to provide function theoretic description of when u and φ induce a bounded or compact weighted composition operator on various function spaces. Recently, there has been a great interest in studying weighted composition operators and other related product-type operators on Bloch-type spaces and other function spaces on $\mathbb{D}$, such as [1, 3, 4, 5, 6, 7, 9, 12, 13, 15, 16, 18, 20, 21, 22, 23, 28, 41, 42]. For weighted composition operators between Bloch type spaces in the polydisc, see [11, 32] for example. Composition operators, extended Cesàro operators and other operators between $\mathscr{B}_\mu$ and $\mathscr{B}_\omega$ on $\mathbb{B}$ were studied in [2, 8, 10, 14, 17, 19, 24, 26, 27, 29, 30, 31, 33, 36, 37, 38, 39], for example.

In [38], Zhang and Xiao studied the boundedness and compactness of $uC_\varphi:\mathscr{B}_\mu\to\mathscr{B}_\omega$. In [37], Zhang and Li gave some other necessary and sufficient conditions of when $uC_\varphi:\mathscr{B}_\mu\to\mathscr{B}_\omega$ is bounded or compact. In this paper, we investigate the essential norm of the operator $uC_\varphi:\mathscr{B}_\mu\to\mathscr{B}_\omega$. Recall that the essential norm of $T:X\to Y$, denoted by $\|T\|_{e,X\to Y}$, is defined by

$$\|T\|_{e,X\to Y} = \inf\{\|T-K\|_{X\to Y} : K \text{ is a compact operator from } X \text{ to } Y\}.$$

Constants are denoted by C, they are positive and may differ from one occurrence to the next. We say that $A\lesssim B$ if there exists a constant C such that $A\leqslant CB$. The symbol $A\approx B$ means that $A\lesssim B\lesssim A$.

2. Auxiliary results

In this section, we give some auxiliary results which will be used in proving the main result of this paper. They are incorporated in the lemmas which follow.

LEMMA 1. [38, Lemma 2.3] *Suppose* μ *is normal on* $[0,1)$. *Then there exists* $\mu_* \in H(\mathbb{D})$, *such that*

(i) for any $t \in [0,1)$, $\mu_*(t) \in \mathbb{R}^+$, *and* $\mu_*(t)$ *is increasing on* $[0,1)$;

(ii) for all $z \in \mathbb{D}$, $|\mu_*(z)| \leqslant \mu_*(|z|)$;

(iii) $0 < \inf_{t\in[0,1)} \mu(t)\mu_*(t) \leqslant \sup_{t\in[0,1)} \mu(t)\mu_*(t) < \infty$;

(iv) for all $z = (z_1, z_2, \cdots, z_n) \in \mathbb{B}$ *and* $0 \leqslant r < 1$, $\mu(|z|)|\mu_*'(rz_1)| \lesssim \frac{1}{1-r|z_1|}$.

In the rest of the paper, we will always use μ_* to denote the analytic function related to μ in Lemma 1.

To study the compactness, we need the following lemma.

LEMMA 2. [35, Lemma 2.10] *Suppose that* ω *and* μ *are normal on* $[0,1)$. *If* $T : \mathscr{B}_\mu \to \mathscr{B}_\omega$ *is linear and bounded, then* T *is compact if and only if whenever* $\{f_k\}$ *is bounded in* $\mathscr{B}_\mu$ *and* $f_k \to 0$ *uniformly on compact subsets of* $\mathbb{B}$, $\lim_{k\to\infty} \|Tf_k\|_{\mathscr{B}_\omega} = 0$.

LEMMA 3. [5, Lemma 2] *Suppose* μ *is normal on* $[0,1)$. *Then the following statements hold.*

(i) There exists a $\delta \in (0,1)$, *such that* μ *is decreasing on* $[\delta,1)$, $\lim_{t\to 1} \mu(t) = 0$.

(ii) Fix $\alpha > 1$, $\beta \in (0,1)$. *When* $t \in (0,1), s \in (\beta,1)$,

$$\mu(t) \approx \mu(t^\alpha) \approx \frac{1}{\mu_*(t)}, \quad \int_0^{s^\alpha} \frac{1}{\mu(t)} dt \approx \int_0^s \frac{1}{\mu(t)} dt.$$

(iii) For any $z \in \mathbb{D}$, $|\int_0^z \mu_*(\eta)d\eta| \lesssim \int_0^{|z|} \mu_*(t)dt$. *If* $|\eta| \leqslant |z|$, $\mu(|z|)|\mu_*(\eta)| \lesssim 1$.

LEMMA 4. [38, Lemmas 2.2] *Suppose* μ *is normal and* $f \in \mathscr{B}_\mu$. *Then*

$$|f(z)| \lesssim \left(1 + \int_0^{|z|} \frac{1}{\mu(t)} dt\right) \|f\|_{\mathscr{B}_{\mu,1}}.$$

The following lemma is something which should be known to experts, but we give a proof of it for the benefit of the reader.

