Mathematical
Inequalities
& Applications
Volume 22, Number 1 (2019), 77–96 doi:10.7153/mia-2019-22-06

ON MULTILINEAR COMMUTATORS OF MARCINKIEWICZ INTEGRALS IN VARIABLE EXPONENT LEBESGUE AND HERZ TYPE SPACES

LIWEI WANG AND LISHENG SHU

(*Communicated by I. Perić*)

Abstract. Based on some pointwise estimates, we establish the boundedness of multilinear commutators of Marcinkiewicz integrals in variable exponent Lebesgue spaces, which in turn are used to obtain some boundedness results for such operators in variable exponent Herz and Herz-Morrey spaces. Further, we consider the boundedness in variable exponent Herz-type Hardy spaces applying the atomic decomposition and generalization of the BMO norms.

1. Introduction

Suppose that $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$ $(n \geqslant 2)$ equipped with the normalized Lebesgue measure $d\sigma(x')$. Let $\Omega \in L^1(\mathbb{S}^{n-1})$ be homogeneous of degree zero and satisfy

$$\int_{\mathbb{S}^{n-1}} \Omega(x')d\sigma(x') = 0, \tag{1}$$

where $x' = x/|x|$ for any $x \neq 0$.

Stein in [41] introduced the n-dimensional Marcinkiewicz integral operator

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_{\Omega,t}(f)(x)|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}},$$

where

$$F_{\Omega,t}(f)(x) = \int_{|x-y|\leqslant t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy.$$

Moreover, he proved that if Ω satisfies a Lip_γ $(0 < \gamma \leqslant 1)$ condition on $\mathbb{S}^{n-1}$, *i.e.*,

$$|\Omega(x') - \Omega(y')| \leqslant C|x' - y'|^\gamma, \qquad x', y' \in \mathbb{S}^{n-1}, \tag{2}$$

Mathematics subject classification (2010): 42B20, 42B35, 46E30.
Keywords and phrases: Marcinkiewicz integrals, multilinear commutators, variable exponent, Herz-type spaces.
This research is supported by Pre-research Project of the National Natural Science Foundations of China (Grant No: 2019yyzr14) and University NSR Project of Anhui Province (KJ2016A760).

then μ_Ω is of type (p,p) for $1<p\leqslant 2$ and of weak type $(1,1)$. Benedek, Calderón and Panzone in [2] showed that μ_Ω is of type (p,p) for $1<p<\infty$ provided that Ω is continuously differentiable in $x\neq 0$. Subsequently, Torchinsky, Ding *etc.* made important progress on this operator, see [9, 10, 23, 24, 25, 26, 27, 28, 29, 43] for its recent development.

A locally integrable function b is said to be a BMO function, if it satisfies

$$\|b\|_* := \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B|}\int_B |b(y)-b_B|\,dy < \infty,$$

where and in the sequel B is ball centered at x and radius of r, $b_B=\frac{1}{|B|}\int_B b(t)dt$ and $\|b\|_*$ is the norm in $\mathrm{BMO}(\mathbb{R}^n)$. For $b\in \mathrm{BMO}(\mathbb{R}^n)$, $[b,\mu_\Omega]$, the commutator of the Marcinkiewicz integral operator, is then defined by

$$\mu_{\Omega,b}(f)=[b,\mu_\Omega]f := b\mu_\Omega(f)-\mu_\Omega(bf).$$

Torchinsky and Wang [43] showed that for $1<p<\infty$, $[b,\mu_\Omega]: L^p(\omega)\to L^p(\omega)$ if $\Omega\in Lip_\gamma(\mathbb{S}^{n-1})$ $(0<\gamma\leqslant 1)$ and ω is a weight in the Muckenhoupt A_p class (see [13] for the definition).

Given a vector $\vec{b}=(b_1,b_2,\ldots,b_m)$, where $b_j's$ are suitable functions. Motivated by the work of Pérez and Trujillo-González [37] on multilinear operators, we define the multilinear commutators of the Marcinkiewicz integrals by

$$\mu_{\Omega,\vec{b}}(f)(x)=\left(\int_0^\infty\left|\int_{|x-y|\leqslant t}\prod_{i=1}^m (b_i(x)-b_i(y))\frac{\Omega(x-y)}{|x-y|^{n-1}}f(y)dy\right|^2\frac{dt}{t^3}\right)^{\frac12}, \qquad (3)$$

where $m\in\mathbb{N}$. Clearly, if $m=1$ and $b_i=b$, then the operators $\mu_{\Omega,\vec{b}}$ coincide with the commutator $[b,\mu_\Omega]$. In the case of $\Omega\in Lip_\gamma(\mathbb{S}^{n-1})$ $(0<\gamma\leqslant 1)$ and $b_i\in \mathrm{BMO}(\mathbb{R}^n)$, Zhang in [51] proved that $\mu_{\Omega,\vec{b}}$ are bounded on $L^p(\omega)$ for $1<p<\infty$ when $\omega\in A_p$ and established a weighted weak $L(\log L)$-type estimate when $p=1$ and $\omega\in A_1$. We refer to [14, 34] for an extensive study of multilinear operators.

In recent years, following the fundamental work of Kováčik and Rákosník [21], function spaces with variable exponent have attracted a great attention in connection with problems of the boundedness of classical operators (such as maximal, potential and Calderón-Zygmund operators *etc.*) on those spaces, which in turn were motivated by the treatment of recent problems in fluid dynamics, image restoration and PDE with non-standard growth conditions, see for instance [3, 7, 15, 39, 48, 49, 50, 52].

Unfortunately the variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ and the classical cases have some undesired properties. For example, the variable $L^{p(\cdot)}(\mathbb{R}^n)$ spaces are not translation invariant. As a consequence, the variable exponent Lebesgue spaces are not rearrangement invariant Banach spaces, and so neither good-λ techniques nor rearrangement inequalities may be applied for a generalization of some standard results in classical Lebesgue spaces to the case of $L^{p(\cdot)}(\mathbb{R}^n)$, see [5, 8] for futher details.

Karlovich and Lerner in [20] proved that $[b,T]$, the commutator of a standard Calderón-Zygmund singular integral operator T and a BMO function b, is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, which improves a celebrated result by Coifman, Rochberg and Weiss in [4]. Recently, Xu [47] made a futher step and showed that the multilinear commutators with vector symbol $\vec{b} = (b_1, b_2, \dots, b_m)$ as defined by Pérez and Trujillo-González in [37] enjoy the same $L^{p(\cdot)}(\mathbb{R}^n)$ estimates when $b_i \in \mathrm{BMO}(\mathbb{R}^n)$, $i = 1, 2, \dots, m$. These results inspire us to ask whether the multilinear operators $\mu_{\Omega,\vec{b}}$ have the similar mapping properties in variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$ spaces? Our first result (see Theorem 1 below) will give an affirmative answer to this question.

Herz spaces $\dot{K}_p^{\alpha,q}(\mathbb{R}^n)$ and $K_p^{\alpha,q}(\mathbb{R}^n)$ (see [32] for the definition) have been playing a central role in harmonic analysis and PDE. For instance, they are good substitutes of the ordinary Hardy spaces when considering the boundedness of non-translation invariant singular integral operators, they also appear in the characterization of multiplier on Hardy spaces and in the regularity theory for elliptic and parabolic equations in divergence form, see [32, 38]. The generalized Herz spaces $\dot{K}_{p(\cdot)}^{\alpha,q}(\mathbb{R}^n)$ and $K_{p(\cdot)}^{\alpha,q}(\mathbb{R}^n)$ with variable exponent $p(\cdot)$ were recently studied by Izuki [16, 17]. Simultaneously, he has given some basic lemmas on generalization of the BMO norms to get the boundedness of classical operators and their commutators on such spaces. For the time being, the theory of variable Herz spaces and their generalization spaces (*e.g.* variable Herz-Morrey spaces $M\dot{K}_{q,p(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)$ as studied in [19]) are widely developed, one can consult [1, 12, 33, 40] for its development and applications.

On the other hand, the variable exponent Herz-type Hardy spaces, as well as their atomic decomposition characterizations, have been intensively studied by a significant number of authors [12, 45]. Using these decompositions, they also established the boundedness results for some singular integrals on such spaces.

