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COMPLETE MONOTONICITY AND INEQUALITES INVOLVING GURLAND'S RATIOS OF GAMMA FUNCTIONS

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Abstract. In this paper, by a comparison inequality for an auxiliary function with two parameters, we present necessary and sufficient conditions for four classes of ratios involving gamma function to be logarithmically completely monotonic. These not only greatly generalize and improve certain known results, but also yield many new inequalities for gamma, psi and polygamma functions.

1. Introduction

A function f is called completely monotonic (for short, CM) on an interval I if f has derivatives of all orders on I and satisfies

$$(-1)^k (f(x))^{(k)} \geqslant 0$$

for all $k \geqslant 0$ on I (see [1], [2]). A function f is called logarithmically completely monotonic (for short, LCM) on an interval I if f has derivatives of all orders on I and its logarithm $\ln f$ satisfies

$$(-1)^k (\ln f(x))^{(k)} \geqslant 0$$

for all $k \in \mathbb{N}$ on I (see [3], [4]). The notion of completely monotonic function in several variables can refer to the recent paper [5].

The celebrated Bernstein–Widder's theorem [2, p. 161, Theorem 12b] states that $f(x)$ is completely monotonic on $(0,\infty)$ if and only if

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t),$$

where $\alpha(t)$ is nonnegative measure such that integral converges on $(0,\infty)$.

For convenience, we denote the sets of the completely and logarithmically completely monotonic functions on I by $\mathscr{C}[I]$ and $\mathscr{L}[I]$, respectively.

The classical Euler's gamma function Γ is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \tag{1}$$

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for $x > 0$, and its logarithmic derivative $\psi(x) = \Gamma'(x)/\Gamma(x)$ is known as the psi or digamma function, while ψ', ψ'', ... are called polygamma functions.

Over the past decades, the complete monotonicity of certain ratios of gamma functions has been researched by many mathematicians, see for example, [4]–[22].

We now focus on a special ratio of gamma functions

$$T(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma((x+y)/2)^2}, \quad x,\ y > 0,$$

some properties of which can be found in [8]. An interesting relation between $T(u,v)$ and the modified Bessel function $I_v(x)$ and $K_v(x)$ was established in recent papers [23], [24]. A generalization of Gurland's ratio was given in [25] (see also [7]) by

$$\frac{\Gamma(x+a)\Gamma(x+b)}{\Gamma(x+c)\Gamma(x+d)}$$

with $a-c = d-b$.

In particular, we see that for $n \in \mathbb{N}$,

$$T(n,n+1) = \frac{1}{n}\frac{\Gamma(n+1)^2}{\Gamma(n+1/2)^2} = \frac{1}{n\pi}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2 = \frac{1}{n\pi}\frac{1}{W_n^2},$$

where W_n is the Wallis ratio [26]. In probability theory and their applications, the ratio

$$T(x,x+2u) = \frac{\Gamma(x)\Gamma(x+2u)}{\Gamma(x+u)^2}, \quad x,\ x+2u > 0$$

is related to the variance of an estimator involving gamma distribution, and satisfies the inequality

$$T(x,x+2u) = \frac{\Gamma(x)\Gamma(x+2u)}{\Gamma(x+u)^2} > 1 + \frac{u^2}{x} \tag{2}$$

for $x, x+2u > 0$ (see [27]). More properties including complete monotonicity of $T(x,x+2u)$ were given in [8]. While the ratio

$$T\left(\frac{1}{p},\frac{3}{p}\right) = \frac{\Gamma(1/p)\Gamma(3/p)}{\Gamma(2/p)^2}, \quad p > 0,$$

also known as generalized Gaussian ratio or Mallat ratio [28], appears in the form of ratio of the variance and squared absolute expectation of a generalized Gaussian random variable with the shape parameter p [29], and has some interesting applications in the field of image recognition [30]. Some new properties of $T(1/p,3/p)$ have been presented in a recent paper [31].