LEMMA 5. *Suppose μ is normal, $0<r,s<1$ and $f\in H(\mathbb{B})$. Then, for all $|z|\leqslant s$,*

$$|\nabla f(z)|\leqslant \frac{2n}{1-s}\max_{|z|\leqslant\frac{1+s}{2}}|f(z)| \text{ and } |f(z)-f(rz)|\leqslant\frac{2n(1-r)}{1-s}\max_{|z|\leqslant\frac{1+s}{2}}|f(z)|.$$

Proof. Set $z=(z_1,z_2,\cdots,z_n)\in\mathbb{B}$ such that $|z|\leqslant s$. For $i=1,2,\cdots,n$, let $\Gamma_{z,i}=\{\eta\in\mathbb{D};|\eta-z_i|=\frac{1-s}{2}\}$, and

$$\lambda(z,i,\eta)=(z_1,\cdots,z_{i-1},\eta,z_{i+1},\cdots,z_n),\eta\in\Gamma_{z,i}.$$

Since $f\in H(\mathbb{B})$, $\frac{\partial f}{\partial z_i}\in H(\mathbb{B})$. Taking f as a one complex variable function about the i-th component of z, by Cauchy's integral formula, we have

$$\begin{aligned}\left|\frac{\partial f}{\partial z_i}(z)\right| &= \frac{1}{2\pi}\left|\int_{\Gamma_{z,i}}\frac{f(\lambda(z,i,\eta))}{(\eta-z_i)^2}d\eta\right| \\ &= \frac{1}{\pi(1-s)}\left|\int_0^{2\pi} f(\lambda(z,i,z_i+\frac{1-s}{2}e^{\mathrm{i}\theta}))e^{-\mathrm{i}\theta}d\theta\right| \\ &\leqslant \frac{2}{1-s}\max_{|z|\leqslant\frac{1+s}{2}}|f(z)|.\end{aligned}$$

Here we use the change $\eta=z_i+\frac{1-s}{2}e^{\mathrm{i}\theta}(0\leqslant\theta\leqslant 2\pi)$. Then,

$$|\nabla f(z)|\leqslant\frac{2n}{1-s}\max_{|z|\leqslant\frac{1+s}{2}}|f(z)|.$$

When $|z|\leqslant s$,

$$\begin{aligned}|f(z)-f(rz)| &= \left|\int_r^1\frac{df(tz)}{dt}dt\right| = \left|\int_r^1\langle(\nabla f)(tz),\overline{z}\rangle dt\right| \\ &\leqslant (1-r)\sup_{|z|\leqslant s}|\nabla f(z)|\leqslant\frac{2n(1-r)}{1-s}\max_{|z|\leqslant\frac{1+s}{2}}|f(z)|.\end{aligned}$$

The proof is complete. □

3. Main result and proof

For simplicity, let $J\varphi(z)$ denote the Jacobian matrix of φ, that is

$$J\varphi(z)=\begin{pmatrix}\frac{\partial\varphi_1}{\partial z_1}(z) & \frac{\partial\varphi_1}{\partial z_2}(z) & \cdots & \frac{\partial\varphi_1}{\partial z_n}(z)\\ \frac{\partial\varphi_2}{\partial z_1}(z) & \frac{\partial\varphi_2}{\partial z_2}(z) & \cdots & \frac{\partial\varphi_2}{\partial z_n}(z)\\ \cdots & \cdots & \cdots & \cdots\\ \frac{\partial\varphi_n}{\partial z_1}(z) & \frac{\partial\varphi_n}{\partial z_2}(z) & \cdots & \frac{\partial\varphi_n}{\partial z_n}(z)\end{pmatrix}.$$

Therefore

$$\nabla f(\varphi(z))=(\nabla f)(\varphi(z))J\varphi(z),$$

and

$$J\varphi(z)u^T = \left(\sum_{k=1}^{n}\frac{\partial\varphi_1}{\partial z_k}(z)u_k, \sum_{k=1}^{n}\frac{\partial\varphi_2}{\partial z_k}(z)u_k, \cdots, \sum_{k=1}^{n}\frac{\partial\varphi_n}{\partial z_k}(z)u_k\right)^T.$$

Here $u = (u_1, u_2, \cdots, u_n)$ is a row vector as we stipulated in the introduction.

THEOREM 1. *Suppose μ, ω are normal, $u \in H(\mathbb{B})$ and φ is a holomorphic self-map of $\mathbb{B}$ such that $uC_\varphi : \mathscr{B}_\mu \to \mathscr{B}_\omega$ is bounded. Then the following statements hold.*

(i) When $\int_0^1 \frac{1}{\mu(t)}dt < \infty$,

$$\|uC_\varphi\|_{e,\mathscr{B}_\mu\to\mathscr{B}_\omega} \approx \limsup_{|\varphi(z)|\to 1}\sup_{w\in\mathbb{C}^n\setminus\{0\}} M_1(z,w).$$

(ii) When $\int_0^1 \frac{1}{\mu(t)}dt = \infty$,

$$\|uC_\varphi\|_{e,\mathscr{B}_\mu\to\mathscr{B}_\omega} \approx \limsup_{|\varphi(z)|\to 1}\sup_{w\in\mathbb{C}^n\setminus\{0\}} M_1(z,w) + \limsup_{|\varphi(z)|\to 1} M_2(z).$$

Here

$$M_1(z,w) = \frac{\omega(|z|)|u(z)|}{\mu(|\varphi(z)|)}\left\{\frac{(1-|\varphi(z)|^2)|J\varphi(z)w^T|^2 + |\langle\varphi(z), J\varphi(z)w^T\rangle|^2}{(1-|z|^2)|w|^2 + |\langle z,w\rangle|^2}\right\}^{\frac{1}{2}},$$

and

$$M_2(z) = \omega(|z|)|\nabla u(z)|\left(1 + \int_0^{|\varphi(z)|}\frac{1}{\mu(t)}dt\right).$$