Motivated by the results mentioned above, another purpose of this article is to prove the boundedness of the operators $\mu_{\Omega,\vec{b}}$ in variable exponent Herz-type spaces, which includes variable exponent Herz, Herz-Morrey and the atomic Herz-Hardy spaces.

We usually denote cubes in $\mathbb{R}^n$ by Q, $|Q|$ is the Lebesgue measure of Q. χ_E is a characteristic function of a measurable set $E \subset \mathbb{R}^n$. Let $B_l = \{x \in \mathbb{R}^n : |x| \leqslant 2^l\} (l \in \mathbb{Z})$ and $B := B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$. f_B means the integral average of f on B, namely, $f_B = \frac{1}{|B|}\int_B f(x)dx$. $p'(\cdot)$ denotes the conjugate exponent defined by $1/p(\cdot) + 1/p'(\cdot) = 1$. By $\mathscr{S}'(\mathbb{R}^n)$ we denote the space of tempered distributions. For $x \in \mathbb{R}$, we denote by $[x]$ the largest integer less than or equal to x. The letter C stands for a positive constant, which may vary from line to line. The expression $f \lesssim g$ means that $f \leqslant Cg$, and $f \approx g$ means $f \lesssim g \lesssim f$.

2. Preliminaries and lemmas

We begin with a brief and necessarily incomplete review of the variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$, see [5, 8] for more information.

Let $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ be a measurable function. We assume that

$$1 \leqslant p_- \leqslant p(x) \leqslant p_+ < \infty,$$

where and in the sequel

$$p_- := \text{ess inf}_{x\in\mathbb{R}^n} p(x), \quad p_+ := \text{ess sup}_{x\in\mathbb{R}^n} p(x).$$

By $L^{p(\cdot)}(\mathbb{R}^n)$ we denote the set of all measurable functions f on $\mathbb{R}^n$ such that

$$\rho_{p(\cdot)}(f) := \int_{\mathbb{R}^n} |f(x)|^{p(x)}dx < \infty.$$

This is a Banach space with respect to the Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \inf\{\mu > 0 : \rho_{p(\cdot)}(f/\mu) \leqslant 1\}.$$

Obviously, this norm has the following property

$$\||f|^\nu\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \|f\|^\nu_{L^{\nu p(\cdot)}(\mathbb{R}^n)}, \quad \nu \geqslant 1/p_-.$$

Given an open set $\Omega \subset \mathbb{R}^n$, the space $L^{p(\cdot)}_{\text{loc}}(\Omega)$ is defined by

$$L^{p(\cdot)}_{\text{loc}}(\Omega) = \{f : f \in L^{p(\cdot)}(K) \text{ for all compact subsets } K \subset \Omega\}.$$

For simplicity, we use the notation

$$\mathscr{P}(\mathbb{R}^n) := \{p(\cdot) : \ p_- > 1 \text{ and } p_+ < \infty\}$$

and

$$\mathscr{B}(\mathbb{R}^n) := \{p(\cdot) \in \mathscr{P}(\mathbb{R}^n) : \ M \text{ is bounded on } L^{p(\cdot)}(\mathbb{R}^n)\},$$

where M is the Hardy-Littlewood maximal operator defined by

$$Mf(x) = \sup_{Q\ni x} \frac{1}{|Q|} \int_Q |f(y)|dy.$$

We say a measurable function $\phi : \mathbb{R}^n \to [1,\infty)$ is globally log-Hölder continuous if it satisfies

$$|\phi(x) - \phi(y)| \leqslant \frac{-C}{\log(|x-y|)}, \quad |x-y| \leqslant 1/2,, \tag{4}$$

$$|\phi(x) - \phi(y)| \leqslant \frac{C}{\log(e+|x|)}, \quad |y| \geqslant |x|, \tag{5}$$

for any $x,y \in \mathbb{R}^n$. The set of $p(\cdot)$ satisfying (4) and (5) is denoted by $LH(\mathbb{R}^n)$. It is well-known that if $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)\bigcap LH(\mathbb{R}^n)$, then the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, thus we have $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$, see [6, 35].

When $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n)$, the generalized Hölder inequality holds in the form

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leqslant r_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}, \tag{6}$$

with $r_p = 1 + 1/p_- - 1/p_+$, see [21, Theorem 2.1].

The following Lemmas 1 and 2 are due to Izuki [18, Page 203].

LEMMA 1. *Suppose* $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$, *then we have*

$$\frac{1}{|B|}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leqslant C.$$

LEMMA 2. *Suppose* $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$, *then we have for all measurable subsets* $E \subset B$,

$$\frac{\|\chi_E\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leqslant C\left(\frac{|E|}{|B|}\right)^{\delta_1}, \quad \frac{\|\chi_E\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \leqslant C\left(\frac{|E|}{|B|}\right)^{\delta_2},$$

where δ_1, δ_2 *are constants with* $0 < \delta_1, \delta_2 < 1$.

REMARK 1. We would like to stress that everywhere below the constants δ_1 and δ_2 are always the same as in Lemma 2.

LEMMA 3. *Suppose* $p_i(\cdot), p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, $i = 1, 2, \ldots, m$, *so that*

$$\frac{1}{p(x)} = \sum_{i=1}^{m} \frac{1}{p_i(x)},$$

where $m \in \mathbb{N}$. *Then for all* $f_i \in L^{p_i(\cdot)}(\mathbb{R}^n)$, *we have*

$$\left\|\prod_{i=1}^{m} f_i\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant C \prod_{i=1}^{m} \|f_i\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}.$$

LEMMA 4. *Suppose* $p(\cdot) \in LH(\mathbb{R}^n)$ *and* $0 < p_- \leqslant p(x) \leqslant p_+ < \infty$.
(i) For all balls (or cubes) $|B| \leqslant 2^n$ *and any* $x \in B$, *we have*

$$\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \approx |B|^{1/p(x)}.$$

(ii) For all balls (or cubes) $|B| \geqslant 1$, *we have*

$$\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \approx |B|^{1/p_\infty},$$

where $p_\infty := \lim_{x \to \infty} p(x)$.

The proofs of Lemmas 3 and 4 can be found in [5] and [8], respectively. Combining Lemma 3, Lemma 4 and Lemma 3 in [18, Page 464], a simple computation shows that

LEMMA 5. *Suppose* $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$, $b_i \in \mathrm{BMO}(\mathbb{R}^n)$, $i = 1, 2, \ldots, m$, $k > j$ $(k, j \in \mathbb{N})$, *then we have*

$$\sup_{B \subset \mathbb{R}^n} \frac{1}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \left\|\prod_{i=1}^{m}(b_i - (b_i)_B)\chi_B\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \approx \prod_{i=1}^{m} \|b_i\|_*$$

and

$$\left\|\prod_{i=1}^{m}(b_i - (b_i)_{B_j})\chi_{B_k}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant C(k-j)^m \prod_{i=1}^{m} \|b_i\|_* \|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

REMARK 2. We remark that Lemma 5 is a generalization of the well known properties for $\mathrm{BMO}(\mathbb{R}^n)$ spaces (see [42]), and is also a generalized version of Izuki's result in [17, Page 204].

3. Boundedness on variable exponent Lebesgue spaces

We first recall the duality and density in variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$, and some pointwise estimates for sharp maximal functions.

For $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, the spaces $L^{p(\cdot)}(\mathbb{R}^n)$ can be endowed with the Orlicz type norm

$$\|f\|^0_{L^{p(\cdot)}(\mathbb{R}^n)} := \sup\left\{ \int_{\mathbb{R}^n} |f(x)g(x)|dx : g \in L^{p'(\cdot)}(\mathbb{R}^n), \quad \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leqslant 1 \right\}.$$

This norm, as pointed out in [21], is equivalent to the Luxemburg-Nakano norm, that is

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant \|f\|^0_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant r_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}, \tag{7}$$

where $r_p = 1 + 1/p_- - 1/p_+$.

By L^∞_c we denote the set of all bounded functions f with compact support. From [21, Theorem 2.11] (see also [20, Lemma 2.2]), we get the following.

