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ON STRONGLY GENERALIZED CONVEX FUNCTIONS OF HIGHER ORDER

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Abstract. In this paper, we have introduced the notion of strongly generalized convex functions of higher order. We derived new integral inequalities of Hermite-Hadamard and Hermite-Hadamard-Féjer type for the class of strongly generalized convex functions of higher order. The results of Awan *et al.* [M. U. AWAN, M. A. NOOR, K. I. NOOR AND F. SAFDAR, *On strongly generalized convex functions*, Filomat **31**, 18 (2017), 5783–5790] are the special case of the results obtained in this paper.

1. Introduction

Karamardian [9] introduced strongly convex function in 1969. However, there are references citing Polyak [13] has introduced strongly convex functions as a generalization of convex functions, see [11, 12]. Karamardian [9] established the relationship between strongly convex functions and strongly monotone maps. It is well known that every differentiable function is strongly convex if and only if its gradient map is strongly monotone [9]. Karamardian [9] also showed that every bidifferentiable function is strongly convex if and only if its Hessian matrix is strongly positive definite.

Lin and Fukushima [10] introduced strongly convex functions of higher order to simplify the study of mathematical programs with equilibrium constraints. Obviously, strong convexity of higher order is a generalization of strong convexity, the function $\psi(x) = x^4$ is strongly convex of order 4, but not strongly convex of order 2 on $\mathbb{R}$, see [10]. Lin and Fukushima [10] have established that the optimal solution of MPEC under strong convexity of higher order is same as the optimal solution of penalized problem. Further, Lin and Fukushima [10] have shown that the higher order strong convexity of a function is equivalent to higher order strong monotonicity of the gradient map of the function.

Azócar *et al.* [2] derived an appropriate counterpart of the Féjer inequalities and presented also refinement of Hermite-Hadamard inequalities for strongly convex functions. Gordji *et al.* [6] introduced the concept of η-convex/(φ-convex [7]) functions as a generalization of convex functions and obtained the Hermite-Hadamard, Féjer, Jensen and Slater type inequalities for η-convex functions. Further, Delavar and Sen [4] have

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given some applications for Hermite-Hadamard-Féjer type integral inequalities for differentiable η-convex functions.

Recently, Awan *et al*. [1] introduced the notion of strongly η-convex functions and formulated some new integral inequalities of Hermite-Hadamard type for strongly η-convex functions. For more details, one can refer to [3, 5].

Motivated by Awan *et al*. [1] and Lin and Fukushima [10], we introduce in this paper the concept of strongly η-convex functions of higher order, as a generalization of the strongly η-convex functions. We investigate the Hermite-Hadamard and Hermite-Hadamard-Féjer type inequalities for strongly η-convex functions of higher order. Special cases are also investigated. These results extend and unify several results established in literature.

2. Preliminaries

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. We denote the usual inner product by $\langle\cdot,\cdot\rangle$ and for $x \in \mathbb{R}^n$, $\|\cdot\|$ denote the norm defined by

$$\|x\| = \left(\sum_{i=1}^{n} {x_i}^2\right)^{1/2}.$$

It is basic knowledge in mathematical analysis that a function $\psi : X \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be convex on $X \neq \phi$ if

$$\psi(tx + (1-t)y) \leqslant t\psi(x) + (1-t)\psi(y),$$

for all $x, y \in X$ and $t \in [0,1]$.

DEFINITION 1. [10] A function $\psi : X \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be strongly convex with order $\sigma > 0$ on a convex set $X \subseteq \mathbb{R}^n$ if there exist a constant $c > 0$, such that

$$\psi(tx + (1-t)y) \leqslant t\psi(x) + (1-t)\psi(y) - ct(1-t)\|x-y\|^{\sigma},$$

for any $x, y \in X$ and any $t \in [0,1]$.

When $\sigma = 2$, this property reduces to the strong convexity in the ordinary sense.