Proof. In [38], Zhang and Xiao proved that $uC_\varphi : \mathscr{B}_\mu \to \mathscr{B}_\omega$ is bounded if and only if

$$\sup_{z\in\mathbb{B}, w\in\mathbb{C}^n\setminus\{0\}} M_1(z,w) < \infty \quad \text{and} \quad \sup_{z\in\mathbb{B}} M_2(z) < \infty. \tag{2}$$

Since $1 \in \mathscr{B}_\mu$, we have $u \in \mathscr{B}_\omega$. By (1), Lemmas 1 and 4, we get

$$\omega(|z|)|u(z)| \lesssim \left(\omega(|z|) + \int_0^{|z|}\omega(|z|)\omega_*(t)dt\right)\|u\|_{\mathscr{B}_{\omega,1}} \lesssim \|u\|_{\mathscr{B}_\omega}. \tag{3}$$

The upper estimate of $\|uC_\varphi\|_{e,\mathscr{B}_\mu\to\mathscr{B}_\omega}$.

For all $k \in \mathrm{N}_+$, let $\rho_k = 1 - \frac{1}{k+1}$ and $f \in \mathscr{B}_\mu$. By Lemma 4, we have

$$|(uC_{\rho_k\varphi}f)(0)| = |u(0)f(\rho_k\varphi(0))| \tag{4}$$

$$\lesssim |u(0)|\left(1 + \int_0^{|\varphi(0)|}\frac{1}{\mu(t)}dt\right)\|f\|_{\mathscr{B}_{\mu,1}} \approx \|f\|_{\mathscr{B}_\mu}. \tag{5}$$

From Lemma 4 and (3), we get

$$\omega(|z|)|\nabla(uC_{\rho_k\varphi}f)(z)| \leqslant \|u\|_{\mathscr{B}_{\omega,1}}|f(\rho_k\varphi(z))| + \omega(|z|)|u(z)||\nabla(f(\rho_k\varphi(z)))| \tag{6}$$

$$\lesssim \|f\|_{\mathscr{B}_\mu}\|u\|_{\mathscr{B}_\omega}\left(1 + \int_0^{\rho_k}\frac{dt}{\mu(t)} + \frac{1}{\mu(\rho_k)}\right). \tag{7}$$

By (5) and (7), $uC_{\rho_k\varphi}:\mathscr{B}_\mu\to\mathscr{B}_\omega$ is bounded for each $k\in\mathbb{N}$. By (3), (4), (6), Lemmas 2 and 5, $uC_{\rho_k\varphi}$ is compact for each $k\in\mathbb{N}$.

For any $f\in\mathscr{B}_\mu$ such that $\|f\|_{\mathscr{B}_\mu}\leqslant 1$, let

$$g_{k,f}(z)=f(z)-f(\rho_k z).$$

Then $\|g_{k,f}\|_{\mathscr{B}_{\mu,1}}\lesssim 1$. Since f is uniformly continuous on the compact subsets of $\mathbb{D}$, $\{g_{k,f}\}$ converges to 0 uniformly on compact subsets of $\mathbb{B}$. After a calculation, we have

$$|g_{k,f}(\varphi(z))|=\left|\int_{\rho_k}^{1}\langle(\nabla f)(t\varphi(z)),\overline{\varphi(z)}\rangle dt\right|\lesssim\int_{\rho_k|\varphi(z)|}^{|\varphi(z)|}\frac{1}{\mu(t)}dt. \tag{8}$$

Then,

$$\lim_{k\to\infty}\sup_{\|f\|_{\mathscr{B}_\mu}\leqslant 1}|u(0)||g_{k,f}(\varphi(0))|=0. \tag{9}$$

Fix $s\in(0,1)$. After a calculation and by (8),

$$\begin{aligned}
&\frac{\omega(|z|)|\langle\nabla(uC_\varphi g_{k,f})(z),\overline{w}\rangle|}{\sqrt{(1-|z|^2)|w|^2+|\langle z,w\rangle|^2}}\\
&\leqslant\frac{\omega(|z|)|\langle\nabla u(z),\overline{w}\rangle||g_{k,f}(\varphi(z))|}{\sqrt{(1-|z|^2)|w|^2+|\langle z,w\rangle|^2}}+\frac{\omega(|z|)|u(z)||\langle(\nabla g_{k,f})(\varphi(z))J\varphi(z),\overline{w}\rangle|}{\sqrt{(1-|z|^2)|w|^2+|\langle z,w\rangle|^2}} &(10)\\
&\lesssim\|u\|_{\mathscr{B}_\omega}\int_{\rho_k|\varphi(z)|}^{|\varphi(z)|}\frac{1}{\mu(t)}dt+\frac{M_1(z,w)\mu(|\varphi(z)|)|\langle(\nabla g_{k,f})(\varphi(z)),\overline{J\varphi(z)w^T}\rangle|}{\sqrt{(1-|\varphi(z)|^2)|J\varphi(z)w^T|^2+|\langle\varphi(z),J\varphi(z)w^T\rangle|^2}} &(11)\\
&\leqslant\|u\|_{\mathscr{B}_\omega}\int_{\rho_k|\varphi(z)|}^{|\varphi(z)|}\frac{1}{\mu(t)}dt+M_1(z,w)\|g_{k,f}\|_{\mathscr{B}_\mu}. &(12)
\end{aligned}$$

