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A NEW VARIABLE EXPONENT PICONE IDENTITY AND APPLICTIONS

TINGFU FENG AND JUNQIANG HAN

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Abstract. In this paper, we derive a new variable exponent Picone identity for $p(x)$-Laplacian, which contains some known Picone identities. As applications, a strict monotonicity of principal eigenvalues with respect to domains for the eigenvalue problems to $p(x)$-Laplace equation, a variable exponent Barta type inequality, a variable exponent Hardy type inequality with weight, a Sturmian comparison principle to $p(x)$-Laplace equation and a Liouville type theorem to $p(x)$-Laplace system are shown.

1. Introduction and main results

In recent years, variable exponent elliptic equations and systems with $p(x)$ growth conditions which arise from the image restoration and decomposition [6, 8, 10, 23], electrotheological fluids [4, 5, 17, 21, 22] and nonlinear elasticity theory [26] etc., have been considerably studied. A prototypical operator is so called $p(x)$-Laplacian

$$\Delta_{p(x)}u = div\left(|\nabla u|^{p(x)-2}\nabla u\right), \quad p(x) > 1;$$

if $p(x) = p =$ constant, it becomes the usual p-Laplacian

$$\Delta_p u = div\left(|\nabla u|^{p-2}\nabla u\right), \quad p > 1.$$

Růžička [21] pointed out that p-Laplacian has the p homogeneity but $p(x)$-Laplacian is nonhomogeneous. It reflects that $p(x)$-Laplacian has the more complex nonlinearity. Since there is no strict equivalence relation between the norm $\|u\|_{p(x)}$ and $p(x)$ modular $\int_\Omega |u|^{p(x)}dx$ on the variable Lebesgue space $L^{p(x)}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ $(n \geqslant 3)$ is a bounded domain with the Lipschitz continuous boundary $\partial\Omega$, those efficient methods to p-Laplacian are fail to $p(x)$-Laplacian. However, there exists some good inequality relations between $\|u\|_{p(x)}$ and $\int_\Omega |u|^{p(x)}dx$:

(i) if $\|u\|_{p(x)} \geqslant 1$, then $\|u\|_{p(x)}^{p^-} \leqslant \int_\Omega |u|^{p(x)}dx \leqslant \|u\|_{p(x)}^{p^+}$;

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(ii) if $\|u\|_{p(x)} < 1$, then $\|u\|_{p(x)}^{p^+} \leqslant \int_\Omega |u|^{p(x)} dx \leqslant \|u\|_{p(x)}^{p^-}$,
where

$$p^- = ess \inf_{x\in\Omega} p(x), \quad p^+ = ess \sup_{x\in\Omega} p(x).$$

These inequalities play a very important role in the study of energy functionals on $L^{p(x)}(\Omega)$, eigenvalue problems [15], existence and uniqueness of solutions [7, 14, 27], multiplicity of solutions [18] to variable exponent elliptic equations. Also see Harjulehto et al. [16] for the symposium of $p(x)$-Laplace equations with non-standard growth and therein references.

In 1910, Picone [20] considered the homogeneous linear second order differential system

$$\begin{cases} (a_1(x)u')' + b_1(x)u = 0, \\ (a_2(x)v')' + b_2(x)v = 0, \end{cases}$$

where u and v are differentiable functions in x, and proved the following identity: for $v(x) \neq 0$,

$$\left(\frac{u}{v}\left(a_1u'v - a_2uv'\right)\right)' = (b_2 - b_1)u^2 + (a_1 - a_2)u'^2 + a_2\left(u' - v'\frac{u}{v}\right)^2; \tag{1}$$

then a Sturmian comparison principle under the conditions $a_1(x) > a_2(x), b_2(x) > b_1(x)$, and the oscilation theorem of solutions via (1) were obtained. Allegretto [1] generalized (1) to Laplacian Δ: for differentiable functions $v > 0$ and $u \geqslant 0$,

$$\left(\nabla u - \frac{u}{v}\nabla v\right)^2 = |\nabla u|^2 + \frac{u^2}{v^2}|\nabla v|^2 - 2\frac{u}{v}\nabla v \cdot \nabla u = |\nabla u|^2 - \nabla\left(\frac{u^2}{v}\right)\nabla v. \tag{2}$$

