THE EXACT CONSTANT FOR THE $\ell_1 - \ell_2$ NORM INEQUALITY

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Abstract. A fundamental inequality for Hilbert spaces is the $\ell_1 - \ell_2$-norm inequality which gives that for any $x \in \mathbb{H}^n$, $\|x\|_1 \leqslant \sqrt{n}\|x\|_2$. But this is a strict inequality for all but vectors with constant modulus for their coefficients. We will give a trivial method to compute, for each x, the constant c for which $\|x\|_1 = c\sqrt{n}\|x\|_2$. Since this inequality is one of the most used results in Hilbert space theory, we believe this will have unlimited applications in the field. We will also show some variations of this result.

1. Introduction

The $\ell_1 - \ell_2$-norm inequality which gives that for any $x \in \mathbb{H}^n$, $\|x\|_1 \leqslant \sqrt{n}\|x\|_2$. But this is a strict inequality for all but vectors with constant modulus for their coefficients. We will give a trivial method to compute, for each x, the constant c for which $\|x\|_1 = c\sqrt{n}|x\|_2$. Since this is one of the most fundamental and most used inequalities in Hilbert space theory, we believe this will have broad application in the field. We will also show some variations of this result. For a background in this area see [1, 2].

2. The $\ell_1 - \ell_2$-norm inequality

We need a definition.

DEFINITION 1. A vector of the form $x = \frac{1}{\sqrt{n}}(c_1, c_2, \ldots, c_n) \in \mathbb{H}^n$, with $|c_i| = 1$ for all $i = 1, 2, \ldots, n$ will be called a *constant modulus vector*.

THEOREM 1. *Let* $x = (a_1, a_2, \ldots, a_n) \in \mathbb{H}^n$, *a real or complex Hilbert space. The following are equivalent:*

1. We have

$$\|x\|_1 = \left(1 - \frac{c_x}{2}\right)\sqrt{n}\|x\|_2.$$

2. *We have*

$$\sum_{i=1}^{n} \left| \frac{|a_i|}{\|x\|_2} - \frac{1}{\sqrt{n}} \right|^2 = c_x.$$

3. *The infimum of the distance from $\frac{x}{\|x\|_2}$ to the constant modulus vectors is $\sqrt{c_x}$.*

In particular,

$$\|x\|_1 \leqslant \sqrt{s}\|x\|_2,$$

if and only if

$$\left(1 - \frac{c_x}{2} \right) \sqrt{n} \leqslant \sqrt{s},$$

if and only if

$$1 - \frac{c_x}{2} \leqslant \sqrt{\frac{s}{n}}.$$

Proof. $(1) \Leftrightarrow (2)$: We compute:

$$\sum_{i=1}^{n} \left| \frac{|a_i|}{\|x\|_2} - \frac{1}{\sqrt{n}} \right|^2 = \frac{1}{\|x\|_2^2} \sum_{i=1}^{n} |a_i|^2 + \sum_{i=1}^{n} \frac{1}{n} - \frac{2}{\sqrt{n}\|x\|_2} \sum_{i=1}^{n} |a_i|$$

$$= 2 \left(1 - \frac{1}{\sqrt{n}\|x\|_2} \sum_{i=1}^{n} |a_i| \right) = c_x.$$

if and only if

$$\frac{1}{\sqrt{n}\|x\|_2} \sum_{i=1}^{n} |a_i| = 1 - \frac{c_x}{2},$$

if and only if

$$\sum_{i=1}^{n} |a_i| = \left(1 - \frac{c_x}{2} \right) \sqrt{n}\|x\|_2.$$

$(1) \Leftrightarrow (3)$: We compute:

$$\inf \left\{ \sum_{i=1}^{n} \left| \frac{a_i}{\|x\|_2} - \frac{c_i}{\sqrt{n}} \right|^2 : |c_i| = 1 \right\}$$

$$= \inf \left\{ \frac{1}{\|x\|_2^2} \sum_{i=1}^{n} |a_i|^2 + \sum_{i=1}^{n} \left| \frac{c_i}{\sqrt{n}} \right|^2 - 2\frac{1}{\|x\|_2\sqrt{n}} Re \sum_{i=1}^{n} a_i \bar{c}_i : |c_i| = 1 \right\}$$

$$= 2 - \frac{2}{\|x\|_2\sqrt{n}} \sum_{i=1}^{n} |a_i|.$$

The equality occurs when $\frac{1}{\sqrt{n}}(c_1, c_2, \ldots, c_n)$ is a constant modulus vector with $c_i = \frac{a_i}{|a_i|}$ if $a_i \neq 0$.