Inspired by the above comments, the first aim of this paper is to find the necessary and sufficient conditions for the ratios $G_{u,v;p,q}(x)/G_{u,v;r,s}(x)$ and $T_{p,q}(x)/T_{r,s}(x)$ to be complete monotonicity, where

$$G_{u,v;p,q}(x)=\begin{cases}\left[\dfrac{\Gamma(x+p+u)\Gamma(x+q+v)}{\Gamma(x+p+v)\Gamma(x+q+u)}\right]^{1/((p-q)(u-v))} & \text{if } (p-q)(u-v)\neq 0,\\ \exp\left[\dfrac{\psi(x+p+u)-\psi(x+p+v)}{u-v}\right] & \text{if } p=q,\, u\neq v,\\ \exp\left[\dfrac{\psi(x+p+u)-\psi(x+q+u)}{p-q}\right] & \text{if } p\neq q,\, u=v,\\ \exp\left[\psi'(x+p+u)\right] & \text{if } p=q,\, u=v\end{cases} \tag{3}$$

for $x>-\min(p,q)-\min(u,v)$,

$$T_{p,q}(x)=G_{p/2,q/2;p/2,q/2}(x)^{1/4}=\begin{cases}\left(\dfrac{\Gamma(x+p)\Gamma(x+q)}{\Gamma(x+(p+q)/2)^2}\right)^{1/(p-q)^2} & \text{if } p\neq q,\\ \exp\left[\frac{1}{4}\psi'(x+p)\right] & \text{if } p=q\end{cases} \tag{4}$$

for $x>-\min(p,q)$. All complete monotonicity results are proved in Section 3.

REMARK 1. It is clear that $G_{u,v;p,q}(x)^{(p-q)(u-v)}$ is a generalization of $T(x,x+2\beta)$.

The second aim is to obtain some new inequalities for the Gurland's ratio, which is presented in Section 4.

2. An important auxiliary function

In order to prove our results, we need to study some properties of an important auxiliary function. This function denoted by $y_{p,q}:\mathbb{R}\longrightarrow\mathbb{R}$ is defined, for $p,q\in\mathbb{R}$, by

$$y_{p,q}(t)=\begin{cases}\dfrac{e^{-pt}-e^{-qt}}{q-p} & \text{if } p\neq q,\\ te^{-pt} & \text{if } p=q.\end{cases} \tag{5}$$

It is easy to check that $y_{p,q}(t)$ has the following two simple properties.

PROPERTY 1. *We have*

$$y_{p,q}(t)=t\int_0^1 \exp(-ptu-qt(1-u))\,du, \tag{6}$$

$$y_{p,q}(t)=\begin{cases}e^{-(p+q)t/2}\dfrac{\sinh[(p-q)t/2]}{(p-q)/2} & \text{if } p\neq q,\\ te^{-pt} & \text{if } p=q.\end{cases} \tag{7}$$

PROPERTY 2. *Let $p,q,t \in \mathbb{R}$. Then $y_{p,q}(t)$ satisfies that*
(i) $y_{p,q}(t) > (<) 0$ for $t > (< 0)$;
(ii) $y_{p,q}(t) = y_{q,p}(t)$;
(iii) $e^{-\rho t} y_{p-\rho,q-\rho}(t) = y_{p,q}(t)$ for any $\rho \in \mathbb{R}$.

In order to prove Property 3, we need the following lemma.

LEMMA 1. ([20, Theorem 2.1]) *Let $p,q \in \mathbb{R}$ and $H_{p,q}$ be defined on $(0,\infty)$ by*

$$H_{p,q}(t) = \begin{cases} \left(\frac{q}{p}\frac{\sinh(pt)}{\sinh(qt)}\right)^{1/(p-q)} & \text{if } pq(p-q) \neq 0, \\ \left(\frac{\sinh(pt)}{pt}\right)^{1/p} & \text{if } p \neq 0,\ q = 0, \\ \left(\frac{\sinh(qt)}{qt}\right)^{1/q} & \text{if } p = 0,\ q \neq 0, \\ e^{t\coth(pt)-1/p} & \text{if } p = q,\ pq \neq 0, \\ 1 & \text{if } p = q = 0. \end{cases} \tag{8}$$

Then the function $t \mapsto t^{-1}\ln H_{p,q}(t)$ is strictly increasing (decreasing) from $(0,\infty)$ onto $(0,(p+q)/(|p|+|q|))$ $(((p+q)/(|p|+|q|),0))$ and concave (convex) on $(0,\infty)$ if $p+q > (<) 0$.