Allegretto and Huang [2] extended (2) to p-Laplacian: for differentiable functions $v > 0$ and $u \geqslant 0$,

$$|\nabla u|^p + (p-1)\frac{u^p}{v^p}|\nabla v|^p - p\frac{u^{p-1}}{v^{p-1}}|\nabla v|^{p-2}\nabla v \cdot \nabla u = |\nabla u|^p - \nabla\left(\frac{u^p}{v^{p-1}}\right)|\nabla v|^{p-2}\nabla v, \tag{3}$$

and established the Sturmian comparison principle, a Liouville's theorem, a Hardy inequality and some profound results to p-Laplace equations and systems. An extension of (3) to p-sub-Laplacian on the Heisenberg group sees Niu, Zhang and Wang [19].

Recently, a nonlinear Picone identity for Laplacian was proved by Tyagi [24]. Bal [9] generalized Tyagi's result with $\alpha = 1$ to p-Laplacian. Furthermore, Dwivedi [12] obtained a Picone identity for p-biharmonic operator. Afterward, this result was extended to the Heisenberg group by Dwivedi and Tyagi [13]. Allegretto [3] considered the Rayleigh quotient problem

$$\mathrm{Q}(u) = \frac{\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx}{\int_\Omega \frac{|u|^{p(x)}}{p(x)} dx}, \quad \text{for} \quad 0 < u \in C_0^\infty(\Omega),$$

which corresponds to the functional

$$I(u)=\int_{\Omega}\frac{|\nabla u|^{p(x)}}{p(x)}dx$$

with the Euler-Lagrange equation

$$div\left(|\nabla u|^{p(x)-2}\nabla u\right)=0,$$

and derived a variable exponent Picone identity: for differentiable function $v>0$ on $\overline{\Omega}$ and for any $0\leqslant u\in C_0^\infty(\Omega)$,

$$\begin{aligned}&\frac{|\nabla u|^{p(x)}}{p(x)}-\nabla\left(\frac{u^{p(x)}}{p(x)v^{p(x)-1}}\right)|\nabla v|^{p(x)-2}\nabla v\\&=\frac{|\nabla u|^{p(x)}}{p(x)}-\left(\frac{u}{v}\right)^{p(x)-1}|\nabla v|^{p(x)-2}\nabla v\cdot\nabla u+\frac{p(x)-1}{p(x)}\left(\frac{u}{v}\right)^{p(x)}|\nabla v|^{p(x)}.\end{aligned}\tag{4}$$

where $\nabla v\cdot\nabla p(x)\equiv 0$. A similar Picone identity to (4) was also found by Yoshida [25].

In this paper, we derive another variable exponent Picone identity different from (4), which contains some known Picone identities and can be used to give some new applications not seen in [3]. Our main result is the following:

THEOREM 1. *Let $v>0$ be a differentiable function in $\overline{\Omega}$ and $0\leqslant u\in C_0^1(\Omega)$, and denote*

$$\begin{aligned}L(u,v)=&|\nabla u|^{p(x)}-\frac{u^{p(x)}\ln\frac{u}{v}}{v^{p(x)-1}}|\nabla v|^{p(x)-2}\nabla v\cdot\nabla p(x)-p(x)\frac{u^{p(x)-1}}{v^{p(x)-1}}|\nabla v|^{p(x)-2}\nabla v\cdot\nabla u\\&+(p(x)-1)\frac{u^{p(x)}}{v^{p(x)}}|\nabla v|^{p(x)},\end{aligned}\tag{5}$$

$$R(u,v)=|\nabla u|^{p(x)}-\nabla\left(\frac{u^{p(x)}}{v^{p(x)-1}}\right)|\nabla v|^{p(x)-2}\nabla v.\tag{6}$$

Then

$$R(u,v)=L(u,v).\tag{7}$$

Moreover, there holds

$$L(u,v)\geqslant 0,$$

if $\nabla v\cdot\nabla p(x)\equiv 0$. Furthermore, $L(u,v)=0$ a.e. in Ω if and only if

$$\nabla\left(\frac{u}{v}\right)=0$$

a.e. in Ω.