When $|\varphi(z)|\leqslant s$, by Lemma 5 and (8), we have

$$|\nabla(g_{k,f})(\varphi(z))|\leqslant\frac{2n}{1-s}\max_{|\eta|\leqslant\frac{1+s}{2}}|g_{k,f}(\eta)|\lesssim\frac{2n}{1-s}\max_{|\eta|\leqslant\frac{1+s}{2}}\int_{\rho_k|\eta|}^{|\eta|}\frac{1}{\mu(t)}dt.$$

So, when $|\varphi(z)|\leqslant s$,

$$\begin{aligned}
&\frac{\mu(|\varphi(z)|)|\langle\nabla g_{k,f}(\varphi(z)),\overline{J\varphi(z)w^T}\rangle|}{\sqrt{(1-|\varphi(z)|^2)|J\varphi(z)w^T|^2+|\langle\varphi(z),J\varphi(z)w^T\rangle|^2}}\\
&\leqslant\frac{\mu(|\varphi(z)|)|\nabla(g_{k,f})(\varphi(z))||J\varphi(z)w^T|}{\sqrt{1-|\varphi(z)|^2}|J\varphi(z)w^T|}=\frac{\mu(|\varphi(z)|)|\nabla(g_{k,f})(\varphi(z))|}{\sqrt{1-|\varphi(z)|^2}}\\
&\leqslant C(\mu,s)\max_{|\eta|\leqslant\frac{1+s}{2}}\int_{\rho_k|\eta|}^{|\eta|}\frac{1}{\mu(t)}dt. &(13)
\end{aligned}$$

Since $\int_0^t\frac{1}{\mu(\eta)}d\eta$ is uniformly continuous on $[0,(1+s)/2]$, by (2), (11) and (13), we get

$$\lim_{k\to\infty}\sup_{\|f\|_{\mathscr{B}_\mu}\leqslant 1}\sup_{\substack{|\varphi(z)|\leqslant s\\ w\in\mathbb{C}^n\setminus\{0\}}}\frac{\omega(|z|)|\langle\nabla(uC_\varphi g_{k,f})(z),\overline{w}\rangle|}{\sqrt{(1-|z|^2)|w|^2+|\langle z,w\rangle|^2}}=0. \tag{14}$$

When $\int_0^1 \frac{1}{\mu(t)}dt < \infty$, $\int_0^t \frac{1}{\mu(\eta)}d\eta$ is uniformly continuous on $[0,1)$. By (9), (12) and (14),

$$\begin{aligned}
&\|uC_\varphi\|_{e,\mathscr{B}_\mu\to\mathscr{B}_\omega} \leqslant \limsup_{k\to\infty}\|uC_\varphi - uC_{\rho_k\varphi}\|_{\mathscr{B}_\mu\to\mathscr{B}_\omega} \\
&= \limsup_{k\to\infty}\sup_{\|f\|_{\mathscr{B}_\mu}\leqslant 1}\|(uC_\varphi)g_{k,f}\|_{\mathscr{B}_\omega} \\
&\leqslant \limsup_{k\to\infty}\sup_{\|f\|_{\mathscr{B}_\mu}\leqslant 1}\left(\sup_{|\varphi(z)|\leqslant s} + \sup_{s<|\varphi(z)|<1}\right)\sup_{w\in\mathbb{C}^n\setminus\{0\}}\frac{\omega(|z|)|\langle\nabla(uC_\varphi g_{k,f})(z),\overline{w}\rangle|}{\sqrt{(1-|z|^2)|w|^2+|\langle z,w\rangle|^2}} \\
&\leqslant \limsup_{k\to\infty}\sup_{\|f\|_{\mathscr{B}_\mu}\leqslant 1}\sup_{s<|\varphi(z)|<1}\sup_{w\in\mathbb{C}^n\setminus\{0\}} M_1(z,w)\|g_{k,f}\|_{\mathscr{B}_\mu} \\
&\lesssim \sup_{s<|\varphi(z)|<1}\sup_{w\in\mathbb{C}^n\setminus\{0\}} M_1(z,w).
\end{aligned} \tag{15}$$

By letting $s\to 1$, we have

$$\|uC_\varphi\|_{e,\mathscr{B}_\mu\to\mathscr{B}_\omega} \lesssim \limsup_{|\varphi(z)|\to 1}\sup_{w\in\mathbb{C}^n\setminus\{0\}} M_1(z,w).$$

Next we discuss the case of $\int_0^1 \frac{1}{\mu(t)}dt = \infty$. If $|\varphi(z)| > s$, by (1), (10), part of (12) and Lemma 4, we have

$$\begin{aligned}
&\sup_{w\in\mathbb{C}^n\setminus\{0\}}\frac{\omega(|z|)|\langle\nabla(uC_\varphi g_{k,f})(z),\overline{w}\rangle|}{\sqrt{(1-|z|^2)|w|^2+|\langle z,w\rangle|^2}} \\
&\lesssim M_2(z)\|g_{k,f}\|_{\mathscr{B}_\mu} + \sup_{w\in\mathbb{C}^n\setminus\{0\}} M_1(z,w)\|g_{k,f}\|_{\mathscr{B}_\mu}.
\end{aligned} \tag{16}$$