PROPERTY 3. *For $u,v,r,s \in \mathbb{R}$, let $y_{u,v}$ be defined on $(0,\infty)$ by (5). Then the comparison inequality $y_{u,v}(t) \geqslant y_{r,s}(t)$ holds for all $t > 0$ if and only if*

$$u+v \leqslant r+s \quad \text{and} \quad \min(u,v) \leqslant \min(r,s). \tag{9}$$

Proof. Let $p = |u-v|/2$, $q = |r-s|/2$. In the case of $(u-v)(r-s) \neq 0$, we use the hyperbolic function representation (6) to obtain

$$\begin{aligned} h(t) &= \frac{1}{t}\ln\frac{y_{u,v}(t)}{y_{r,s}(t)} = \frac{r+s-(u+v)}{2} + \frac{1}{t}\ln\left(\frac{|r-s|}{|u-v|}\frac{\sinh|(u-v)t/2|}{\sinh|(r-s)t/2|}\right) \\ &= \frac{r+s-(u+v)}{2} + \frac{1}{t}\ln\left(\frac{q}{p}\frac{\sinh(pt)}{\sinh(qt)}\right) \\ &= \frac{r+s-(u+v)}{2} + (p-q)\frac{\ln H_{p,q}(t)}{t}, \end{aligned}$$

which is also true for $(u-v)(r-s) = 0$.

Due to $p,q \geqslant 0$, by Lemma 1 we see that $t \mapsto t^{-1}\ln H_{p,q}(t)$ is strictly increasing from $(0,\infty)$ onto $(0,1)$, which implies that $t \mapsto h(t)$ is increasing (decreasing) on $(0,\infty)$ if $p \geqslant (\leqslant) q$. Consequently, $h(t) \geqslant 0$ for all $t > 0$ if and only if $h(0^+) \geqslant 0$ and

$h(\infty) \geqslant 0$. A simple computation yields

$$\begin{aligned} h\left(0^{+}\right) &= \frac{r+s-(u+v)}{2}, \\ h(\infty) &= \frac{r+s-(u+v)}{2} + (p-q) = \frac{r+s-(u+v)}{2} + \frac{|u-v|-|r-s|}{2} \\ &= \min(r,s) - \min(u,v). \end{aligned}$$

which proves the desired assertion. □

3. Completely monotonicity of Gurland's ratio

In this section, by using properties of $y_{u,v}(t)$ presented in Section 2, we establish the necessary and sufficient conditions for the functions related to Gurland's ratio, that is, both $\ln\left(G_{u,v;p,q}/G_{u,v;r,s}\right)$ and $\ln\left(T_{p,q}/T_{r,s}\right)$ are completely monotonic.

THEOREM 1. *For fixed $p,q,r,s,u,v \in \mathbb{R}$, let $\rho = \min(p,q,r,s) + \min(u,v)$ and let the function $G_{u,v,p,q}$ be defined on $(-\min(p,q) - \min(u,v), \infty)$ by (3). Then $\ln\left(G_{u,v,p,q}/G_{u,v,r,s}\right) \in \mathscr{C}\left[(-\rho,\infty)\right]$ if and only if*

$$p+q \leqslant r+s \ \text{ and } \ \min(p,q) \leqslant \min(r,s). \tag{10}$$

Proof. We first give the following integral representation

$$\ln G_{u,v;p,q}(x) = \int_0^\infty y_{p,q}(t)\, y_{u,v}(t) \frac{e^{-xt}}{t\left(1-e^{-t}\right)} dt. \tag{11}$$

To this end, using the integral representation of $\ln\Gamma(z)$ [32, p.258, (6.1.50)]

$$\ln\Gamma(z) = \int_0^\infty \left[(z-1)e^{-t} - \frac{e^{-t}-e^{-zt}}{1-e^{-t}}\right] \frac{dt}{t} \qquad (\mathrm{Re}(z) > 0) \tag{12}$$

yields that, for $(p-q)(u-v) \neq 0$,

$$\begin{aligned} \ln G_{u,v;p,q}(x) &= \frac{\ln\Gamma(x+p+u) + \ln\Gamma(x+q+v) - \ln\Gamma(x+p+v) - \ln\Gamma(x+q+u)}{(p-q)(u-v)} \\ &= \int_0^\infty \frac{e^{-t(p+u)} + e^{-t(q+v)} - e^{-t(p+v)} - e^{-t(q+u)}}{(p-q)(u-v)} \frac{e^{-xt}}{t\left(1-e^{-t}\right)} dt \\ &= \int_0^\infty \frac{e^{-tp}-e^{-tq}}{q-p} \frac{e^{-tu}-e^{-tv}}{v-u} \frac{e^{-xt}}{t\left(1-e^{-t}\right)} dt \\ &= \int_0^\infty y_{p,q}(t)\, y_{u,v}(t) \frac{e^{-xt}}{t\left(1-e^{-t}\right)} dt, \end{aligned}$$