REMARK 1. If $p(x)=2$ in (5) and (6), we have (2); if $p(x)=p$ in (5) and (6), it follows (3).

This paper is organized as follows: The proof of Theorem 1 is given in Section 2; Section 3 is devoted to applications of Theorem 1 including a strict monotonicity of principal eigenvalues with respect to domains for the eigenvalue problems to $p(x)$-Laplace equation, a variable exponent Barta type inequality, a variable exponent Hardy type inequality with weight, a Sturmian comparison principle to $p(x)$-Laplace equation and a Liouville type theorem to $p(x)$-Laplace system.

2. Proof of Theorem 1

Proof of Theorem 1. We see with a direct computation that

$$\begin{aligned}
R(u,v) &= |\nabla u|^{p(x)} - \frac{v^{p(x)-1}\nabla\left(u^{p(x)}\right) - u^{p(x)}\nabla\left(v^{p(x)-1}\right)}{\left[v^{p(x)-1}\right]^2}|\nabla v|^{p(x)-2}\nabla v\\
&= |\nabla u|^{p(x)} - \frac{\nabla\left(u^{p(x)}\right)}{v^{p(x)-1}}|\nabla v|^{p(x)-2}\nabla v + \frac{u^{p(x)}\nabla\left(v^{p(x)-1}\right)}{\left[v^{p(x)-1}\right]^2}|\nabla v|^{p(x)-2}\nabla v\\
&= |\nabla u|^{p(x)} - \frac{u^{p(x)}\ln u\nabla p(x) + p(x)u^{p(x)-1}\nabla u}{v^{p(x)-1}}|\nabla v|^{p(x)-2}\nabla v\\
&\quad + \frac{u^{p(x)}\left(v^{p(x)-1}\ln v\nabla p(x) + (p(x)-1)\,v^{p(x)-2}\nabla v\right)}{\left[v^{p(x)-1}\right]^2}|\nabla v|^{p(x)-2}\nabla v\\
&= |\nabla u|^{p(x)} - \frac{u^{p(x)}\ln u}{v^{p(x)-1}}|\nabla v|^{p(x)-2}\nabla v\cdot\nabla p(x) - p(x)\frac{u^{p(x)-1}}{v^{p(x)-1}}|\nabla v|^{p(x)-2}\nabla v\cdot\nabla u\\
&\quad + \frac{u^{p(x)}\ln v}{v^{p(x)-1}}|\nabla v|^{p(x)-2}\nabla v\cdot\nabla p(x) + (p(x)-1)\,\frac{u^{p(x)}}{v^{p(x)}}|\nabla v|^{p(x)}\\
&= L(u,v),
\end{aligned}$$

which proves (7). Next we check $L(u,v)\geqslant 0$. Rewriting $L(u,v)$ by

$$\begin{aligned}
L(u,v) = p(x)&\left[\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{p(x)-1}{p(x)}\left(\left(\frac{u}{v}|\nabla v|\right)^{p(x)-1}\right)^{\frac{p(x)}{p(x)-1}}\right]\\
&- p(x)\frac{u^{p(x)-1}}{v^{p(x)-1}}|\nabla v|^{p(x)-1}|\nabla u|\\
&+ p(x)\frac{u^{p(x)-1}}{v^{p(x)-1}}|\nabla v|^{p(x)-2}\left\{|\nabla v|\,|\nabla u| - \nabla v\cdot\nabla u\right\}\\
&- \frac{u^{p(x)}\ln\frac{u}{v}}{v^{p(x)-1}}|\nabla v|^{p(x)-2}\nabla v\cdot\nabla p(x)\\
:=&I + II + III,
\end{aligned} \tag{8}$$