LEMMA 6. *Suppose $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, then L^∞_c is dense in $L^{p(\cdot)}(\mathbb{R}^n)$ and in $L^{p'(\cdot)}(\mathbb{R}^n)$.*

For $\delta > 0$ and $f \in L^1_{loc}(\mathbb{R}^n)$, we define

$$M_\delta(f)(x) = M(|f|^\delta)^{1/\delta}(x) = \left(\sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|^\delta dy \right)^{1/\delta}.$$

Given a function $f \in L^\delta_{loc}(\mathbb{R}^n)$, set also

$$f^\sharp_\delta(x) = \sup_{Q \ni x} \inf_{c \in \mathbb{R}} \left(\frac{1}{|Q|} \int_Q |f(y) - c|^\delta dy \right)^{1/\delta},$$

where the supremums are taken over all cubes $Q \subset \mathbb{R}^n$ containing x.

The non-increasing rearrangement of a measurable function f on $\mathbb{R}^n$ is defined as

$$f^*(s) := \inf\left\{ \sigma > 0 : |\{t \in \mathbb{R}^n : |f(t)| > \sigma\}| \leqslant s \right\}, \quad s > 0,$$

and for a fixed $\lambda \in (0,1)$, the local sharp maximal function $M^\sharp_\lambda f$ is given by

$$M^\sharp_\lambda(f)(x) = \sup_{Q \ni x} \inf_{c \in \mathbb{R}} ((f-c)\chi_Q)^*(\lambda|Q|).$$

The next lemma is due to [20, Proposition 2.3].

LEMMA 7. *Suppose* $\lambda \in (0,1)$, $\delta > 0$ *and* $f \in L^{\delta}_{loc}(\mathbb{R}^n)$, *then we have*

$$M^{\sharp}_{\lambda}(f)(x) \leqslant (1/\lambda)^{1/\delta} f^{\sharp}_{\delta}(x), \qquad x \in \mathbb{R}^n.$$

A function Φ defined on $[0,\infty)$ is said to be a Young function, if Φ is a continuous, nonnegative, strictly increasing and convex function with $\lim_{t\to 0^+} \Phi(t)/t = \lim_{t\to 0^+} t/\Phi(t) = 0$. We define the Φ-average of a function f over a cube Q by

$$\|f\|_{\Phi,Q} = \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q \Phi\left(\frac{|f(y)|}{\lambda}\right) dy \leqslant 1\right\}.$$

Associated to this Φ-average, we define the maximal operator M_{Φ} by

$$M_{\Phi}(f)(x) := \sup_{Q \ni x} \|f\|_{\Phi,Q}.$$

When $\Phi(t) = t\log^r(e+t)$ $(r \geqslant 1)$, we denote M_{Φ} by $M_{L(\log L)^r}$. It is well-known that if $m \in \mathbb{N}$, then $M_{L(\log L)^m} \approx M^{m+1}$, the $m+1$ iterations of the Hardy-Littlewood maximal operator M, see [36, Page 179].

Ding, Lu and Zhang [11] established the following pointwise estimates for the sharp function of μ_{Ω}, which generalizes the ones obtained by Torchinsky and Wang [43].

LEMMA 8. *Let* $0 < \delta < 1$ *and* f, $\mu_{\Omega}(f)$ *be both locally integrable. Then there exsits a positive constant* C, *independet of* f *and* x, *such that*

$$(\mu_{\Omega}(f))^{\sharp}_{\delta}(x) \leqslant CMf(x), \qquad x \in \mathbb{R}^n.$$

For the multilinear commutators $\mu_{\Omega,\vec{b}}$, there holds a similar piontwise estimate. To state it, we first introduce some notations.

As in [37], given any positive integer m, for all $1 \leqslant j \leqslant m$, we denote by C^m_j the family of all finite subset $\sigma = \{\sigma(1),\sigma(2),\ldots,\sigma(j)\}$ of $\{1,2,\ldots,m\}$ of j different elements. For any $\sigma \in C^m_j$, we associate the complementary sequence σ' given by $\sigma' = \{1,2,\ldots,m\}\backslash\sigma$.

Suppose $\vec{b} = (b_1,b_2,\ldots,b_m)$ and $\sigma = \{\sigma(1),\sigma(2),\ldots,\sigma(j)\} \in C^m_j$. Denote $\vec{b}_{\sigma} = \{b_{\sigma(1)},b_{\sigma(2)},\ldots,b_{\sigma(j)}\}$, $b_{\sigma} = b_{\sigma(1)}b_{\sigma(2)}\cdots b_{\sigma(j)}$ and $\|b_{\sigma}\| = \prod_{j\in\sigma}\|b\|_*$. In the case $\sigma = \{1,2,\ldots,m\}$, we denote $\|b_{\sigma}\|$ by $\|\vec{b}\|$.

For any $\sigma = \{\sigma(1),\sigma(2),\ldots,\sigma(j)\} \in C^m_j$, we define

$$\mu_{\Omega,\vec{b}_{\sigma}}(f)(x) = \left(\int_0^{\infty}\left|\int_{|x-y|\leqslant t}\prod_{i=1}^{j}(b_{\sigma(i)}(x) - b_{\sigma(i)}(y))\frac{\Omega(x-y)}{|x-y|^{n-1}}f(y)dy\right|^2\frac{dt}{t^3}\right)^{\frac{1}{2}}.$$

If $\sigma = \{1,2,\ldots,m\}$, then we understand $\mu_{\Omega,\vec{b}_{\sigma}} = \mu_{\Omega,\vec{b}}$ and $\mu_{\Omega,\vec{b}_{\sigma'}} = \mu_{\Omega}$.

We now mention an immediate consequence of Proposition 2.4 in [51].

LEMMA 9. *Suppose* $\Omega \in Lip_\gamma(\mathbb{S}^{n-1})$ $(0<\gamma\leqslant 1)$ *and* $0<\delta<\varepsilon<1$. *Let* $\mu_{\Omega,\vec{b}}$ *be as in* (3). *Then for any* $f \in L_c^\infty$, *there exists a constant* $C>0$, *depending only on* δ *and* ε, *such that*

$$(\mu_{\Omega,\vec{b}}f)_\delta^\sharp(x) \leqslant C\Big\{\|\vec{b}\|M_{L(\log L)^m}f(x)+\sum_{j=1}^m\sum_{\sigma\in C_j^m}\|b_\sigma\|M_\varepsilon(\mu_{\Omega,\vec{b}_{\sigma'}}f)(x)\Big\}.$$

We also need the following result from Lerner [22, Theorem 1].

LEMMA 10. *Suppose* $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *and let* f *be a measurable function with* $f^*(+\infty)=0$, *then*

$$\int_{\mathbb{R}^n}|f(x)g(x)|dx \leqslant c_n\int_{\mathbb{R}^n}M^\sharp_{\lambda_n}f(x)Mg(x)dx,$$

where constants c_n, λ_n *depend only on dimension* n.

We now state the main result of this section.

THEOREM 1. *Suppose* $\Omega \in Lip_\gamma(\mathbb{S}^{n-1})$ $(0<\gamma\leqslant 1)$ *and let* $p(\cdot)\in\mathscr{B}(\mathbb{R}^n)$, $\vec{b}=(b_1,b_2,\ldots,b_m)$ *and* $b_i\in \mathrm{BMO}(\mathbb{R}^n)$, $i=1,2,\ldots,m$. *Then the multilinear commutators* $\mu_{\Omega,\vec{b}}$ *as in* (3) *are bounded on* $L^{p(\cdot)}(\mathbb{R}^n)$.

REMARK 3. Clearly, this result is a generalized version of [44, Theorem 1]. However, it should be pointed that in the proof of Theorem 1 we use some ideas from [20] and [47].