Similar to (15), by (8), (9), (14) and (16), we have

$$\|uC_\varphi\|_{e,\mathscr{B}_\mu\to\mathscr{B}_\omega} \lesssim \sup_{s<|\varphi(z)|<1} M_2(z) + \sup_{s<|\varphi(z)|<1}\sup_{w\in\mathbb{C}^n\setminus\{0\}} M_1(z,w).$$

By letting $s\to 1$, we get

$$\|uC_\varphi\|_{e,\mathscr{B}_\mu\to\mathscr{B}_\omega} \lesssim \limsup_{|\varphi(z)|\to 1} M_2(z) + \limsup_{|\varphi(z)|\to 1}\sup_{w\in\mathbb{C}^n\setminus\{0\}} M_1(z,w).$$

The lower estimate of $\|uC_\varphi\|_{e,\mathscr{B}_\mu\to\mathscr{B}_\omega}$. For arbitrary $\varepsilon > 0$, there are $\{z^k\}_{k=1}^\infty \subset \mathbb{B}$ and $\{w^k\}_{k=1}^\infty \subset \mathbb{C}^n\setminus\{0\}$ such that

$$r_k = |\varphi(z^k)| \to 1 \text{ as } k\to\infty,$$

and

$$\lim_{k\to\infty} M_1(z^k,w^k) > \limsup_{|\varphi(z)|\to 1}\sup_{w\in\mathbb{C}^n\setminus\{0\}} M_1(z,w) - \varepsilon. \tag{17}$$

Without loss of generality, suppose $1-\frac{1}{2k^2} < r_k < 1$ and $J\varphi(z^k)(w^k)^T \neq 0$.

Let $e_1 = (1,0,\cdots,0)$. There is a unitary transformation U_k satisfying $\varphi(z^k) = r_k e_1 U_k$. Assume $\{f_k\} \subset \mathscr{B}_\mu$ and $g_k(z) = f_k(zU_k^H)$. Then

$$|z| = |zU_k^H| = |zU_k|, \nabla g_k(z) = (\nabla f_k)(zU_k^H)(U_k^H)^T = (\nabla f_k)(zU_k^H)\overline{U_k}. \tag{18}$$

Since $z \in \mathbb{C}$ is a row vector, we have

$$|\overline{U_k} z^T| = |(\overline{U_k} z^T)^T| = |zU_k^H| = |z|.$$

Therefore,

$$|J\varphi(z^k)(w^k)^T| = |\overline{U_k} J\varphi(z^k)(w^k)^T|, \tag{19}$$

and

$$\begin{aligned}\nabla(g_k \circ \varphi)(z^k) &= (\nabla g_k)(\varphi(z^k))J\varphi(z^k) = (\nabla f_k)(\varphi(z^k)U_k^H)\overline{U_k}J\varphi(z^k) \\ &= (\nabla f_k)(r_k e_1 U_k U_k^H)\overline{U_k}J\varphi(z^k) = (\nabla f_k)(r_k e_1)\overline{U_k}J\varphi(z^k).\end{aligned} \tag{20}$$

Let

$$v^k = (v_1^k, v_2^k, \cdots, v_n^k) = (\overline{U_k} J\varphi(z^k)(w^k)^T)^T.$$

By (19), (20) and $\varphi(z^k) = r_k e_1 U_k$, we have

$$\begin{aligned}&\frac{\omega(|z^k|)|u(z^k)||\langle\nabla(g_k\circ\varphi)(z^k), \overline{w^k}\rangle|}{\sqrt{(1-|z^k|^2)|w^k|^2 + |\langle z^k, w^k\rangle|^2}} \\ &= M_1(z^k,w^k)\frac{\mu(|\varphi(z^k)|)|\langle\nabla(g_k\circ\varphi)(z^k), \overline{w^k}\rangle|}{\sqrt{(1-|\varphi(z^k)|^2)|J\varphi(z^k)(w^k)^T|^2 + |\langle\varphi(z^k), J\varphi(z^k)(w^k)^T\rangle|^2}} \\ &= M_1(z^k,w^k)\frac{\mu(|r_k|)|\langle(\nabla f_k)(r_k e_1)\overline{U_k}J\varphi(z^k), (\overline{w^k})^T\rangle|}{\sqrt{(1-|r_k|^2)|\overline{U_k}J\varphi(z^k)(w^k)^T|^2 + |\langle r_k e_1 U_k, J\varphi(z^k)(w^k)^T\rangle|^2}} \\ &= M_1(z^k,w^k)\frac{\mu(|r_k|)|\langle(\nabla f_k)(r_k e_1), \overline{v^k}\rangle|}{\sqrt{(1-|r_k|^2)|v^k|^2 + |\langle r_k e_1, v^k\rangle|^2}}.\end{aligned} \tag{21}$$

There exists $\{f_k\} \subset \mathscr{B}_\mu$ satisfying the following conditions (we will give an example later).

(a) $\|f_k\|_{\mathscr{B}_\mu} \lesssim 1$;

(b) $\{f_k(z)\}$ converges to 0 uniformly on compact subsets of $\mathbb{B}$;

(c)

$$\liminf_{k\to\infty} \frac{\mu(|r_k|)|\langle(\nabla f_k)(r_k e_1), \overline{v^k}\rangle|}{\sqrt{(1-|r_k|^2)|v^k|^2 + |\langle r_k e_1, v^k\rangle|^2}} \gtrsim 1;$$

(d) $\lim_{k\to\infty} |f_k(r_k e_1)| = 0$.