where

$$I = p(x)\left[\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{p(x)-1}{p(x)}\left(\left(\frac{u}{v}|\nabla v|\right)^{p(x)-1}\right)^{\frac{p(x)}{p(x)-1}}\right]$$

$$- p(x)\frac{u^{p(x)-1}}{v^{p(x)-1}}|\nabla v|^{p(x)-1}|\nabla u|,$$

$$II = p(x)\frac{u^{p(x)-1}}{v^{p(x)-1}}|\nabla v|^{p(x)-2}\{|\nabla v||\nabla u| - \nabla v\cdot\nabla u\},$$

$$III = -\frac{u^{p(x)}\ln\frac{u}{v}}{v^{p(x)-1}}|\nabla v|^{p(x)-2}\nabla v\cdot\nabla p(x).$$

Recalling Young's inequality (see [11, 21]): for $a \geqslant 0,\ b \geqslant 0,\ p(x) > 1,\ q(x) > 1$ and $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$,

$$ab \leqslant \frac{a^{p(x)}}{p} + \frac{b^{q(x)}}{q},$$

and the equality holds if and only if $a^{p(x)} = b^{q(x)}$, we now take $a = |\nabla u|$ and $b = \left(\frac{u}{v}|\nabla v|\right)^{p(x)-1}$ to follow

$$p(x)|\nabla u|\left(\frac{u}{v}|\nabla v|\right)^{p(x)-1} \leqslant p(x)\left[\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{p(x)-1}{p(x)}\left(\left(\frac{u}{v}|\nabla v|\right)^{p(x)-1}\right)^{\frac{p(x)}{p(x)-1}}\right],$$

and so $I \geqslant 0$. Clearly, $II \geqslant 0$ in virtue of $|\nabla v||\nabla u| - \nabla v\cdot\nabla u \geqslant 0$. By $\nabla v\cdot\nabla p(x) \equiv 0$, we immediately have $III \equiv 0$. Hence $L(u,v) \geqslant 0$ from (8).

If $\nabla\left(\frac{u}{v}\right) = 0$, then $u = cv$ and then $L(u,v) = 0$. Now we conclude that $L(u,v) = 0$ implies $\nabla\left(\frac{u}{v}\right) = 0$. In fact, if $L(u,v)(x_0) = 0, x_0 \in \Omega$, we consider two cases respectively.

(a) If $u(x_0) \neq 0$, then $I = 0$, $II = 0$ and $III = 0$. It shows by $I = 0$,

$$|\nabla u| = \frac{u}{v}|\nabla v|; \tag{9}$$

and by $II = 0$,

$$\nabla u = c\nabla v, \tag{10}$$

where c is a positive constant. Putting (10) into (9) yields $u = cv$, namely $\nabla\left(\frac{u}{v}\right) = 0$.

(b) If $u(x_0) = 0$, denote $S = \{x \in \Omega\,|\,u(x) = 0\}$ and then $\nabla u = 0$ a.e. in S. Thus

$$\nabla\left(\frac{u}{v}\right) = \frac{v\nabla u - u\nabla v}{v^2} = 0.$$

3. Some applications

Before giving some applications, we first describe variable exponent Lebesgue spaces and Sobolev spaces, see [11, 15, 22]. Assume that $p(x):\overline{\Omega}\to(1,\infty)$ is a Lipschitz continuous function. A variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined by

$$L^{p(x)}(\Omega)=\left\{u;\ \int_{\Omega}|u(x)|^{p(x)}dx<\infty\right\}$$

with the norm

$$\|u\|_{L^{p(x)}(\Omega)}=\|u\|_{p(x)}=\inf\left\{\lambda>0;\int_{\Omega}\left|\frac{u(x)}{\lambda}\right|dx\leqslant 1\right\}.$$

A variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega)=\left\{u\in L^{p(x)}(\Omega);\nabla u\in L^{p(x)}(\Omega)\right\}$$

with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)}=\|u\|_{1,p(x)}=\|u\|_{p(x)}+\|\nabla u\|_{p(x)}.$$

Denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^{\infty}(\Omega)$ in $W^{1,p(x)}(\Omega)$ under the norm

$$\|u\|_{W_0^{1,p(x)}(\Omega)}=\|\nabla u\|_{p(x)}.$$

It is well known that $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are both separable and reflexive Banach spaces.