Proof of Theorem 1. Let $f\in L_c^\infty$ and $g\in L^{p'(\cdot)}(\mathbb{R}^n)\subset L^1_{\mathrm{loc}}(\mathbb{R}^n)$. We show Theorem 1 by indction on m. For $m=1$, the same argument as in [44, Page 1098] gives

$$\|\mu_{\Omega,b}f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant \|\mu_{\Omega,b}f\|^0_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Suppose now that the Theorem 1 is true for $m-1$. We will show that it is true for m. By Theorem 1.4 in [51, Page 1389], $\mu_{\Omega,\vec{b}}$ satisfies the conditions of Lemmas 9. Thus, from Lemmas 9, 7, 8 and the generalized Hölder inequality (6), it follows that

$$\begin{aligned}\int_{\mathbb{R}^n}|(\mu_{\Omega,\vec{b}}f)(x)g(x)|dx &\leqslant C\int_{\mathbb{R}^n}M^\sharp_{\lambda_n}(\mu_{\Omega,\vec{b}}f)(x)Mg(x)dx\\ &\leqslant C\int_{\mathbb{R}^n}(\mu_{\Omega,\vec{b}}f)^\sharp_\delta(x)Mg(x)dx\\ &\leqslant C\int_{\mathbb{R}^n}\Big\{\|\vec{b}\|M_{L(\log L)^m}f(x)+\sum_{j=1}^m\sum_{\sigma\in C_j^m}\|b_\sigma\|M_\varepsilon(\mu_{\Omega,\vec{b}_{\sigma'}}f)(x)\Big\}Mg(x)dx\\ &\leqslant C\int_{\mathbb{R}^n}\Big\{\|\vec{b}\|M_{L(\log L)^m}f(x)+\sum_{j=1}^m\sum_{\sigma\in C_j^m}\|b_\sigma\|M_{L(\log L)^{m-j}}f(x)\Big\}Mg(x)dx\\ &\leqslant C\prod_{j=1}^m\|b\|_*\int_{\mathbb{R}^n}\sum_{j=1}^m M_{L(\log L)^{m-j}}f(x)Mg(x)dx\\ &\leqslant C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}.\end{aligned} \tag{8}$$

Combining (8) and (7), we get that

$$\|\mu_{\Omega,\vec{b}}f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant \|\mu_{\Omega,\vec{b}}f\|^0_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

By Lemma 6, this concludes the proof of Theorem 1. □

4. Boundedness on variable exponent Herz-Morrey spaces

Let $B_k = \{x \in \mathbb{R}^n : |x| \leqslant 2^k\}$, $R_k = B_k \backslash B_{k-1}$ and $\chi_k = \chi_{R_k}$ be the characteristic function of the set R_k for $k \in \mathbb{Z}$.

DEFINITION 1. Let $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, $0 < q \leqslant \infty$ and $\alpha \in \mathbb{R}$. The homogeneous variable exponent Herz space $\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$ consists of all $f \in L^{p(\cdot)}_{loc}(\mathbb{R}^n \backslash \{0\})$ satisfying

$$\|f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} := \left(\sum_{k\in\mathbb{Z}} 2^{\alpha kq}\|f\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{1/q} < \infty,$$

with the usual modification when $q = \infty$.

DEFINITION 2. Let $\lambda \geqslant 0$, $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, $0 < q \leqslant \infty$ and $\alpha \in \mathbb{R}$. The homogeneous variable exponent Herz-Morrey space $M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ consists of all $f \in L^{p(\cdot)}_{loc}(\mathbb{R}^n \backslash \{0\})$ satisfying

$$\|f\|_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} := \sup_{L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L} 2^{\alpha kq}\|f\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{1/q} < \infty,$$

with the usual modification when $q = \infty$.

REMARK 4. It obviously follows that $M\dot{K}^{\alpha,0}_{q,p(\cdot)}(\mathbb{R}^n) = \dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$ and $\dot{K}^{0,q}_{p(\cdot)}(\mathbb{R}^n) = L^{p(\cdot)}(\mathbb{R}^n)$. If $p(\cdot)$ is constant, then $\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$ and $M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ coincide with the classical Herz and Herz-Morrey spaces, respectively (see [30, 32]).

Now we present the main results of this section.

THEOREM 2. *Suppose* $\Omega \in Lip_\gamma(\mathbb{S}^{n-1})$ $(0 < \gamma \leqslant 1)$ *and let* $\vec{b} = (b_1, b_2, \ldots, b_m)$, $b_i \in BMO(\mathbb{R}^n)$, $i = 1, 2, \ldots, m$. *If* $\lambda > 0$, $p(\cdot) \in \mathscr{P}(\mathbb{R}^n) \bigcap LH(\mathbb{R}^n)$, $0 < q < \infty$ *and* $\lambda - n\delta_1 < \alpha < n\delta_2$, *where* $0 < \delta_1, \delta_2 < 1$ *are the constants appearing in Lemma 2. Then the multilinear commutators* $\mu_{\Omega,\vec{b}}$ *are bounded on* $M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$.

In fact, Theorem 2 remains valid also in the particular case $\lambda = 0$, namely, in the framework of variable exponent Herz spaces. More precisely, we have

THEOREM 3. *Suppose* $\Omega \in Lip_\gamma(\mathbb{S}^{n-1})$ $(0 < \gamma \leqslant 1)$ *and let* $p(\cdot)$, q, $\vec{b}$ *be as in Theorem 2. If* $-n\delta_1 < \alpha < n\delta_2$, *where* $0 < \delta_1, \delta_2 < 1$ *are the constants appearing in Lemma 2. Then the multilinear commutators* $\mu_{\Omega,\vec{b}}$ *are bounded on* $\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$.

REMARK 5. We note that Theorem 3 generalizes the corresponding result in [46, Page 178] to the case of multilinear operators. Since its proof is essentially a repetition of the proof of Theorem 2, we will omit the details.

Proof of Theorem 2. Let $f \in M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$. Write

$$f(x) = \sum_{j=-\infty}^{\infty} f(x)\chi_j(x) = \sum_{j=-\infty}^{\infty} f_j(x).$$

Then, we have

$$\begin{aligned}
\|\mu_{\Omega,\vec{b}}(f)\|^q_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} &= \sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{k\alpha q} \|\mu_{\Omega,\vec{b}}(f)\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\leqslant C \sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{k\alpha q} \Big(\sum_{j=-\infty}^{k-3} \|\mu_{\Omega,\vec{b}}(f_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \Big)^q \\
&\quad + C \sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{k\alpha q} \Big(\sum_{j=k-2}^{k+2} \|\mu_{\Omega,\vec{b}}(f_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \Big)^q \\
&\quad + C \sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{k\alpha q} \Big(\sum_{j=k+3}^{\infty} \|\mu_{\Omega,\vec{b}}(f_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \Big)^q \\
&:= V_1 + V_2 + V_3.
\end{aligned}$$

To estimate V_1, note that if $x \in R_k$, $y \in R_j$ and $j \leqslant k-3$, then $|x-y| \sim |x|$. Since $\Omega \in \mathrm{Lip}_\gamma(\mathbb{S}^{n-1}) \subset L^\infty(\mathbb{S}^{n-1})$, by the Minkowski inequality, we have

$$\begin{aligned}
&\left(\int_0^{|x|} \left| \int_{|x-y|\leqslant t} \prod_{i=1}^m (b_i(x)-b_i(y)) \frac{\Omega(x-y)}{|x-y|^{n-1}} f_j(y)dy \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}} \\
&\leqslant \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} \int_{R_j} \frac{|f_j(y)| \prod_{i=1}^m |b_i(x)-b_i(y)|}{|x-y|^{n-1}} \left(\int_{|x-y|\leqslant t, |x|\geqslant t} \frac{dt}{t^3} \right)^{\frac{1}{2}} dy \\
&\leqslant C \int_{R_j} \frac{|f_j(y)| \prod_{i=1}^m |b_i(x)-b_i(y)|}{|x-y|^{n-1}} \frac{|y|^{\frac{1}{2}}}{|x-y|^{\frac{3}{2}}} dy \\
&\leqslant C 2^{-kn} \int_{R_j} |f_j(y)| \prod_{i=1}^m |b_i(x)-b_i(y)| dy.
\end{aligned} \tag{9}$$

Similarly, we derive the estimate

$$\begin{aligned}
&\left(\int_{|x|}^{\infty} \left| \int_{|x-y|\leqslant t} \prod_{i=1}^m (b_i(x)-b_i(y)) \frac{\Omega(x-y)}{|x-y|^{n-1}} f_j(y)dy \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}} \\
&\leqslant C \int_{R_j} \frac{|f_j(y)| \prod_{i=1}^m |b_i(x)-b_i(y)|}{|x-y|^{n-1}} \left(\int_{|x|}^{\infty} \frac{dt}{t^3} \right)^{\frac{1}{2}} dy
\end{aligned} \tag{10}$$

$$\leqslant C\int_{R_j}\frac{|f_j(y)|\prod_{i=1}^{m}|b_i(x)-b_i(y)|}{|x-y|^{n-1}}\frac{1}{|x|}dy$$
$$\leqslant C2^{-kn}\int_{R_j}|f_j(y)|\prod_{i=1}^{m}|b_i(x)-b_i(y)|dy.$$