DEFINITION 2. [1] A function $\psi : X \subset \mathbb{R} \longrightarrow \mathbb{R}$ is said to be η-convex function with respect to $\eta : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, if

$$\psi(tx + (1-t)y) \leqslant \psi(y) + t\eta(\psi(x), \psi(y)), \quad \forall x, y \in X, \ \ t \in [0,1].$$

THEOREM 1. [6] *Suppose that* $\psi : [a,b] \longrightarrow \mathbb{R}$ *is a* η*-convex function such that* η *is bounded from above on* $\psi([a,b]) \times \psi([a,b])$. *Then,*

$$\begin{aligned}\psi\left(\frac{a+b}{2}\right) - \frac{M_\eta}{2} &\leqslant \frac{1}{b-a}\int_a^b \psi(x)dx \\ &\leqslant \frac{\psi(a)+\psi(b)}{2} + \frac{\eta(\psi(a),\psi(b))+\eta(\psi(b),\psi(a))}{4} \\ &\leqslant \frac{\psi(a)+\psi(b)}{2} + \frac{M_\eta}{2},\end{aligned}$$

where M_η *is upper bound of* η.

THEOREM 2. [6] (Hermite-Hadamard-Féjer left Inequality). *Suppose that* $\psi : [a,b] \longrightarrow \mathbb{R}$ *is a* η*-convex function, such that* η *is bounded from above on* $\psi([a,b]) \times \psi([a,b])$. *Also suppose that* $\xi : [a,b] \longrightarrow \mathbb{R}^+$ *is integrable and symmetric with respect to* $\frac{a+b}{2}$. *Then,*

$$\begin{aligned}\psi\left(\frac{a+b}{2}\right)\int_a^b \xi(x)dx - \frac{1}{2}\int_a^b \eta(\psi(a+b-x),\psi(x))\xi(x)dx \\ \leqslant \int_a^b \psi(x)\xi(x)dx.\end{aligned}$$

THEOREM 3. [6] (Hermite-Hadamard-Féjer Right Inequality). *Suppose that* $\psi : [a,b] \longrightarrow \mathbb{R}$ *is a* η*-convex function, such that* η *is bounded from above on* $\psi([a,b]) \times \psi([a,b])$. *Also suppose that* $\xi : [a,b] \longrightarrow \mathbb{R}^+$ *is integrable and symmetric with respect to* $\frac{a+b}{2}$. *Then,*

$$\begin{aligned}\int_a^b \psi(x)\xi(x)dx \leqslant &\frac{\psi(a)+\psi(b)}{2}\int_a^b \xi(x)dx \\ &+ \frac{\eta(\psi(a),\psi(b))+\eta(\psi(b),\psi(a))}{2(b-a)}\int_a^b (b-x)\xi(x)dx.\end{aligned}$$

Now, we define the η-convex and strongly η-convex function of higher order in $\mathbb{R}^n$.

DEFINITION 3. A function $\psi : X \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be η-convex function with respect to $\eta : \psi(X) \times \psi(X) \longrightarrow \mathbb{R}$, if

$$\psi(tx+(1-t)y) \leqslant \psi(y) + t\eta(\psi(x),\psi(y)), \quad \forall x,y \in X, \quad t \in [0,1].$$

DEFINITION 4. A function $\psi : X \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be strongly η-convex function of order $\sigma > 0$ with respect to $\eta : \psi(X) \times \psi(X) \longrightarrow \mathbb{R}$ and modulus $c > 0$, if

$$\psi(tx+(1-t)y) \leqslant \psi(y) + t\eta(\psi(x),\psi(y)) - ct(1-t)\|x-y\|^\sigma, \quad \forall x,y \in X, \quad t \in [0,1]. \tag{1}$$

EXAMPLE 1. $X=\mathbb{R}^+$, $\psi(x)=4x$, $\eta(x,y)=\exp(x-y)^4+x$. Then, ψ is strongly η-convex function of order 4 with modulus 1.

When $\sigma=2$, this property reduces to the strongly η-convex function in the ordinary sense. But if σ not equal to 2, they are different.

EXAMPLE 2. $X=\mathbb{R}^+\cup\{0\}$, $\psi(x)=x$, $\eta(x,y)=(x-y)^4+x+y$. Then, ψ is strongly η-convex function of order 4 with modulus 1 and is not strongly η-convex function of order 2.

REMARK 1. If $X\subset\mathbb{R}$ and $x=y$ in (1), then (1) reduces to Remark 1.4 of [1].