We consider the Dirichlet problem of $p(x)$-Laplacian

$$\begin{cases}-\Delta_{p(x)}u=\lambda|u|^{p(x)-2}u, & x\in\Omega,\\ u=0, & x\in\partial\Omega.\end{cases}\tag{11}$$

DEFINITION 1. Let $\lambda\in\mathbb{R}$ and $u\in W_0^{1,p(x)}(\Omega)$, (u,λ) is called a solution to (11) if for any $\phi\in W_0^{1,p(x)}(\Omega)$,

$$\int_{\Omega}|\nabla u|^{p(x)-2}\nabla u\nabla\phi dx=\lambda\int_{\Omega}|u|^{p(x)-2}u\phi dx.$$

If (u,λ) is a solution to (11) and $u\neq 0$, we call that λ is an eigenvalue to (11) and u is an eigenfunction corresponding to λ.

For a solution (u,λ) to (11) and $u\neq 0$, it is easy to yields

$$\lambda=\frac{\int_{\Omega}|\nabla u|^{p(x)}dx}{\int_{\Omega}|u|^{p(x)}dx},\tag{12}$$

and $\lambda>0$. From (12), the principal eigenvalue to (11) is defined by

$$\lambda_1 = \inf_{u \in W_0^{1,p(x)}(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^{p(x)} dx}{\int_\Omega |u|^{p(x)} dx}$$

and existence of λ_1 was obtained by Fan, Zhang and Zhao [15].

Using Theorem 1, we can obtain the strict monotonicity of principal eigenvalues with respect to domains, which enrich the results on principal eigenvalues.

PROPOSITION 1. *Suppose that $\Omega_1 \subset \Omega_2$ and $\Omega_1 \neq \Omega_2$. If both $\lambda_1(\Omega_1)$ and $\lambda_1(\Omega_2)$ exist, and u_1 and u_2 are positive eigenfunctions corresponding to $\lambda_1(\Omega_1)$ and $\lambda_1(\Omega_2)$, respectively, satisfying*

$$\begin{cases} -\Delta_{p(x)} u_1 = \lambda_1(\Omega_1) |u_1|^{p(x)-2} u_1, & x \in \Omega_1, \\ u_1 > 0, & x \in \Omega_1, \\ u_1 = 0, & x \in \partial\Omega_1, \end{cases} \tag{13}$$

and

$$\begin{cases} -\Delta_{p(x)} u_2 = \lambda_1(\Omega_2) |u_2|^{p(x)-2} u_2, & x \in \Omega_2, \\ u_2 > 0, & x \in \Omega_2, \\ u_2 = 0, & x \in \partial\Omega_2, \end{cases} \tag{14}$$

with $\nabla u_1 \cdot \nabla p(x) \equiv 0$, $\nabla u_2 \cdot \nabla p(x) \equiv 0$, then

$$\lambda_1(\Omega_1) > \lambda_1(\Omega_2). \tag{15}$$

Proof. It follows from (13), (14) and (7) that

$$\begin{aligned} 0 &\leqslant \int_\Omega L(u_1,u_2)dx = \int_\Omega R(u_1,u_2)dx \\ &= \int_{\Omega_1} |\nabla u_1|^{p(x)} dx - \int_{\Omega_1} \nabla \left(\frac{{u_1}^{p(x)}}{{u_2}^{p(x)-1}} \right) |\nabla u_2|^{p(x)-2} \nabla u_2 dx \\ &= \int_{\Omega_1} |\nabla u_1|^{p(x)} dx + \int_{\Omega_1} \frac{{u_1}^{p(x)}}{{u_2}^{p(x)-1}} \Delta_{p(x)} u_2 dx \\ &= \int_{\Omega_1} |\nabla u_1|^{p(x)} dx - \lambda_1(\Omega_2) \int_{\Omega_1} {u_1}^{p(x)} dx \\ &= (\lambda_1(\Omega_1) - \lambda_1(\Omega_2)) \int_{\Omega_1} {u_1}^{p(x)} dx, \end{aligned}$$

which gives

$$\lambda_1(\Omega_1) - \lambda_1(\Omega_2) \geqslant 0.$$

Noting that $\lambda_1(\Omega_1) \neq \lambda_1(\Omega_2)$ because of $\Omega_1 \subset \Omega_2$ and $\Omega_1 \neq \Omega_2$, this concludes (15). □

The next is a variable exponent Barta type inequality.