Let $\vec{\lambda}=(\lambda_1,\lambda_2,\ldots,\lambda_m)$, $\lambda_i=(b_i)_{B_j}$, $i=1,2,\ldots,m$. From (9), (10) and the generalized Hölder inequality (6), it follows that

$$\begin{aligned}
|\mu_{\Omega,\vec{b}}(f_j)(x)| &\leqslant \left(\int_0^{|x|}\left|\int_{|x-y|\leqslant t}\prod_{i=1}^{m}(b_i(x)-b_i(y))\frac{\Omega(x-y)}{|x-y|^{n-1}}f_j(y)dy\right|^2\frac{dt}{t^3}\right)^{\frac{1}{2}}\\
&\quad+\left(\int_{|x|}^{\infty}\left|\int_{|x-y|\leqslant t}\prod_{i=1}^{m}(b_i(x)-b_i(y))\frac{\Omega(x-y)}{|x-y|^{n-1}}f_j(y)dy\right|^2\frac{dt}{t^3}\right)^{\frac{1}{2}}\\
&\leqslant C2^{-kn}\int_{R_j}\prod_{i=1}^{m}|b_i(x)-b_i(y)||f_j(y)|dy\\
&\leqslant C2^{-kn}\sum_{i=0}^{m}\sum_{\sigma\in C_i^m}|(b(x)-\vec{\lambda})_\sigma|\int_{R_j}|(b(y)-\vec{\lambda})_{\sigma'}|\cdot|f_j(y)|dy\\
&\leqslant C2^{-kn}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\sum_{i=0}^{m}\sum_{\sigma\in C_i^m}|(b(x)-\vec{\lambda})_\sigma|\|(b(\cdot)-\vec{\lambda})_{\sigma'}\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Gathering this, Lemmas 1, 2 and 5, we get that

$$\begin{aligned}
&\|(\mu_{\Omega,\vec{b}}f_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\leqslant C2^{-kn}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\sum_{i=0}^{m}\sum_{\sigma\in C_i^m}\|(b(\cdot)-\vec{\lambda})_\sigma\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|(b(\cdot)-\vec{\lambda})_{\sigma'}\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\\
&\leqslant C(k-j)^m\prod_{i=1}^{m}\|b_i\|_*2^{-kn}\|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\leqslant C(k-j)^m\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_k}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}\\
&\leqslant C(k-j)^m2^{(j-k)n\delta_2}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Thus we arrive at

$$V_1\leqslant C\sup_{L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}\left(\sum_{j=-\infty}^{k-3}2^{j\alpha}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}(k-j)^m2^{(k-j)(\alpha-n\delta_2)}\right)^q.$$

If $0<q\leqslant 1$, applying the well-known inequality

$$\left(\sum_{i=1}^{\infty}a_i\right)^q\leqslant\sum_{i=1}^{\infty}a_i^q,\quad a_i>0,\quad i=1,2,\ldots,\tag{11}$$

we deduce that

$$\begin{aligned} V_1 &\leqslant C\sup_{L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L}\sum_{j=-\infty}^{k-3} 2^{j\alpha q}(k-j)^{mq}2^{(k-j)(\alpha-n\delta_2)q}\|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\leqslant C\sup_{L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{j=-\infty}^{L-3} 2^{j\alpha q}\|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\sum_{k=j+3}^{L}(k-j)^{mq}2^{(k-j)(\alpha-n\delta_2)q} \\ &\leqslant C\|f\|^q_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

If $1<q<\infty$, since $\alpha-n\delta_2<0$, Hölder's inequality implies that

$$\begin{aligned} V_1 &\leqslant C\sup_{L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L}\Big(\sum_{j=-\infty}^{k-3} 2^{j\alpha q}\|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}2^{(k-j)(\alpha-n\delta_2)q/2}\Big) \\ &\quad\times\Big(\sum_{j=-\infty}^{k-3}(k-j)^{mq'}2^{(k-j)(\alpha-n\delta_2)q'/2}\Big)^{\frac{q}{q'}} \\ &\leqslant C\sup_{L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L}\Big(\sum_{j=-\infty}^{k-3} 2^{j\alpha q}\|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}2^{(k-j)(\alpha-n\delta_2)q/2}\Big) \\ &\leqslant C\sup_{L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{j=-\infty}^{L-3} 2^{j\alpha q}\|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\sum_{k=j+3}^{L}2^{(k-j)(\alpha-n\delta_2)q/2} \\ &\leqslant C\sup_{L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{j=-\infty}^{L-3} 2^{j\alpha q}\|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\leqslant C\|f\|^q_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

For V_2, by the boundedness of $\mu_{\Omega,\vec{b}}$ on $L^{p(\cdot)}(\mathbb{R}^n)$, we get immediately that

$$V_2\leqslant C\sup_{L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{k\alpha q}\|f_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\leqslant C\|f\|^q_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.$$

For V_3, similar to the estimate of V_1, we have

$$\begin{aligned} |\mu_{\Omega,\vec{b}}(f_j)(x)| &\leqslant \left(\int_0^{|y|}\left|\int_{|x-y|\leqslant t}\prod_{i=1}^{m}(b_i(x)-b_i(y))\frac{\Omega(x-y)}{|x-y|^{n-1}}f_j(y)dy\right|^2\frac{dt}{t^3}\right)^{\frac{1}{2}} \\ &\quad+\left(\int_{|y|}^{\infty}\left|\int_{|x-y|\leqslant t}\prod_{i=1}^{m}(b_i(x)-b_i(y))\frac{\Omega(x-y)}{|x-y|^{n-1}}f_j(y)dy\right|^2\frac{dt}{t^3}\right)^{\frac{1}{2}}. \end{aligned}$$

Note that if $x\in R_k$, $y\in R_j$ and $j\geqslant k+3$, then $|x-y|\sim|y|$. As argued before, by the Minkowski inequality, we obtain

$$\begin{aligned}
&\left(\int_0^{|y|}\left|\int_{|x-y|\leqslant t}\prod_{i=1}^m(b_i(x)-b_i(y))\frac{\Omega(x-y)}{|x-y|^{n-1}}f_j(y)dy\right|^2\frac{dt}{t^3}\right)^{\frac12}\\
&\leqslant C\int_{R_j}\frac{|f_j(y)|\prod_{i=1}^m|b_i(x)-b_i(y)|}{|x-y|^{n-1}}\left(\int_{|x-y|\leqslant t,|y|\geqslant t}\frac{dt}{t^3}\right)^{\frac12}dy\\
&\leqslant C\int_{R_j}\frac{|f_j(y)|\prod_{i=1}^m|b_i(x)-b_i(y)|}{|x-y|^{n-1}}\frac{|x|^{\frac12}}{|x-y|^{\frac32}}dy\\
&\leqslant C2^{-jn}\int_{R_j}|f_j(y)|\prod_{i=1}^m|b_i(x)-b_i(y)|dy.
\end{aligned}\tag{12}$$

Similarly, we have

$$\begin{aligned}
&\left(\int_{|y|}^{\infty}\left|\int_{|x-y|\leqslant t}\prod_{i=1}^m(b_i(x)-b_i(y))\frac{\Omega(x-y)}{|x-y|^{n-1}}f_j(y)dy\right|^2\frac{dt}{t^3}\right)^{\frac12}\\
&\leqslant C\int_{R_j}\frac{|f_j(y)|\prod_{i=1}^m|b_i(x)-b_i(y)|}{|x-y|^{n-1}}\left(\int_{|y|}^{\infty}\frac{dt}{t^3}\right)^{\frac12}dy\\
&\leqslant C\int_{R_j}\frac{|f_j(y)|\prod_{i=1}^m|b_i(x)-b_i(y)|}{|x-y|^{n-1}}\frac{1}{|y|}dy\\
&\leqslant C2^{-jn}\int_{R_j}|f_j(y)|\prod_{i=1}^m|b_i(x)-b_i(y)|dy.
\end{aligned}\tag{13}$$