By (18), $\{g_k\}$ also satisfies (a) and (b).

Suppose $K:\mathscr{B}_\mu\to\mathscr{B}_\omega$ is compact. By Lemma 2, $\lim_{k\to\infty}\|Kg_k\|_{\mathscr{B}_\omega}=0$. Letting $k\to\infty$, by

$$\|(uC_\varphi-K)g_k\|_{\mathscr{B}_\omega}\geqslant\frac{\omega(|z^k|)|u(z^k)||\langle(\nabla(g_k\circ\varphi))(z^k),\overline{w^k}\rangle|}{\sqrt{(1-|z^k|^2)|w^k|^2+|\langle z^k,w^k\rangle|^2}}-|g_k(\varphi(z^k))|\|u\|_{\mathscr{B}_\omega}-\|Kg_k\|_{\mathscr{B}_\omega},$$

and (21), (c) and (d) , we have

$$\limsup_{k\to\infty}\|(uC_\varphi-K)g_k\|_{\mathscr{B}_\omega}\gtrsim\limsup_{k\to\infty}M_1(z^k,w^k).\tag{22}$$

Because K is arbitrary and $\|g_k\|_{\mathscr{B}_\mu}\lesssim 1$,

$$\|uC_\varphi\|_{e,\mathscr{B}_\mu\to\mathscr{B}_\omega}\gtrsim\limsup_{k\to\infty}M_1(z^k,w^k).$$

Since ε is arbitrary and (17), we have

$$\|uC_\varphi\|_{e,\mathscr{B}_\mu\to\mathscr{B}_\omega}\gtrsim\limsup_{|\varphi(z)|\to1}\sup_{w\in\mathbb{C}^n\setminus\{0\}}M_1(z,w).\tag{23}$$

When $\int_0^1\frac{1}{\mu(t)}dt=\infty$. Let $\{z^k\}$ be an arbitrary sequence in $\mathscr{B}$ such that $r_k=|\varphi(z_k)|\to1$ as $k\to\infty$ and

$$h_k(z)=\frac{1}{\int_0^{r_k^2}\mu_*(t)dt}\left(\int_0^{\langle z,\varphi(z^k)\rangle}\mu_*(\eta)d\eta\right)^2.$$

Then from the assumption, and by Lemmas 1 and 3, h_k converges to 0 uniformly on compact subsets of $\mathbb{B}$. After a calculation, we have

$$\nabla h_k(z)=\frac{2\mu_*(\langle z,\varphi(z^k)\rangle)}{\int_0^{r_k^2}\mu_*(t)dt}\left(\int_0^{\langle z,\varphi(z^k)\rangle}\mu_*(\eta)d\eta\right)\left(\overline{\varphi_1(z^k)},\overline{\varphi_2(z^k)},\cdots,\overline{\varphi_n(z^k)}\right).$$

Since $|\langle z,\varphi(z^k)\rangle|\leqslant|z||\varphi(z^k)|=r_k|z|$, by (1), Lemmas 1 and 3, we have

$$\|h_k\|_{\mathscr{B}_\mu}\approx\|h_k\|_{\mathscr{B}_\mu,1}=\sup_{z\in\mathbb{B}}\mu(|z|)|\nabla h_k(z)|\leqslant\sup_{z\in\mathbb{B}}\frac{2\mu(|z|)\mu_*(|z|)}{\int_0^{r_k^2}\mu_*(t)dt}\left|\int_0^{r_k}\mu_*(\eta)d\eta\right|\lesssim1,$$

and

$$\begin{aligned}|\nabla(h_k\circ\varphi)(z^k)|&\approx\sup_{w\in\mathbb{C}^n\setminus\{0\}}\frac{|\langle\nabla h_k(\varphi(z^k))J\varphi(z^k),\overline{w}\rangle|}{\sqrt{(1-|z^k|^2)|w|^2+|\langle z^k,w\rangle|^2}}\\&=\sup_{w\in\mathbb{C}^n\setminus\{0\}}\frac{2\mu_*(|\varphi(z^k)|^2)|\langle\overline{\varphi(z^k)},\overline{J\varphi(z^k)w^T}\rangle|}{\sqrt{(1-|z^k|^2)|w|^2+|\langle z^k,w\rangle|^2}}\\&\lesssim\sup_{w\in\mathbb{C}^n\setminus\{0\}}\frac{1}{\mu(|\varphi(z^k)|)}\left\{\frac{(1-|\varphi(z^k)|^2)|J\varphi(z^k)w^T|^2+|\langle\varphi(z^k),J\varphi(z^k)w^T\rangle|^2}{(1-|z^k|^2)|w|^2+|\langle z^k,w\rangle|^2}\right\}^{\frac12}.\end{aligned}$$

Then, by $\int_0^1 \frac{1}{\mu(t)} = \infty$, we obtain

$$\begin{aligned}\|uC_\varphi h_k\|_{\mathscr{B}_{\omega,1}} &\geqslant \omega(|z^k|)|\nabla u(z^k)||h_k(\varphi(z^k))| - \omega(|z^k|)|u(z^k)||\nabla(h_k\circ\varphi)(z^k)| \\ &\gtrsim \omega(|z^k|)|\nabla u(z^k)|\int_0^{|\varphi(z^k)|^2}\mu_*(t)dt - \sup_{w\in\mathbb{C}^n\setminus\{0\}} M_1(z^k,w) \\ &\gtrsim M_2(z^k) - \sup_{w\in\mathbb{C}^n\setminus\{0\}} M_1(z^k,w).\end{aligned}$$