PROPOSITION 2. *Suppose that $u \in W_0^{1,p(x)}(\Omega)$ ia a positive solution to (11). Then for any differentiable function $v > 0$ in $\overline{\Omega}$ with $\Delta_{p(x)} v \in C\left(\overline{\Omega}\right)$ and $\nabla v \cdot \nabla p(x) \equiv 0$, we have*

$$\lambda_1 \geqslant \inf_{x\in\Omega} \frac{-\Delta_{p(x)} v}{v^{p(x)-1}}. \tag{16}$$

Proof. Using (7), it leads to

$$\begin{aligned} 0 &\leqslant \int_\Omega L(u,v)dx = \int_\Omega R(u,v)dx \\ &= \int_\Omega |\nabla u|^{p(x)} dx - \int_\Omega \nabla\left(\frac{u^{p(x)}}{v^{p(x)-1}}\right)|\nabla v|^{p(x)-2}\nabla v dx \\ &= \int_\Omega |\nabla u|^{p(x)} dx + \int_\Omega \frac{u^{p(x)}}{v^{p(x)-1}} \Delta_{p(x)} v dx, \end{aligned}$$

which implies

$$\int_\Omega |\nabla u|^{p(x)} dx \geqslant \int_\Omega u^{p(x)} \left[\frac{-\Delta_{p(x)} v}{v^{p(x)-1}}\right] dx \geqslant \int_\Omega u^{p(x)} dx \, \underset{x\in\Omega}{inf} \left[\frac{-\Delta_{p(x)} v}{v^{p(x)-1}}\right],$$

namely,

$$\frac{\int_\Omega |\nabla u|^{p(x)} dx}{\int_\Omega u^{p(x)} dx} \geqslant \inf_{x\in\Omega} \left[\frac{-\Delta_{p(x)} v}{v^{p(x)-1}}\right].$$

The proof of (16) is ended. □

PROPOSITION 3. *If a differentiable function $v > 0$ in $\overline{\Omega}$ with $\nabla v \cdot \nabla p(x) \equiv 0$, satisfies*

$$-\Delta_{p(x)} v \geqslant \lambda g(x) v^{p(x)-1} \tag{17}$$

for some $\lambda > 0$ and a weight function $g(x)$, then for any $0 \leqslant u \in C_0^1(\Omega)$, there holds

$$\int_\Omega |\nabla u|^{p(x)} dx \geqslant \lambda \int_\Omega g(x) u^{p(x)} dx. \tag{18}$$

Proof. By (17) and (7), we know

$$\begin{aligned} 0 &\leqslant \int_\Omega L(u,v)dx = \int_\Omega R(u,v)dx \\ &= \int_\Omega |\nabla u|^{p(x)} dx - \int_\Omega \nabla\left(\frac{u^{p(x)}}{v^{p(x)-1}}\right)|\nabla v|^{p(x)-2}\nabla v dx \\ &= \int_\Omega |\nabla u|^{p(x)} dx + \int_\Omega \frac{u^{p(x)}}{v^{p(x)-1}} \Delta_{p(x)} v dx \\ &\leqslant \int_\Omega |\nabla u|^{p(x)} dx - \lambda \int_\Omega g(x) u^{p(x)} dx, \end{aligned}$$

which gives (18). □

Now we provide a Sturmian comparison principle to $p(x)$-Laplace equation.