Let $\vec{\lambda}=(\lambda_1,\lambda_2,\ldots,\lambda_m)$, $\lambda_i=(b_i)_{B_k}$, $i=1,2,\ldots,m$. Combining (12), (13) and the generalized Hölder inequality (6), we conclude that

$$\begin{aligned}
|\mu_{\Omega,\vec{b}}f_j(x)|&\leqslant C2^{-jn}\int_{R_j}\prod_{i=1}^m|b_i(x)-b_i(y)||f_j(y)|dy\\
&\leqslant C2^{-jn}\sum_{i=0}^m\sum_{\sigma\in C_i^m}|(b(x)-\vec{\lambda})_\sigma|\int_{R_j}|(b(y)-\vec{\lambda})_{\sigma'}|\cdot|f_j(y)|dy\\
&\leqslant C2^{-jn}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\sum_{i=0}^m\sum_{\sigma\in C_i^m}|(b(x)-\vec{\lambda})_\sigma|\|(b(\cdot)-\vec{\lambda})_{\sigma'}\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

This together with Lemmas 1, 2 and 5 gives

$$\begin{aligned}
&\|(\mu_{\Omega,\vec{b}}f_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\leqslant C2^{-jn}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\sum_{i=0}^m\sum_{\sigma\in C_i^m}\|(b(\cdot)-\vec{\lambda})_\sigma\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|(b(\cdot)-\vec{\lambda})_{\sigma'}\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\\
&\leqslant C(j-k)^m\prod_{i=1}^m\|b_i\|_*2^{-jn}\|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}
\end{aligned}$$

$$\leqslant C(j-k)^m \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \frac{\|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}$$
$$\leqslant C(j-k)^m 2^{(k-j)n\delta_1} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Thus we get

$$\begin{aligned}
V_3 &\leqslant C \sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{k\alpha q} \Big(\sum_{j=k+3}^{\infty} (j-k)^m 2^{(k-j)n\delta_1} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \Big)^q \\
&\leqslant C \sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} \Big(\sum_{j=k+3}^{L} 2^{j\alpha} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} (j-k)^m 2^{(k-j)(n\delta_1+\alpha)} \Big)^q \\
&\quad + C \sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} \Big(\sum_{j=L+1}^{\infty} 2^{j\alpha} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} (j-k)^m 2^{(k-j)(n\delta_1+\alpha)} \Big)^q \\
&:= V_{31} + V_{32}.
\end{aligned}$$

If $1 < q < \infty$, since $\alpha + n\delta_1 > 0$, by Hölder's inequality, we obtain

$$\begin{aligned}
V_{31} &\leqslant C \sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} \Big(\sum_{j=k+3}^{L} 2^{j\alpha q} \|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} 2^{(k-j)(\alpha+n\delta_1)q/2} \Big) \\
&\quad \times \Big(\sum_{j=k+3}^{L} (j-k)^{mq'} 2^{(k-j)(\alpha+n\delta_1)q'/2} \Big)^{\frac{q}{q'}} \\
&\leqslant C \sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} \Big(\sum_{j=k+3}^{L} 2^{j\alpha q} \|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} 2^{(k-j)(\alpha+n\delta_1)q/2} \Big) \\
&\leqslant C \sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{L} 2^{j\alpha q} \|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \sum_{k=-\infty}^{j-3} 2^{(k-j)(\alpha+n\delta_1)q/2} \\
&\leqslant C \|f\|^q_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

For V_{32}, in view of $\alpha + n\delta_1 - \lambda > 0$, we derive that

$$\begin{aligned}
V_{32} &\leqslant C \sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} \Big(\sum_{j=L+1}^{\infty} 2^{j\alpha q} \|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} 2^{(k-j)(\alpha+n\delta_1+\lambda)q/2} \Big) \\
&\quad \times \Big(\sum_{j=L+1}^{\infty} (j-k)^{mq'} 2^{(k-j)(\alpha+n\delta_1-\lambda)q'/2} \Big)^{\frac{q}{q'}} \\
&\leqslant C \sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} \sum_{j=L+1}^{\infty} 2^{(k-j)(\alpha+n\delta_1+\lambda)q/2} 2^{j\lambda q} 2^{-j\lambda q} \Big(\sum_{\ell=-\infty}^{j} 2^{\ell\alpha q} \|f_\ell\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \Big) \\
&\leqslant C \|f\|^q_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} \sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{k\lambda q} \sum_{j=L+1}^{\infty} 2^{(k-j)(\alpha+n\delta_1-\lambda)q/2} \\
&\leqslant C \|f\|^q_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

If $0 < q \leqslant 1$, by inequality (11), we have

$$
\begin{aligned}
V_{31} \leqslant & C \sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{k\alpha q} \sum_{j=k+3}^{L} (j-k)^{mq} 2^{(k-j)n\delta_1 q} \|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \\
\leqslant & C \sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{L} 2^{j\alpha q} \|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \sum_{k=-\infty}^{j-3} (j-k)^{mq} 2^{(k-j)(\alpha+n\delta_1)q} \\
\leqslant & C \|f\|^q_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

For V_{32}, we have

$$
\begin{aligned}
V_{32} \leqslant & C \sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{k\alpha q} \sum_{j=L+1}^{\infty} (j-k)^{mq} 2^{(k-j)n\delta_1 q} \|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \\
\leqslant & C \sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{k\alpha q} \sum_{j=L+1}^{\infty} (j-k)^{mq} 2^{(k-j)n\delta_1 q} 2^{-j\alpha q} 2^{j\lambda q} \\
& \times 2^{-j\lambda q} \Big(\sum_{\ell=-\infty}^{j} 2^{\ell\alpha q} \|f_\ell\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \Big) \\
\leqslant & C \|f\|^q_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} \sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{k\lambda q} \sum_{j=L+1}^{\infty} (j-k)^{mq} 2^{(k-j)(\alpha+n\delta_1-\lambda)q} \\
\leqslant & C \|f\|^q_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

This completes the proof of Theorem 2. □

5. Boundedness on variable exponent Herz-type Hardy spaces

Note that in Theorem 3 for the range of α, we have the restriction $\alpha < n\delta_2$. It is natural to ask what will happen if $\alpha \geqslant n\delta_2$. The main purpose of this section is to further study the mapping properties of the multilinear commutators $\mu_{\Omega,\vec{b}}$ in this situation. Before stating the main result, we give some definitions.

DEFINITION 3. Suppose $\alpha \geqslant n\delta_2$, $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ and non-negative integer $s \geqslant [\alpha - n\delta_2]$. Let $b_i (i = 1, 2, \ldots, m)$ be alocally integrable function and $\vec{b} = (b_1, b_2, \ldots, b_m)$. A function $a(x)$ on $\mathbb{R}^n$ is said to be a central $(\alpha, p(\cdot), s; \vec{b})$-atom, if it satisfies

(i) $\operatorname{supp} a \subset B(0,r) := \{x \in \mathbb{R}^n : |x| < r\}$.

(ii) $\|a\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant |B(0,r)|^{-\frac{\alpha}{n}}$.

(iii) $\displaystyle\int_{\mathbb{R}^n} x^\beta a(x) \prod_{i\in\sigma} b_i(x) dx = 0, \quad for\ |\beta| \leqslant s, \sigma \in C_j^m, j = 0, 1, \ldots, m.$

REMARK 6. It is easy to see that if $p(x) \equiv p$ is constant, then taking $\delta_2 = 1 - \frac{1}{p}$ we can get the classical case, see [31].

A temperate distribution f is said to belong to $H\dot{K}^{\alpha,q,s}_{p(\cdot),\vec{b}}(\mathbb{R}^n)$, if it can be written as

$$f=\sum_{j=-\infty}^{\infty}\lambda_j a_j, \text{ in the sense of } \mathscr{S}'(\mathbb{R}^n),$$

where a_j is a central $(\alpha,p(\cdot),s;\vec{b})$-atom with support contained in B_j, $\lambda_j\in\mathbb{R}$ and $\sum|\lambda_j|^q<\infty$. Moreover,

$$\|f\|_{H\dot{K}^{\alpha,q,s}_{p(\cdot),\vec{b}}(\mathbb{R}^n)}\approx\inf\left(\sum_{j=-\infty}^{\infty}|\lambda_j|^q\right)^{\frac{1}{q}},$$

where the infimum is taken over all above decompositions of f.

Our main result in this section can be stated as follows.