LEMMA 1. [8] *If $\psi^{(n)}$ for $n\in\mathbb{N}$ exists and is integrable on $[a,b]$, then*

$$\frac{\psi(a)+\psi(b)}{2}-\frac{1}{b-a}\int_a^b\psi(x)dx-\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}\psi^{(k)}(a)$$
$$=\frac{(b-a)^n}{2n!}\int_0^1 t^{n-1}(n-2t)\psi^{(n)}(ta+(1-t)b)dt.$$

THEOREM 4. *Let $\psi: X\subseteq\mathbb{R}^n\longrightarrow\mathbb{R}$ be a differentiable strongly η-convex function of order $\sigma>0$. If ψ has minimum at y, then*

$$\eta(\psi(x),\psi(y))-c\|x-y\|^\sigma\geqslant 0. \tag{2}$$

Proof. Since ψ has minimum at y, then $\nabla\psi(y)=0$ and the condition

$$\langle\nabla\psi(y),x-y\rangle\geqslant 0 \tag{3}$$

is satisfied automatically.

We know that ψ is strongly η-convex of order $\sigma>0$, then

$$\psi(tx+(1-t)y)\leqslant\psi(y)+t\eta(\psi(x),\psi(y))-ct(1-t)\|x-y\|^\sigma.$$

Dividing above inequality by t and taking limit $t\longrightarrow 0$ on both sides, we have

$$\langle\nabla\psi(y),x-y\rangle\leqslant\eta(\psi(x),\psi(y))-c\|x-y\|^\sigma. \tag{4}$$

From (3) and (4), we have $\eta(\psi(x),\psi(y))-c\|x-y\|^\sigma\geqslant 0$.

This completes the proof. □

REMARK 2. When $X\subset\mathbb{R}$ and $\sigma=2$, then above theorem reduces to Theorem 1.6 of [1].

3. Main results

In this section, we derive our main results.

THEOREM 5. *Let* $\psi : [a,b] \longrightarrow \mathbb{R}$ *be strongly* η*-convex function of order* $\sigma > 0$ *with modulus* $c > 0$. *If* $\eta(.,.)$ *is bounded from above on* $\psi([a,b]) \times \psi([a,b])$, *then*

$$\begin{aligned}\psi\left(\frac{a+b}{2}\right) - \frac{M_\eta}{2} + \frac{c}{4(\sigma+1)}\|b-a\|^\sigma &\leqslant \frac{1}{b-a}\int_a^b \psi(x)dx \\ &\leqslant \frac{\psi(a)+\psi(b)}{2} + \frac{\eta(\psi(a),\psi(b))+\eta(\psi(b),\psi(a))}{4} - \frac{c}{6}\|b-a\|^\sigma \\ &\leqslant \frac{\psi(a)+\psi(b)}{2} + \frac{M_\eta}{2} - \frac{c}{6}\|b-a\|^\sigma,\end{aligned}$$

where M_η *is upper bound of* η.

Proof. Since ψ is strongly η-convex of order σ, then

$$\begin{aligned}\psi\left(\frac{a+b}{2}\right) &= \psi\left(\frac{1}{2}\left(\frac{a+b-t(b-a)}{2}\right) + \frac{1}{2}\left(\frac{a+b+t(b-a)}{2}\right)\right) \\ &\leqslant \psi\left(\frac{a+b+t(b-a)}{2}\right) \\ &\quad + \frac{1}{2}\eta\left(\psi\left(\frac{a+b-t(b-a)}{2}\right), \psi\left(\frac{a+b+t(b-a)}{2}\right)\right) - \frac{c}{4}t^\sigma\|b-a\|^\sigma \\ &\leqslant \psi\left(\frac{a+b+t(b-a)}{2}\right) + \frac{M_\eta}{2} - \frac{c}{4}t^\sigma\|b-a\|^\sigma.\end{aligned}$$

This implies

$$\psi\left(\frac{a+b}{2}\right) - \frac{M_\eta}{2} + \frac{c}{4}t^\sigma\|b-a\|^\sigma \leqslant \psi\left(\frac{a+b+t(b-a)}{2}\right). \tag{5}$$

Similarly,

$$\psi\left(\frac{a+b}{2}\right) - \frac{M_\eta}{2} + \frac{c}{4}t^\sigma\|b-a\|^\sigma \leqslant \psi\left(\frac{a+b-t(b-a)}{2}\right). \tag{6}$$