which is clearly also valid for $(p-q)(u-v) \neq 0$. It then follows that

$$\ln \frac{G_{u,v,p,q}(x)}{G_{u,v,r,s}(x)} = \int_0^\infty \left[y_{p,q}(t) - y_{r,s}(t)\right] y_{u,v}(t) \frac{e^{-xt}}{t\left(1-e^{-t}\right)} dt. \tag{13}$$

From the integral representation and Property 2, we have

$$\ln \frac{G_{u,v,p,q}(x)}{G_{u,v,r,s}(x)} = \int_0^\infty \left[y_{p-\rho,q-\rho}(t) - y_{r-\rho,s-\rho}(t)\right] y_{u,v}(t) \frac{e^{-(x+\rho)t}}{t(1-e^{-t})} dt.$$

By Bernstein–Widder's theorem [2, p. 161, Theorem 12b] and Property 3, the desired result follows. □

The integral representation (13) together with $y_{0,0}(t) = t$ and (iii) of Property 2 gives

$$\begin{aligned} \ln \frac{G_{0,0,p,q}(x)}{G_{0,0,r,s}(x)} &= \int_0^\infty y_{p,q}(t)\, y_{0,0}(t) \frac{e^{-xt}}{t(1-e^{-t})} dt \\ &= \int_0^\infty \left[y_{p-\rho,q-\rho}(t) - y_{r-\rho,s-\rho}(t)\right] \frac{e^{-(x+\rho)t}}{1-e^{-t}} dt, \end{aligned}$$

which, by Bernstein–Widder's theorem [2, p. 161, Theorem 12b] and Property 3, implies the following corollary.

COROLLARY 1. *For fixed $p,q,r,s \in \mathbb{R}$, let $\rho = \min(p,q,r,s)$ and let $G_{u,v,p,q}$ be defined on $(-\min(p,q), \infty)$ by (3). Then the function $\ln\left(G_{0,0,p,q}/G_{0,0,r,s}\right) \in \mathscr{C}[(-\rho,\infty)]$ if and only if the inequalities (10) are satisfied.*

Denoted by

$$W_{u,v}(x) = \begin{cases} \left(\dfrac{\Gamma(x+u)}{\Gamma(x+v)}\right)^{1/(u-v)} & \text{if } u \neq v, \\ \exp[\psi(x+u)] & \text{if } u = v \end{cases} \tag{14}$$

for $x > -\min(u,v)$. It is evident that

$$\ln \frac{G_{0,0,p,q}(x)}{G_{0,0,r,s}(x)} = \frac{d}{dx}\left[\ln \frac{W_{p,q}(x)}{W_{r,s}(x)}\right] \quad \text{and} \quad \lim_{x\to\infty} \ln \frac{W_{p,q}(x)}{W_{r,s}(x)} = 0,$$

where the second equality follows from $\Gamma(x+p)/\Gamma(x+q) \sim x^{p-q}$ as $x \to \infty$. Then by Corollary 1 we immediately get

COROLLARY 2. *For fixed $p,q,r,s \in \mathbb{R}$, let $\rho = \min(p,q,r,s)$ and let $W_{u,v}(x)$ be defined on $(-\min(u,v), \infty)$ by (14). Then $\ln(W_{p,q}/W_{r,s}) \in \mathscr{C}[(-\rho,\infty)]$ if and only if*

$$p+q \geqslant r+s \quad \text{and} \quad \min(p,q) \geqslant \min(r,s).$$

REMARK 2. Corollary 2 was first given in [21], which unifies and improves some known results established in [6], [7], [13, Theorem 1], [14, Theorem 1].