THEOREM 4. *Suppose $\Omega\in Lip_\gamma(\mathbb{S}^{n-1})$ $(0<\gamma\leqslant 1)$ and let $\vec{b}=(b_1,b_2,\dots,b_m)$, $b_i\in BMO(\mathbb{R}^n)$, $i=1,2,\dots,m$. If $p(\cdot)\in\mathscr{P}(\mathbb{R}^n)\bigcap LH(\mathbb{R}^n)$, $0<q<\infty$, $0<\varepsilon\leqslant 1$ and $n\delta_2\leqslant\alpha<n\delta_2+\varepsilon$, where δ_2 is the constant appearing in Lemma 2. Then the multilinear commutators $\mu_{\Omega,\vec{b}}$ map $H\dot{K}^{\alpha,q,0}_{p(\cdot),\vec{b}}(\mathbb{R}^n)$ into $\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$.*

Proof of Theorem 4. Let a_j be a central $(\alpha,p(\cdot),0;\vec{b})$-atom with support contained in B_j. We first restrict $0<q\leqslant 1$. In this case, it suffices to show $\|\mu_{\Omega,\vec{b}}a_j\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)}\leqslant C$. We write

$$\begin{aligned}\|\mu_{\Omega,\vec{b}}a_j\|^q_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)}&=\sum_{k=-\infty}^{j+2}2^{k\alpha q}\|\chi_k\mu_{\Omega,\vec{b}}a_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}+\sum_{k=j+3}^{\infty}2^{k\alpha q}\|\chi_k\mu_{\Omega,\vec{b}}a_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\\&:=I+J.\end{aligned}$$

We treat I first. Using the $L^{p(\cdot)}(\mathbb{R}^n)$-boundedness of $\mu_{\Omega,\vec{b}}$, we have

$$I\leqslant C\sum_{k=-\infty}^{j+2}2^{k\alpha q}\|a_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\leqslant C\sum_{k=-\infty}^{j+2}2^{(k-j)\alpha q}\leqslant C.$$

To evaluate J, let $\lambda_i=(b_i)_{B_j}$. If $x\in R_k$, $y\in B_j$ and $k\geqslant j+3$, then $2|y|<|x|$. By the Minkowski inequality and the vanishing condition of a_j, we deduce that

$$\begin{aligned}\mu_{\Omega,\vec{b}}(a_j)(x)&=\left(\int_0^\infty\left|\int_{|x-y|\leqslant t}\prod_{i=1}^m(b_i(x)-b_i(y))\frac{\Omega(x-y)}{|x-y|^{n-1}}a_j(y)dy\right|^2\frac{dt}{t^3}\right)^{\frac{1}{2}}\\&\leqslant\int_{B_j}\left(\int_{|x-y|\leqslant t}\left|\prod_{i=1}^m(b_i(x)-b_i(y))\frac{\Omega(x-y)}{|x-y|^{n-1}}a_j(y)\right|^2\frac{dt}{t^3}\right)^{\frac{1}{2}}dy\\&\leqslant C\int_{B_j}\left(\int_{|x-y|\leqslant t}\left|\prod_{i=1}^m(b_i(x)-b_i(y))\left(\frac{1}{|x-y|^{n-1}}-\frac{1}{|x|^{n-1}}\right)a_j(y)\right|^2\frac{dt}{t^3}\right)^{\frac{1}{2}}dy\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\int_{B_j}\prod_{i=1}^{m}|b_i(x)-b_i(y)||a_j(y)|\frac{|y|}{|x|^{n+1}}dy\\
&\leqslant C2^{(j-k)-kn}\int_{B_j}\prod_{i=1}^{m}|b_i(x)-b_i(y)||a_j(y)|dy.
\end{aligned}
$$

Thus, by the generalized Hölder inequality (6), Lemmas 1, 2 and 5, we have

$$
\begin{aligned}
&\|\chi_k\mu_{\Omega,\vec{b}}a_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\leqslant C2^{(j-k)-kn}\|a_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\sum_{i=0}^{m}\sum_{\sigma\in C_i^m}\|(b(\cdot)-\vec{\lambda})_\sigma\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|(b(\cdot)-\vec{\lambda})_{\sigma'}\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\\
&\leqslant C(k-j)^m\prod_{i=1}^{m}\|b_i\|_*2^{(j-k)-j\alpha}2^{-kn}\|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\\
&\leqslant C(k-j)^m2^{(j-k)-j\alpha}\frac{\|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}\\
&\leqslant C(k-j)^m2^{(j-k)(1+n\delta_2)-j\alpha}.
\end{aligned}
$$

Thus, in view of $1+n\delta_2-\alpha\geqslant\varepsilon+n\delta_2-\alpha>0$, we have

$$
J\leqslant C\sum_{k=j+3}^{\infty}(k-j)^{mq}2^{(j-k)(1+n\delta_2-\alpha)q}\leqslant C.
$$

Now let $1<q<\infty$ and $f=\sum_{j=-\infty}^{\infty}\lambda_j a_j$. For convenience below we put $\eta=1+n\delta_2-\alpha$, then we have $\eta>0$. By the Minkowski inequality, the Hölder inequality and the $L^{p(\cdot)}(\mathbb{R}^n)$-boundedness of the multilinear commutator $\mu_{\Omega,\vec{b}}$, we obtain

$$
\begin{aligned}
&\|\mu_{\Omega,\vec{b}}f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)}\\
&\leqslant\left\{\sum_{k=-\infty}^{\infty}2^{\alpha kq}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|\|\chi_k\mu_{\Omega,\vec{b}}a_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^q\right\}^{\frac{1}{q}}\\
&\quad+\left\{\sum_{k=-\infty}^{\infty}2^{\alpha kq}\left(\sum_{j=k-2}^{\infty}|\lambda_j|\|\chi_k\mu_{\Omega,\vec{b}}a_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^q\right\}^{\frac{1}{q}}\\
&\leqslant C\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|(k-j)^m2^{(j-k)\eta}\right)^q\right\}^{\frac{1}{q}}+C\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=k-2}^{\infty}|\lambda_j|2^{(k-j)\alpha}\right)^q\right\}^{\frac{1}{q}}\\
&\leqslant C\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|^q2^{(j-k)\eta q/2}\right)\left(\sum_{j=-\infty}^{k-3}(k-j)^{mq'}2^{(j-k)\eta q'/2}\right)^{q/q'}\right\}^{\frac{1}{q}}\\
&\quad+C\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=k-2}^{\infty}|\lambda_j|^q2^{(k-j)\alpha q/2}\right)\left(\sum_{j=k-2}^{\infty}2^{(j-k)\alpha q'/2}\right)^{q/q'}\right\}^{\frac{1}{q}}
\end{aligned}
$$

$$\leqslant C\Big\{\sum_{j=-\infty}^{\infty}|\lambda_j|^q\Big(\sum_{k=j+3}^{\infty}2^{(j-k)\eta q/2}\Big)\Big\}^{\frac{1}{q}}+C\Big\{\sum_{j=-\infty}^{\infty}|\lambda_j|^q\Big(\sum_{k=-\infty}^{j+2}2^{(k-j)\alpha q/2}\Big)\Big\}^{\frac{1}{q}}$$

$$\leqslant C\Big(\sum_{j=-\infty}^{\infty}|\lambda_j|^q\Big)^{\frac{1}{q}}.$$

This completes the proof of Theorem 4. □

Acknowledgement. The authors would like to express their deep gratitude to the referee for giving many valuable comments and suggestions.

REFERENCES

[1] A. ALMEIDA AND D. DRIHEM, *Maximal, potential and singular type operators on Herz spaces with variable exponents*, J. Math. Anal. Appl. **394** (2012), 781–795.

[2] A. BENEDEK, A. CALDERÓN AND R. PANZONE, *Convolution operators on Banach space valued functions*, Proc. Nat. Acad. Sci. USA, **48** (1962), 356–365.

[3] Y. CHEN, S. LEVINE AND R. RAO, *Variable exponent, linear growth functionals in image restoration*, SIAM J. Appl. Math. **66** (2006), 1383–1406.

[4] R. COIFMAN, R. ROCHBERG AND G. WEISS, *Factorization theorems for Hardy spaces in several variables*, Ann. Math. **103** (1976), 611–635.

[5] D. CRUZ-URIBE AND A. FIORENZA, *Variable Lebesgue Spaces: Foundations and Harmonic Analysis, Applied and Numerical Harmonic Analysis*, Birkhäuser, Basel, 2013.

[6] D. CRUZ-URIBE, A. FIORENZA AND C. NEUGEBAUER, *The maximal function on variable L^p spaces*, Ann. Acad. Sci. Fenn. Math. **28** (2003), 223–238.

[7] L. DIENING, *Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$*, Math. Inequal. Appl. **7** (2004), 245–253.