PROPOSITION 4. *Let $k_1(x)$ and $k_2(x)$ be two continuous functions with $k_1(x) < k_2(x)$ on Ω . Assume that there exists a positive function $u \in W_0^{1,p(x)}(\Omega)$ satisfying*

$$\begin{cases} -\Delta_{p(x)}u = k_1(x)|u|^{p(x)-2}u, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{19}$$

Then any nontrivial solution v with $\nabla v \cdot \nabla p(x) \equiv 0$ to the following equation

$$-\Delta_{p(x)}v = k_2(x)|v|^{p(x)-2}v, x \in \Omega, \tag{20}$$

must change sign.

Proof. Suppose that v does not change sign; without loss of generality, let $v > 0$ in $\overline{\Omega}$. By (19), (20) and (7), we observe

$$\begin{aligned} 0 &\leqslant \int_\Omega L(u,v)dx = \int_\Omega R(u,v)dx \\ &= \int_\Omega |\nabla u|^{p(x)}dx - \int_\Omega \nabla\left(\frac{u^{p(x)}}{v^{p(x)-1}}\right)|\nabla v|^{p(x)-2}\nabla v dx \\ &= \int_\Omega |\nabla u|^{p(x)}dx + \int_\Omega \frac{u^{p(x)}}{v^{p(x)-1}}\Delta_{p(x)}v dx \\ &= \int_\Omega (k_1(x) - k_2(x))\, u^{p(x)}dx \\ &< 0, \end{aligned}$$

which is a contradiction. This accomplishes the proof. □

Finally, we exhibit a Liouville type theorem for a variable exponent elliptic system.

PROPOSITION 5. *Let $(u,v) \in W_0^{1,p(x)}(\Omega) \times W_0^{1,p(x)}(\Omega)$ be a pair of positive solutions for the Dirichlet problem to the variable exponent elliptic system*

$$\begin{cases} -\Delta_{p(x)}u = v^{p(x)-1}, & x \in \Omega, \\ -\Delta_{p(x)}v = \frac{\left[v^{p(x)-1}\right]^2}{u^{p(x)-1}}, & x \in \Omega, \\ u > 0, v > 0, & x \in \Omega, \\ u = 0, v > 0, & x \in \partial\Omega, \end{cases} \tag{21}$$

where $\nabla v \cdot \nabla p(x) = 0$. Then $\nabla\left(\frac{u}{v}\right) = 0$ a.e. on Ω.

Proof. For any $\phi, \varphi \in W_0^{1,p(x)}(\Omega)$, it gets by (21) that

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla \phi dx = \int_\Omega v^{p(x)-1}\phi dx, \tag{22}$$

$$\int_\Omega |\nabla v|^{p(x)-2}\nabla v\nabla\varphi dx = \int_\Omega \frac{\left[v^{p(x)-1}\right]^2}{u^{p(x)-1}}\varphi dx. \tag{23}$$

Choosing $\phi = u$ in (22) and $\varphi = \frac{u^{p(x)}}{v^{p(x)-1}}$ and in (23), respectively, we have

$$\int_\Omega |\nabla u|^{p(x)}dx = \int_\Omega v^{p(x)-1}udx = \int_\Omega \nabla\left(\frac{u^{p(x)}}{v^{p(x)-1}}\right)|\nabla v|^{p(x)-2}\nabla v dx,$$

which shows from (7) that

$$\int_\Omega L(u,v)dx = \int_\Omega R(u,v)dx = \int_\Omega |\nabla u|^{p(x)}dx - \int_\Omega \nabla\left(\frac{u^{p(x)}}{v^{p(x)-1}}\right)|\nabla v|^{p(x)-2}\nabla v dx = 0.$$

The conclusion is proved. □

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Tingfu Feng
Department of Applied Mathematics
Northwestern Polytechnical University
Xi'an, shaanxi, 710129, P. R. China
and
Department of Mathematics
Kunming University, Kunming
Yunnan, 650214, P. R. China
e-mail: ftfml@mail.nwpu.edu.cn

Junqiang Han
Department of Applied Mathematics
Northwestern Polytechnical University
Xi'an, shaanxi, 710129, P. R. China
e-mail: southhan@163.com