Now let us take $(p,q) = (u,v) = (u,0)$, $(r,s) = (s+1,s)$ with $u \neq 0$. Then $G_{u,v;p,q}(x)/G_{u,v;r,s}(x)$ can be reduced to

$$\frac{G_{u,0;u,0}(x)}{G_{u,0;s+1,s}(x)} = \left(\frac{x+s}{x+s+u}\right)^{1/u} \left(\frac{\Gamma(x)\Gamma(x+2u)}{\Gamma(x+u)^2}\right)^{1/u^2},$$

$\rho = \min(p,q,r,s) + \min(u,v) = \min(u,0,s) + \min(u,0)$. By Theorem 1, we have

COROLLARY 3. *For fixed $u,s \in \mathbb{R}$ with $u \neq 0$, $\rho = \min(u,0,s) + \min(u,0)$, the function*

$$x \mapsto g_1(x) = u^2 \ln\left(\frac{G_{u,0;u,0}(x)}{G_{u,0;s+1,s}(x)}\right) = \ln\left[\left(\frac{x+s}{x+s+u}\right)^u \frac{\Gamma(x)\Gamma(x+2u)}{\Gamma(x+u)^2}\right] \in \mathscr{C}[(-\rho,\infty)]$$

if and only if $s \geqslant \max((u-1)/2, \min(u,0))$, while $-g_1 \in \mathscr{L}[(-\rho,\infty)]$ if and only if $s \leqslant \min((u-1)/2, \min(u,0))$.

Taking $(u,v) = (r,s) = (1/2,0)$ and $q = 0$ in Theorem 1. Then $\rho = \min(p,q,r,s) + \min(u,v) = \min(p,0)$. We have

COROLLARY 4. *The function*

$$x \mapsto g_2(x) = \begin{cases} \ln\left[\left(\dfrac{\Gamma(x+1/2)^2}{\Gamma(x)\Gamma(x+1)}\right)^2 \left(\dfrac{\Gamma(x)\Gamma(x+p+1/2)}{\Gamma(x+1/2)\Gamma(x+p)}\right)^{1/p}\right] & \text{if } p \neq 0, \\ \ln\left[\left(\dfrac{\Gamma(x+1/2)^2}{\Gamma(x)\Gamma(x+1)}\right)^2 \dfrac{\exp\psi(x+1/2)}{\exp\psi(x)}\right] & \text{if } p = 0 \end{cases}$$

is completely monotonic on $(-\min(p,0),\infty)$ if and only if $p \leqslant 0$. While $-g_2 \in \mathscr{C}(-\min(p,0),\infty)$ if and only if $p \geqslant 1/2$.

REMARK 3. Letting $p = -1/2,\ 1$ in the above corollary, we deduce that the functions

$$x \mapsto \left(1 - \frac{1}{2x}\right)^{-2} \left[\frac{\Gamma(x+1/2)^2}{\Gamma(x)\Gamma(x+1)}\right]^4 \in \mathscr{L}\left[\left(\frac{1}{2},\infty\right)\right],$$

$$x \mapsto \left(1 + \frac{1}{2x}\right)^{-1} \left[\frac{\Gamma(x)\Gamma(x+1)}{\Gamma(x+1/2)^2}\right]^2 \in \mathscr{L}[(0,\infty)],$$

which are equivalent to Theorems 4 and 5 in [7], respectively.

THEOREM 2. *For fixed $p,q,r,s \in \mathbb{R}$, $\rho = \min(p,q,r,s)$, let the function $T_{p,q}$ be defined on $(-\rho,\infty)$ by (4). Then $\ln(T_{p,q}/T_{r,s}) \in \mathscr{C}(-\rho,\infty)$ if and only if the inequalities (10) are satisfied.*

Proof. The relation $T_{p,q}(x) = (1/4)G_{p/2,q/2;p/2,q/2}(x)$ and integral representation (11) yield

$$\ln T_{p,q}(x) = \frac{1}{4}\int_0^\infty y_{p/2,q/2}(t)^2 \frac{e^{-xt}}{t(1-e^{-t})}dt. \tag{15}$$

From this and Property 2, we obtain

$$\ln\frac{T_{p,q}(x)}{T_{r,s}(x)} = \frac{1}{4}\int_0^\infty \left[y_{(p-\rho)/2,(q-\rho)/2}(t)^2 - y_{(r-\rho)/2,(s-\rho)/2}(t)^2\right] \frac{e^{-(x+\rho)t}}{t(1-e^{-t})}dt.$$