[8] L. DIENING, P. HARJULEHTO, P. HÄSTÖ AND M. RŮŽIČKA, *Lebesgue and sobolev spaces with variable exponents*, volume 2017 of Lecture Notes in Mathematics, Springer, Heidelberg, 2011.

[9] Y. DING, D. FAN AND Y. PAN, *L^p-boundedness of Marcinkiewicz integrals with Hardy space function kernel*, Acta. Math. Sin. (Engl. Ser.), **16** (2000), 593–600.

[10] Y. DING, S. LU AND Q. XUE, *Marcinkiewicz integral on Hardy spaces*, Integr. equ. oper. theory **42** (2002), 174–182.

[11] Y. DING, S. LU AND P. ZHANG, *Weighted weak type estimates for commutators of the Marcinkiewicz integrals*, Sci. China Ser. A. **47** (2004), 83–95.

[12] D. DRIHEM AND F. SEGHIRI, *Notes on the Herz-type Hardy spaces of variable smoothness and integrability*, Math. Inequal. Appl. **19** (2016), 145–165.

[13] J. GARCIA-CUERVA AND J. L. RUBIO DE FRANCIA, *Weighted norm inequalities and related topics*, North-Holland, Amsterdam, 1985.

[14] L. GRAFAKOS AND R. H. TORRES, *Multilinear Calderón-Zygmund theory*, Adv. Math. **165** (2002), 124–164.

[15] P. HARJULEHTO, P. HÄSTÖ, Ú. V. LÊ AND M. NUORTIO, *Overview of differential equations with non-standard growth*, Nonlinear Anal. **72** (2010), 4551–4574.

[16] M. IZUKI, *Herz and amalgam spaces with variable exponent, the Haar wavelets and greediness of the wavelet system*, East J. Approx. **15** (2009), 87–109.

[17] M. IZUKI, *Boundedness of commutators on Herz spaces with variable exponent*, Rend. Circ. Mat. Palermo. **59** (2010), 199–213.

[18] M. IZUKI, *Commutators of fractional integrals on Lebesgue and Herz spaces with variable exponent*, Rend. Circ. Mat. Palermo. **59** (2010), 461–472.

[19] M. IZUKI, *Fractional integrals on Herz-Morrey spaces with variable exponent*, Hiroshima Math. J. **40** (2010), 343–355.

[20] A. KARLOVICH AND A. LERNER, *Commutators of singular integrals on generalized L^p spaces with variable exponent*, Publ. Nat. **49** (2005), 111–125.

[21] O. KOVÁČIK AND J. RÁKOSNÍK, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czechoslovak Math. J. **41** (1991), 592–618.

[22] A. LERNER, *Weighted norm inequalities for the local sharp maximal function*, J. Fourier Anal. Appl. **10** (2004), 465–474.

[23] F. LIU, *Integral operators of Marcinkiewicz type on Triebel-Lizorkin spaces*, Math. Nachr. **290** (2017), 75–96.

[24] F. LIU, *On the Triebel-Lizorkin space boundedness of Marcinkiewicz integrals along compound surfaces*, Math. Inequal. Appl. **20** (2017), 515–535.

[25] F. LIU, *A note on Marcinkiewicz integrals associated to surfaces of revolution*, J. Aust. Math. Soc. **104** (2018), 380–402.

[26] F. LIU, Z. FU, Y. ZHENG AND Q. YUAN, *A rough Marcinkiewicz integral along smooth curves*, J. Nonl. Sci. Appl., **9** (2016), 4450–4464.

[27] F. LIU AND H. WU, *L^p bounds for Marcinkiewicz integrals associated to homogeneous mappings*, Monatsh. Math. **181** (2016), 875–906.

[28] F. LIU, H. WU AND D. ZHANG, *L^p bounds for parametric Marcinkiewicz integrals with mixed homogeneity*, Math. Inequal. Appl. **18** (2015), 453–469.

[29] F. LIU AND D. ZHANG, *Parametric Marcinkiewicz integrals associated to polynomials compound curves and extrapolation*, Bull. Korean Math. Soc. **52** (2015), 771–788.

[30] S. LU AND L. XU, *Boundedness of rough singular intergral operators on the homogeneous Morrey-Herz spaces*, Hokkaido Math. J. **34** (2005), 299–314.

[31] S. LU AND D. YANG, *The continuity of commutators on Herz-type spaces*, Michigan Math. J. **44** (1997), 255–281.

[32] S. LU, D. YANG AND G. HU, *Herz type spaces and their applications*, Science Press, Beijing, 2008.

[33] Y. LU AND Y. ZHU, *Boundedness of multilinear Calderón-Zygmund singular operators on Morrey-Herz spaces with variable exponents*, Acta Math. Sin. (Engl. Ser.), **30** (2014), 1180–1194.

[34] Y. MEYER AND R. R. COIFMAN, *Wavelets: Calderón-Zygmund and multilinear Operators*, Cambridge Stud. Adv. Math., Cambridge University Press, 2000.

[35] A. NEKVINDA, *Hardy-littlewood maximal operator in $L^{p(x)}(\mathbb{R}^n)$*, Math. Ineq. Appl. **7** (2004), 255–265.

[36] C. PÉREZ, *Endpoint estimates for commutators of singular integrals*, J. Funct. Anal. **128** (1995), 163–185.

[37] C. PÉREZ AND R. TRUJILLO-GONEÁLEZ, *Sharp weighted estimates for multilinear commutators*, J. London Math. Soc. **65** (2002), 672–692.

[38] M. RAGUSA, *Homogeneous Herz spaces and regularity results*, Nonlinear Anal. **71** (2009), 1909–1914.

[39] M. RŮŽIČKA, *Electrorheological fluids: modeling and mathematical theory*, Springer-Verlag, Berlin, 2000.

[40] S. SAMKO, *Variable exponent Herz spaces*, Mediterr. J. Math. **10** (2013), 2007–2025.

[41] E. M. STEIN, *On the functions of Littlewood-Paley, Lusin and Marcinkiewicz*, Trans. Amer. Math. Soc. **88** (1958), 430–466.

[42] E. M. STEIN, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, Princeton, 1993.

[43] A. TORCHINSKY AND S. WANG, *A note on the Marcinkiewicz integral*, Colloq. Math. **60/61** (1990), 235–243.

[44] H. WANG, Z. FU AND Z. LIU, *Higher order commutators of Marcinkiewicz integrals on variable Lebesgue spaces*, Acta Math. Sci. Ser. A Chin. Ed. **32** (2012), 1092–1101.

[45] H. WANG AND Z. LIU, *The Herz-type Hardy spaces with variable exponent and thier applications*, Taiwanese. J. Math. **16** (2012), 1363–1389.

[46] L. WANG AND L. SHU, *Higher order commutators of Marcinkiewicz integral operator on Herz-Morrey spaces with variable exponent*, J. Math. Res. Appl. **34** (2014), 175–186.

[47] J. XU, *The boundedness of multilinear commutators of singular integrals on Lebesgue spaces with variable exponent*, Czechoslovak Math. J. **57** (2007), 13–27.

[48] X. YAN, D. YANG, W. YUAN AND C. ZHUO, *Variable weak Hardy spaces and their applications*, J. Funct. Anal. **271** (2016), 2822–2887.

[49] D. YANG, C. ZHUO AND W. YUAN, *Triebel-Lizorkin type spaces with variable exponents*, Banach J. Math. Anal. **9** (2015), 146–202.

[50] D. YANG, C. ZHUO AND W. YUAN, *Besov-type spaces with variable smoothness and integrability*, J. Funct. Anal. **269** (2015), 1840–1898.

[51] P. ZHANG, *Weighted estimates for multilinear commutators of Marcinkiewicz integral*, Acta Math. Sin. (Engl. Ser.), **24** (2008), 1387–1400.

[52] P. ZHANG AND J. WU, *Commutators for the maximal function on Lebesgue spaces with variable exponent*, Math. Inequal. Appl. **17** (2014), 1375–1386.

(Received May 11, 2017)

Liwei Wang
School of Mathematics and Physics
Anhui Polytechnic University
Wuhu 241000, The People's Republic of China
e-mail: wangliwei@ahpu.edu.cn

Lisheng Shu
School of Mathematics and Computer Science
Anhui Normal University
Wuhu 241003, The People's Republic of China
e-mail: shulsh@mail.ahnu.edu.cn