If $K:\mathscr{B}_\mu\to\mathscr{B}_\omega$ is compact, by Lemma 2, $\lim_{k\to\infty}\|Kh_k\|_{\mathscr{B}_\omega}=0$. Then

$$\|uC_\varphi - K\|_{\mathscr{B}_\mu\to\mathscr{B}_\omega} \gtrsim \|(uC_\varphi - K)h_k\|_{\mathscr{B}_\omega} \gtrsim M_2(z^k) - \sup_{w\in\mathbb{C}^n\setminus\{0\}} M_1(z^k,w) - \|Kh_k\|_{\mathscr{B}_\omega}.$$

Letting $k\to\infty$, we get

$$\|uC_\varphi - K\|_{\mathscr{B}_\mu\to\mathscr{B}_\omega} \gtrsim \limsup_{k\to\infty} M_2(z^k) - \limsup_{k\to\infty}\sup_{w\in\mathbb{C}^n\setminus\{0\}} M_1(z^k,w).$$

By (23), since $\{z^k\}$ is an arbitrary sequence and K is an arbitrary compact operator, we have

$$\|uC_\varphi\|_{e,\mathscr{B}_\mu\to\mathscr{B}_\omega} \gtrsim \limsup_{|\varphi(z)|\to 1} M_2(z).$$

Finally, we give an example of $\{f_k\}$ satisfying (a)-(d).

Suppose $\int_0^1 \frac{1}{\mu(t)}dt<\infty$. When $\sqrt{(1-r_k^2)(|v_2^k|^2+\cdots+|v_n^k|^2)}<|v_1^k|$, let

$$f_k(z) = \frac{1}{r_k}\int_0^{r_k z_1}\mu_*(\eta)d\eta - \frac{1}{r_k^k}\int_0^{r_k^k z_1}\mu_*(\eta)d\eta.$$

When $\sqrt{(1-r_k^2)(|v_2^k|^2+\cdots+|v_n^k|^2)}\geqslant|v_1^k|$, let $\theta_j^k=\arg v_j^k (j=2,3,\cdots,n)$ and

$$f_k(z) = (\mathrm{e}^{-\mathrm{i}\theta_2^k}z_2+\cdots+\mathrm{e}^{-\mathrm{i}\theta_n^k}z_n)\sqrt{1-r_k^2}\mu_*(r_k z_1).$$

If $\sqrt{(1-r_k^2)(|v_2^k|^2+\cdots+|v_n^k|^2)}<|v_1^k|$, by Lemma 1 we have

$$\mu(|z|)|\nabla f_k(z)| \lesssim 1, \tag{24}$$

and for all $|z|\leqslant r<1$,

$$|f_k(z)| = \left|\frac{1}{r_k}\int_{r_k^k z_1}^{r_k z_1}\mu_*(\eta)d\eta - \left(\frac{1}{r_k^k}-\frac{1}{r_k}\right)\int_0^{r_k^k z_1}\mu_*(\eta)d\eta\right| \leqslant 2(1-r_k^{k-1})\mu_*(r). \tag{25}$$

From $1-\frac{1}{k+1}<1-\frac{1}{2k^2}<r_k<1$, we get

$$\sum_{j=0}^k r_k^j > \sum_{j=0}^k\left(1-\frac{1}{k+1}\right)^j = (k+1)\left(1-(1-\frac{1}{k+1})^{k+1}\right)\to+\infty, \text{ as } k\to+\infty.$$

Since μ is normal, we have $a>0$ such that $\frac{\mu(s)}{(1-s)^a}$ is decreasing on $[\delta,1)$. Then, by Lemma 1, we have

$$\mu(r_k)\mu_*(r_k^{k+1})=\frac{\frac{\mu(r_k)}{(1-r_k)^a}}{\frac{\mu(r_k^{k+1})}{(1-r_k^{k+1})^a}}\frac{(1-r_k)^a}{(1-r_k^{k+1})^a}\mu(r_k^{k+1})\mu_*(r_k^{k+1})\lesssim\frac{1}{(\sum_{j=0}^k r_k^j)^a}\to 0,\text{ as }k\to\infty.$$

Therefore, by Lemmas 1 and 3, we obtain

$$\mu(r_k)(\mu_*(r_k^2)-\mu_*(r_k^{k+1}))\gtrsim 1.$$

Since $\sqrt{(1-r_k^2)(|v_2^k|^2+\cdots+|v_n^k|^2)}<|v_1^k|$, we have

$$\frac{\mu(r_k)|\langle(\nabla f_k)(r_ke_1),\overline{v^k}\rangle|}{\sqrt{(1-|r_k|^2)|v^k|^2+|\langle r_ke_1,v^k\rangle|^2}}>\frac{\mu(r_k)|v_1^k|(\mu_*(r_k^2)-\mu_*(r_k^{k+1}))}{\sqrt{2}|v_1^k|}\gtrsim 1. \tag{26}$$