By using the change of variable technique, we have

$$\begin{aligned}\frac{1}{b-a}\int_a^b \psi(x)dx &= \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} \psi(x)dx + \int_{\frac{a+b}{2}}^b \psi(x)dx\right] \\ &= \frac{1}{2}\int_0^1 \psi\left(\frac{a+b-t(b-a)}{2}\right)dt + \frac{1}{2}\int_0^1 \psi\left(\frac{a+b+t(b-a)}{2}\right)dt \\ &\geqslant \int_0^1 \left[\psi\left(\frac{a+b}{2}\right) - \frac{M_\eta}{2} + \frac{c}{4}t^\sigma\|b-a\|^\sigma\right]dt \\ &= \psi\left(\frac{a+b}{2}\right) - \frac{M_\eta}{2} + \frac{c}{4(\sigma+1)}\|b-a\|^\sigma.\end{aligned}$$

We now prove the right hand side of the theorem. Since ψ is strongly η-convex function of order $\sigma > 0$, we have

$$\psi(ta+(1-t)b) \leqslant \psi(b)+t\eta(\psi(a),\psi(b)) - ct(1-t)\|b-a\|^{\sigma}.$$

Integrating above inequality with respect to t on $[0,1]$, we have

$$\int_0^1 \psi(ta+(1-t)b)dt \leqslant \int_0^1 [\psi(b)+t\eta(\psi(a),\psi(b)) - ct(1-t)\|b-a\|^{\sigma}]dt,$$

$$\frac{1}{b-a}\int_a^b \psi(x)dx \leqslant \psi(b)+\frac{1}{2}\eta(\psi(a),\psi(b)) - \frac{c}{6}\|b-a\|^{\sigma} = P.$$

Similarly,

$$\frac{1}{b-a}\int_a^b \psi(x)dx \leqslant \psi(a)+\frac{1}{2}\eta(\psi(b),\psi(a)) - \frac{c}{6}\|b-a\|^{\sigma} = Q.$$

Therefore,

$$\begin{aligned}\frac{1}{b-a}\int_a^b \psi(x)dx &\leqslant Min\{P,Q\}\\ &\leqslant \frac{\psi(a)+\psi(b)}{2} + \frac{\eta(\psi(a)+\psi(b))+\eta(\psi(b),\psi(a))}{4} - \frac{c}{6}\|b-a\|^{\sigma}\\ &\leqslant \frac{\psi(a)+\psi(b)}{2} + \frac{M_{\eta}}{2} - \frac{c}{6}\|b-a\|^{\sigma}.\end{aligned}$$

This completes the proof. □

REMARK 3. When $\sigma = 2$, then above theorem reduces to Theorem 2.1 of [1]. If we consider $c = 0$, then above theorem reduces to Theorem 1.

Now we establish the result on Féjer type inequality for strongly η-convex function of order $\sigma > 0$.

THEOREM 6. *Let $\psi : [a,b] \longrightarrow \mathbb{R}$ be a strongly η-convex function of order σ, such that $\eta(.,.)$ is bounded above on $\psi([a,b]) \times \psi([a,b])$. Also suppose that ξ : $[a,b] \longrightarrow \mathbb{R}^{+}$ is integrable and symmetric with respect to $\frac{a+b}{2}$, then*

$$\begin{aligned}\psi\left(\frac{a+b}{2}\right)\int_a^b \xi(x)dx + \frac{c}{4}\int_a^b \|(a+b-2x)\|^{\sigma}\xi(x)dx - L_{\eta}(a,b)\\ \leqslant \int_a^b \psi(x)\xi(x)dx\\ \leqslant \frac{\psi(a)+\psi(b)}{2}\int_a^b \xi(x)dx\\ - c\|b-a\|^{(\sigma-2)}\int_a^b (b-x)(x-a)\xi(x)dx + R_{\eta}(a,b),\end{aligned}$$

where

$$L_\eta(a,b) = \frac{1}{2}\int_a^b \eta(\psi(a+b-x), \psi(x))\xi(x)dx$$

and $R_\eta(a,b) = \frac{\eta(\psi(a),\psi(b))+\eta(\psi(b),\psi(a))}{2(b-a)}\int_a^b (b-x)\xi(x)dx$, *respectively.*

Proof. First, we prove the first pair inequality of the theorem.