Thus

$$c_x = 2 - \frac{2}{\|x\|_2 \sqrt{n}} \sum_{i=1}^{n} |a_i| \text{ if and only if (1) holds} \quad \square$$

Now we want to look at an application of the above. For this we need two preliminary results.

THEOREM 2. *Let S be a subspace of $\mathbb{H}^n$ and let P be the orthogonal projection on S. For any $x \in \mathbb{H}^n$, $\frac{Px}{\|Px\|}$ is the closest unit vector in S to x.*

Proof. Let y be a unit vector in S and extend it to be an orthonormal basis $\{y, u_1, u_2, \ldots, u_k\}$ for S. Then

$$Px = \langle x, y \rangle y + \sum_{i=1}^{k} \langle x, u_i \rangle u_i.$$

Hence

$$\|Px\|^2 = |\langle x, y \rangle|^2 + \sum_{i=1}^{k} |\langle x, u_i \rangle|^2 \geqslant |\langle x, y \rangle|^2.$$

Therefore

$$\|Px\| \geqslant |\langle x, y \rangle| \geqslant Re\langle x, y \rangle.$$

Now we have

$$\left\| x - \frac{Px}{\|Px\|} \right\|^2 = \|x\|^2 - 2\|Px\| + 1 \leqslant \|x\|^2 - 2Re\langle x, y \rangle + \|y\|^2 = \|x - y\|^2,$$

which is our claim. $\quad \square$

Next, we examine the $\ell_1 - \ell_2$-norm inequality for subspaces.

THEOREM 3. *Let S be a subspace of $\mathbb{H}^n$ and let P be the projection onto S. The following are equivalent:*

1. *For every unit vector $x \in S$,*

$$\|x\|_1 \leqslant \left(1 - \frac{c}{2}\right) \sqrt{n}.$$

2. *The ℓ_2 distance of any unit vector in S to any constant modulus vector is greater than or equal to $\sqrt{c}$.*

3. *For every constant modulus vector x, we have*

$$\|Px\|_2 \leqslant 1 - \frac{c}{2}.$$

Proof. $(1) \Leftrightarrow (2)$: Let $x = (a_1, a_2, \ldots, a_n)$.

$$\inf\left\{\sum_{i=1}^{n}\left|a_i - \frac{c_i}{\sqrt{n}}\right|^2 : |c_i| = 1\right\} = \inf\left\{\sum_{i=1}^{n}|a_i|^2 + \sum_{i=1}^{n}\frac{1}{n} - \frac{2}{\sqrt{n}}Re\sum_{i=1}^{n}a_i\bar{c}_i : |c_i| = 1\right\}$$

$$= 2 - \frac{2}{\sqrt{n}}\sum_{i=1}^{n}|a_i|.$$

Now,

$$c \leqslant 2 - \frac{2}{\sqrt{n}}\sum_{i=1}^{n}|a_i| \text{ if and only if } \sum_{i=1}^{n}|a_i| \leqslant \left(1 - \frac{c}{2}\right)\sqrt{n}.$$

$(2) \Leftrightarrow (3)$: By Theorem 2, we need to check how close

$$\frac{Px}{\|Px\|} \text{ is to the all one's vector x.}$$

So we compute:

$$\left\|\frac{Px}{\|Px\|} - x\right\|^2 = 2 - \left\langle\frac{Px}{\|Px\|}, x\right\rangle - \left\langle x, \frac{Px}{\|Px\|}\right\rangle = 2 - 2\|Px\|$$

So,

$$c \leqslant \left\|\frac{Px}{\|Px\|} - x\right\|^2 \text{ if and only if } \|Px\| \leqslant 1 - \frac{c}{2}. \quad \square$$

Now we have the second main result. For this recall [1, 2] that if P is a projection on $\mathbb{H}^n$ with orthonormal basis $\{e_i\}_{i=1}^{n}$ then $\sum_{i=1}^{n}\|Pe_i\|^2 = \dim P(\mathbb{H}^n)$.