By Bernstein–Widder's theorem [2, p. 161, Theorem 12b] and Property 3, we see that $\ln\left(T_{p,q}/T_{r,s}\right) \in \mathscr{C}\left[(-\rho,\infty)\right]$ if and only if

$$\frac{p-\rho}{2}+\frac{q-\rho}{2}\leqslant\frac{r-\rho}{2}+\frac{s-\rho}{2}$$

and

$$\min\left(\frac{p-\rho}{2},\frac{q-\rho}{2}\right)\leqslant\min\left(\frac{r-\rho}{2},\frac{s-\rho}{2}\right),$$

that is, the relations (10) are satisfied. □

We note that

$$\frac{T_{p,q}(x)}{T_{0,1}(x)}=\begin{cases} x\dfrac{\Gamma(x+1/2)^2}{\Gamma(x+1)^2}\left(\dfrac{\Gamma(x+p)\Gamma(x+q)}{\Gamma(x+(p+q)/2)^2}\right)^{1/(p-q)^2} & \text{if } p\neq q,\\ x\dfrac{\Gamma(x+1/2)^2}{\Gamma(x+1)^2}\exp\left[\frac{1}{4}\psi'(x+p)\right] & \text{if } p=q \end{cases}$$

$$\frac{T_{p,q}(x)}{T_{r,r+2}(x)}=\begin{cases} \left(1+\dfrac{1}{x+r}\right)^{-1/4}\left(\dfrac{\Gamma(x+p)\Gamma(x+q)}{\Gamma(x+(p+q)/2)^2}\right)^{1/(p-q)^2} & \text{if } p\neq q,\\ \left(1+\dfrac{1}{x+r}\right)^{-1/4}\exp\left[\frac{1}{4}\psi'(x+p)\right] & \text{if } p=q, \end{cases}$$

$$\frac{T_{p,q}(x)}{T_{r,r}(x)}=\begin{cases} \dfrac{1}{\exp\left[\psi'(x+r)/4\right]}\left(\dfrac{\Gamma(x+p)\Gamma(x+q)}{\Gamma(x+(p+q)/2)^2}\right)^{1/(p-q)^2} & \text{if } p\neq q,\\ \exp\left[\frac{1}{4}\psi'(x+p)-\frac{1}{4}\psi'(x+r)\right] & \text{if } p=q. \end{cases}$$

By Theorem 2 the following corollaries are immediate.

COROLLARY 5. *For fixed $p,q\in\mathbb{R}$ with $p\geqslant q$, let the function $T_{p,q}$ be defined on $(-\rho,\infty)$ by (4). Then $\ln\left(T_{p,q}/T_{0,1}\right)\in\mathscr{C}\left[(-\min(q,0),\infty)\right]$ if and only if $q\leqslant \min(0,p,1-p)$, while $\ln\left(T_{0,1}/T_{p,q}\right)\in\mathscr{C}\left[(-\min(q,0),\infty)\right]$ if and only if $\max(0,1-p)\leqslant q\leqslant p$*

COROLLARY 6. *For fixed $p,q,r\in\mathbb{R}$, $\rho=\min(p,q,r)$, let the function $T_{p,q}$ be defined on $(-\rho,\infty)$ by (4). Then $\ln\left(T_{p,q}/T_{r,r+2}\right)\in\mathscr{C}\left[(-\rho,\infty)\right]$ if and only if $r\geqslant \max\left((p+q-2)/2,\min(p,q)\right)$, while $\ln\left(T_{r,r+2}/T_{p,q}\right)\in\mathscr{C}\left[(-\rho,\infty)\right]$ if and only if $r\leqslant \min\left((p+q-2)/2,\min(p,q)\right)$.*

COROLLARY 7. *For fixed $p,q,r\in\mathbb{R}$, $\rho=\min(p,q,r)$, let the function $T_{p,q}$ be defined on $(-\rho,\infty)$ by (4). Then $\ln\left(T_{p,q}/T_{r,r}\right)\in\mathscr{C}\left[(-\rho,\infty)\right]$ if and only if $r\geqslant(p+q)/2$, while $\ln\left(T_{r,r}/T_{p,q}\right)\in\mathscr{C}\left[(-\rho,\infty)\right]$ if and only if $r\leqslant\min(p,q)$.*

4. Some new inequalities for gamma function

Many inequalities for gamma function are derived from the monotonicity or convexity properties of the combinations of gamma functions and some elementary functions, for example, [33], [34], [35], [36], [8], [13], [37], [16], [38]. Ismail et al. [6], [7] further realized that some inequalities for gamma function are the consequences of complete monotonicity of such combinations.