By $\int_0^1\frac{1}{\mu(t)}dt<\infty$ and $1-\frac{1}{2k^2}<r_k<1$, we obtain

$$f_k(r_ke_1)=\frac{1}{r_k}\int_0^{r_k^2}\mu_*(\eta)d\eta-\frac{1}{r_k^k}\int_0^{r_k^{k+1}}\mu_*(t)dt\to 0,\text{ as }k\to\infty. \tag{27}$$

Then we discuss the case of $\sqrt{(1-r_k^2)(|v_2^k|^2+\cdots+|v_n^k|^2)}\geqslant|v_1^k|$. Since $|z_1|^2+|z_2|^2+\cdots+|z_n|^2<1$, we have $|z_j|<\sqrt{1-|z_1|^2}$ when $j=2,\cdots,n$. By Lemma 1,

$$\begin{aligned}\mu(|z|)|\nabla f_k(z)|&\leqslant\sqrt{1-r_k^2}\mu(|z|)\left(r_k|\mu_*'(r_kz_1)|\sum_{j=2}^n|z_j|+(n-1)|\mu_*(r_kz_1)|\right)\\&\lesssim\frac{(n-1)r_k\sqrt{1-|z_1|^2}\sqrt{1-r_k^2}}{1-|r_kz_1|}+(n-1)\sqrt{1-r_k^2}\lesssim 1.\end{aligned} \tag{28}$$

For all $|z|\leqslant r<1$, we have

$$|f_k(z)|\leqslant(n-1)\sqrt{1-r_k^2}\mu_*(r). \tag{29}$$

Since $\sqrt{(1-r_k^2)(|v_2^k|^2+\cdots+|v_n^k|^2)}\geqslant|v_1^k|$, by Lemma 3, we obtain

$$\begin{aligned}\frac{\mu(r_k)|\langle(\nabla f_k)(r_ke_1),\overline{v^k}\rangle|}{\sqrt{(1-|r_k|^2)|v^k|^2+|\langle r_ke_1,v^k\rangle|^2}}&=\frac{\sqrt{1-r_k^2}\mu(r_k)\mu_*(r_k^2)(|v_2^k|+\cdots+|v_n^k|)}{\sqrt{(1-r_k^2)(|v_2^k|^2+\cdots+|v_n^k|^2)+|v_1^k|^2}}\\&\geqslant\frac{\mu(r_k)\mu_*(r_k^2)}{\sqrt{2}}\gtrsim 1.\end{aligned} \tag{30}$$

Obviously,

$$f_k(r_ke_1)=0. \tag{31}$$

From (24) and (28), (a) holds. Since $1-\frac{1}{2k^2}<r_k<1$, we get $\lim_{k\to\infty}(1-r_k^2)=0$ and

$$\lim_{k\to\infty}(1-r_k^{k-1})\leqslant\lim_{k\to\infty}\left(1-\left(1-\frac{1}{2k^2}\right)^{k-1}\right)=0.$$

Then, by (25) and (29), (b) holds. From (26) and (30), we get (c). Using (27) and (31), we see that (d) is true.

Suppose $\int_0^1\frac{1}{\mu(t)}dt=\infty$. When $\sqrt{(1-r_k^2)(|v_2^k|^2+\cdots+|v_n^k|^2)}<|v_1^k|$, let

$$f_k(z)=\frac{\int_0^{r_kz_1}\mu_*(\eta)d\eta\int_{r_k^2z_1}^{r_k(z_1)^2}\mu_*(\eta)d\eta}{\int_0^{r_k^2}\mu_*(t)dt}.$$

When $\sqrt{(1-r_k^2)(|v_2^k|^2+\cdots+|v_n^k|^2)}\geqslant|v_1^k|$, let $\theta_j^k=\arg v_j^k(j=2,3,\cdots,n)$ and

$$f_k(z)=(\mathrm{e}^{-\mathrm{i}\theta_2^k}z_2+\cdots\mathrm{e}^{-\mathrm{i}\theta_n^k}z_n)\sqrt{1-r_k^2}\mu_*(r_kz_1).$$

If $\sqrt{(1-r_k^2)(|v_2^k|^2+\cdots+|v_n^k|^2)}<|v_1^k|$, by Lemmas 1 and 3, we have

$$\mu(|z|)|\nabla f_k(z)|\lesssim 1,\ \sup_{|z|\leqslant r}|f_k(z)|\leqslant\frac{2(\int_0^r\mu_*(t)dt)^2}{\int_0^{r_k^2}\mu_*(t)dt},\tag{32}$$

and

$$\frac{\mu(r_k)|\langle(\nabla f_k)(r_ke_1),\overline{v^k}\rangle|}{\sqrt{(1-|r_k|^2)|v^k|^2+|\langle r_ke_1,v^k\rangle|^2}}=\frac{r_k^2\mu(r_k)|v_1^k|\mu_*(r_k^3)}{\sqrt{(1-r_k^2)(|v_2^k|^2+\cdots+|v_n^k|^2)+|v_1^k|^2}}$$
$$>\frac{r_k^2\mu(r_k)\mu_*(r_k^3)}{\sqrt{2}}\gtrsim 1.\tag{33}$$

If $\sqrt{(1-r_k^2)(|v_2^k|^2+\cdots+|v_n^k|^2)}\geqslant|v_1^k|$, (28), (29) and (30) also hold. By (28) and (32), (a) holds. Using (29) and (32), (b) satisfied. From (30) and (33), we see that (c) is true. $f_k(r_ke_1)=0$ is obvious, thus (d) holds. The proof is complete. □

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