THEOREM 4. *Let S be a s-dimensional subspace of $\mathbb{H}^n$ with orthonormal basis $\{e_i\}_{i=1}^{n}$. If*

$$\|y\|_1 \leqslant \sqrt{s}\|y\|_2, \text{ for all } y \in S,$$

then there is an $I \subset [n]$ with $|I| = s$ and $S = span\{e_i\}_{i \in I}$.

Proof. For any $y \in S$, let c_y be defined in (2) of Theorem 1. Since

$$\|y\|_1 \leqslant \sqrt{s}\|y\|_2, \text{ for all } y \in S,$$

then

$$1 - \frac{c_y}{2} \leqslant \sqrt{\frac{s}{n}}.$$

Set

$$c = \inf\{c_y : y \in S\}$$

then

$$1 - \frac{c}{2} \leqslant \sqrt{\frac{s}{n}} \qquad (1)$$

We will prove: $\{Pe_i\}_{i=1}^n$ is an orthogonal set. This will impliy that there is an $I \subset [n]$ so that $Pe_i = e_i$ for $i \in I$ and $Pe_i = 0$ for $i \in I^c$.

First note that $\{Pe_i\}_{i=1}^n$ is a Parseval frame for S and so

$$\sum_{i=1}^n \|Pe_i\|^2 = s.$$

Assume there are two of these vectors which are not orthogonal. By reindexing, we will assume Pe_1, Pe_2 are not orthogonal. Hence, by replacing Pe_2 by $c_2 Pe_2$ with $|c_2| = 1$ if necessary with $Re\, c_2 \langle Pe_1, Pe_2 \rangle > 0$, we have

$$\|Pe_1 + c_2 Pe_2\|^2 > \|Pe_1\|^2 + \|Pe_2\|^2.$$

Now, by replacing Pe_3 by $c_3 Pe_3$ with $|c_3| = 1$ if necessary, we have

$$\|Pe_1 + c_2 Pe_2 + c_3 Pe_3\|^2 \geqslant \|Pe_1 + Pe_2\|^2 + \|Pe_3\|^2 > \|Pe_1\|^2 + \|Pe_2\|^2 + \|Pe_3\|^3.$$

Continuing, and letting $c_1 = 1$, we have

$$\left\| P \left(\sum_{i=1}^n c_i e_i \right) \right\|^2 > \sum_{i=1}^n \|Pe_i\|^2 = s.$$

It follows from Theorem 3,

$$\sqrt{\frac{s}{n}} < \left\| P \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n c_i e_i \right) \right\| \leqslant 1 - \frac{c}{2},$$

which contradicts Equation (1) above. $\quad\square$

3. An application to $L_p[0,1]$

It was pointed out to us by Bill Johnson that our work has application to Banach space theory. That, in general, when working with finite dimensional ℓ_p, it is better to use the $L_p[0,1]$ normalization. But applying our results, the *nasty $n^{1/2}$ goes away* and the expressions are independent of dimension. What is quite interesting here is the fact that if $p < s$ and $f \in L_1[0,1]$ then we can measure *how peaky f is* by seeing how small $\|f\|_p$ is. What apparently was not realized is that when $p = 1$ and $s = 2$ we get a nice equality instead of an inequality.

THEOREM 5. *Let $f \geqslant 0$ be norm one in $L_2[0,1]$. The following are equivalent:*

1. *We have*

$$\|f\|_1 = \left(1 - \frac{c}{2}\right).$$

2. *We have*

$$\|f - 1\|_2^2 = c.$$

Proof. We use the parallelogram law:

$$4 = \|f-1\|_2^2 + \|f+1\|_2^2$$
$$= \|f-1\|_2^2 + \|f\|_2^2 + 1 + 2\int_0^1 f(t)dt$$
$$= \|f-1\|_2^2 + 2 + 2\|f\|_1.$$

I.e.

$$\|f-1\|_2^2 = 2 - 2\|f\|_1.$$

It follows that

$$\|f-1\|_2^2 = c \text{ if and only if } \|f\|_1 = 1 - \frac{c}{2}. \quad \square$$

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