In this section, we will list some new inequalities for gamma functions by using the (logarithmically) completely monotonicity presented in the third section.

Note that for $(p-q)(u-v) \neq 0$,

$$\ln G_{u,v;p,q}(x) = \frac{\int_q^p \int_v^u \psi'(x+\alpha+\beta)\,d\alpha d\beta}{(p-q)(u-v)},$$

by Theorem 1 we immediately conclude

PROPOSITION 3. *Let* $p,q,r,s,u,v \in \mathbb{R}$ *with* $(p-q)(r-s)(u-v) \neq 0$, $\rho = \min(p,q,r,s) + \min(u,v)$, $n \in \mathbb{N}$. *Then the inequality*

$$(-1)^{n-1} \frac{\int_q^p \int_v^u \psi^{(n)}(x+\alpha+\beta)\,d\alpha d\beta}{(p-q)(u-v)} > (-1)^{n-1} \frac{\int_s^r \int_v^u \psi^{(n)}(x+\alpha+\beta)\,d\alpha d\beta}{(r-s)(u-v)}$$

holds for $x > -\rho$ *if and only if* $p+q \leqslant r+s$ *and* $\min(p,q) \leqslant \min(r,s)$.

It was proved [27]

$$\frac{\Gamma(x)\Gamma(x+2\beta)}{\Gamma(x+\beta)^2} \geqslant 1 + \frac{\beta^2}{x}, \quad x > 0,\ x+2\beta > 0.$$

The following is a direct consequence of Corollary 3.

COROLLARY 8. *For fixed* $u,s \in \mathbb{R}$ *with* $u \neq 0$, $\rho = \min(u,0,s) + \min(u,0)$, *the inequality*

$$\frac{\Gamma(x)\Gamma(x+2u)}{\Gamma(x+u)^2} > \left(1 + \frac{u}{x+s}\right)^u$$

holds for $x > -\rho$ *if and only if* $s \geqslant \max((u-1)/2, \min(u,0))$, *which reverses it for* $x > -\rho$ *if and only if* $s \leqslant \min((u-1)/2, \min(u,0))$. *In particular, by taking* $s = \max((u-1)/2, \min(u,0))$ *and* $s = \min((u-1)/2, \min(u,0))$ *we have*

(i) if $u \in (1,\infty)$, *then*

$$\left(1 + \frac{u}{x+(u-1)/2}\right)^u < \frac{\Gamma(x)\Gamma(x+2u)}{\Gamma(x+u)^2} < \left(1 + \frac{u}{x}\right)^u \quad \textit{for } x > 0;$$

(ii) if $u \in (0,1)$, *then*

$$\left(1 + \frac{u}{x}\right)^u < \frac{\Gamma(x)\Gamma(x+2u)}{\Gamma(x+u)^2} \quad \textit{for } x > 0,$$

$$\frac{\Gamma(x)\Gamma(x+2u)}{\Gamma(x+u)^2} < \left(1 + \frac{u}{x+(u-1)/2}\right)^u \quad \textit{for } x > \frac{1-u}{2};$$

(iii) if $u \in (-1,0)$, then

$$\left(1+\frac{u}{x+u}\right)^u < \frac{\Gamma(x)\Gamma(x+2u)}{\Gamma(x+u)^2} \quad \text{for } x > -2u,$$
$$\frac{\Gamma(x)\Gamma(x+2u)}{\Gamma(x+u)^2} < \left(1+\frac{u}{x+(u-1)/2}\right)^u \quad \text{for } x > \frac{1-3u}{2};$$

(iv) if $u \in (-\infty,-1)$, then

$$\left(1+\frac{u}{x+(u-1)/2}\right)^u < \frac{\Gamma(x)\Gamma(x+2u)}{\Gamma(x+u)^2} < \left(1+\frac{u}{x+u}\right)^u \quad \text{for } x > -2u.$$

REMARK 4. Clearly, this corollary above gives new lower and upper bounds for Gurland's ratio, which extend the Gurland's inequality (2) [27].

Now let us take $(p,q)=(u,0)$, $(r,s)=(s,s)$, $(u,v)=(u,0)$ with $u \neq 0$. Then $G_{u,v;p,q}(x)/G_{u,v;r,s}(x)$ can be reduced to

$$\left(\frac{G_{u,0;u,0}(x)}{G_{u,0;s,s}(x)}\right)^{u^2} = e^{-u[\psi(x+s+u)-\psi(x+s)]}\frac{\Gamma(x)\Gamma(x+2u)}{\Gamma(x+u)^2},$$

and $\rho = \min(p,q,r,s)+\min(u,v) = \min(u,0,s)+\min(u,0)$. By Theorem 1, we have

COROLLARY 9. *For fixed $u,s \in \mathbb{R}$ with $u \neq 0$, $\rho = \min(u,0,s)+\min(u,0)$, the inequality*

$$\frac{\Gamma(x)\Gamma(x+2u)}{\Gamma(x+u)^2} > \exp[u(\psi(x+s+u)-\psi(x+s))]$$

holds for $x > -\rho$ if and only if $s \geqslant u/2$, which reverses it for $x > -\rho$ if and only if $s \leqslant \min(u,0)$. Especially, by putting $s = u/2$ and $s = \min(u,0)$, we have

$$\frac{\exp[u\psi(x+3u/2)]}{\exp[u\psi(x+u/2)]} - \psi(x+u/2)$$
$$< \frac{\Gamma(x)\Gamma(x+2u)}{\Gamma(x+u)^2} < \begin{cases} \dfrac{\exp[u\psi(x+u)]}{\exp[u\psi(x)]} & \text{for } u,x>0, \\[2ex] \dfrac{\exp[u\psi(x+2u)]}{\exp[u\psi(x+u)]} & \text{for } x > -2u > 0. \end{cases}$$

Note that

$$\ln T_{p,q}(x) = \frac{\int_q^p \int_q^p \psi'(x+(\alpha+\beta)/2)\,d\alpha d\beta}{4(p-q)^2} \quad \text{for } p \neq q,$$

from Theorem 2 we obtain

PROPOSITION 4. *For fixed $p,q,r,s \in \mathbb{R}$ with $(p-q)(r-s) \neq 0$, $\rho = \min(p,q,r,s)$, $n \in \mathbb{N}$, both the inequalities*

$$\left[\frac{\Gamma(x+p)\Gamma(x+q)}{\Gamma(x+(p+q)/2)^2}\right]^{1/(p-q)^2} > \left[\frac{\Gamma(x+r)\Gamma(x+s)}{\Gamma(x+(r+s)/2)^2}\right]^{1/(r-s)^2},$$

$$(-1)^n \frac{\int_q^p \int_q^p \psi^{(n+1)}\left[x+(\alpha+\beta)/2\right] d\alpha d\beta}{(p-q)^2} > (-1)^n \frac{\int_s^r \int_s^r \psi^{(n+1)}\left[x+(\alpha+\beta)/2\right] d\alpha d\beta}{(r-s)^2}$$

hold for $x \in (-\rho, \infty)$ *if and only if* $p+q \geqslant r+s$ *and* $\min(p,q) \geqslant \min(r,s)$.

As direct consequences of Corollaries 6 and 7 we conclude that

COROLLARY 10. *For fixed* $p,q,r \in \mathbb{R}$ *with* $p \neq q$, $\rho = \min(p,q,r)$, *the inequality*

$$\frac{\Gamma(x+p)\Gamma(x+q)}{\Gamma(x+(p+q)/2)^2} > \left(1+\frac{1}{x+r}\right)^{(p-q)^2/4}$$

holds for $s > -\rho$ *if and only if* $r \geqslant \max((p+q-2)/2, \min(p,q))$. *It is reversed for* $x > -\rho$ *if and only if* $r \leqslant \min((p+q-2)/2, \min(p,q))$.

COROLLARY 11. *For fixed* $p,q \in \mathbb{R}$ *with* $p \neq q$, $\rho = \min(p,q)$, *the double inequality*

$$\exp\left[\frac{(p-q)^2}{4}\psi'(x+r_1)\right] < \frac{\Gamma(x+p)\Gamma(x+q)}{\Gamma(x+(p+q)/2)^2} < \exp\left[\frac{(p-q)^2}{4}\psi'(x+r_2)\right]$$

holds for $x \in (-\rho, \infty)$ *with the best constants* $r_1 = (p+q)/2$ *and* $r_2 = \min(p,q)$.

REMARK 5. If $p,q > 0$, then by the same method as in [34], the constant r_2 can be improved as $\sqrt{pq}$.

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