Since ψ is strongly η-convex function of order σ, then

$$\begin{aligned}\psi\left(\frac{a+b}{2}\right) =&\psi\left(\frac{1}{2}((1-t)b+ta)+\frac{1}{2}((1-t)a+tb)\right)\\ &\leqslant \psi((1-t)a+tb)+\frac{1}{2}\eta\left(\psi((1-t)b+ta), \psi((1-t)a+tb)\right)\\ &-\frac{c}{4}\|(b-a)(1-2t)\|^{\sigma}.\end{aligned}$$

Since $\xi : [a,b] \longrightarrow \mathbb{R}^+$ is integrable and symmetric with respect to $\frac{a+b}{2}$, then

$$\begin{aligned}&\psi\left(\frac{a+b}{2}\right)\int_a^b \xi(x)dx\\ &= (b-a)\psi\left(\frac{a+b}{2}\right)\int_0^1 \xi((1-t)a+tb)dt\\ &\leqslant (b-a)\int_0^1 \psi((1-t)a+tb)\xi((1-t)a+tb)dt\\ &\quad + \frac{(b-a)}{2}\int_0^1 \eta\left(\psi((1-t)b+ta), \psi((1-t)a+tb)\right)\xi((1-t)a+tb)dt\\ &\quad - \frac{c}{4}(b-a)\int_0^1 \|(b-a)(1-2t)\|^{\sigma}\xi((1-t)a+tb)dt\\ &= \int_a^b \psi(x)\xi(x)dx + \frac{1}{2}\int_a^b \eta\left(\psi(a+b-x), \psi(x)\right)\xi(x)dx\\ &\quad - \frac{c}{4}\int_a^b \|a+b-2x\|^{\sigma}\xi(x)dx.\end{aligned}$$

Next, we prove the second pair inequality of the theorem,

$$\int_a^b \psi(x)\xi(x)dx = (b-a)\int_0^1 \psi(ta+(1-t)b)\xi(ta+(1-t)b)dt.$$

Using the definition of strongly η-convex function of order σ, we have

$$\begin{aligned}\int_a^b \psi(x)\xi(x)dx \leqslant (b-a)\Bigg[&\psi(b)\int_0^1 \xi(ta+(1-t)b)dt\\ &+ \eta(\psi(a), \psi(b))\int_0^1 t\xi(ta+(1-t)b)dt\\ &- c\|b-a\|^{\sigma}\int_0^1 t(1-t)\xi(ta+(1-t)b)dt\Bigg].\end{aligned} \tag{7}$$

Similarly,

$$\begin{aligned}\int_a^b \psi(x)\xi(x)dx \leqslant (b-a)\Big[\psi(a)\int_0^1 \xi(ta+(1-t)b)dt \\ + \eta(\psi(b),\psi(a))\int_0^1 t\xi(ta+(1-t)b)dt \\ - c\|b-a\|^{\sigma}\int_0^1 t(1-t)\xi(ta+(1-t)b)dt\Big].\end{aligned} \tag{8}$$

From (7) and (8), we have

$$\begin{aligned}2\int_a^b \psi(x)\xi(x)dx \leqslant (b-a)[\psi(a)+\psi(b)]\int_0^1 \xi(ta+(1-t)b)dt \\ + (b-a)[\eta(\psi(a),\psi(b))+\eta(\psi(b),\psi(a))]\int_0^1 t\xi(ta+(1-t)b)dt \\ - 2c\|b-a\|^{(\sigma+1)}\int_0^1 t(1-t)\xi(ta+(1-t)b)dt.\end{aligned}$$

Applying change of variable technique in above inequality, we have

$$\begin{aligned}\int_a^b \psi(x)\xi(x)dx \leqslant \frac{[\psi(a)+\psi(b)]}{2}\int_a^b \xi(x)dx \\ + \frac{[\eta(\psi(a),\psi(b))+\eta(\psi(b),\psi(a))]}{2(b-a)}\int_a^b (b-x)\xi(x)dx \\ - c\|b-a\|^{(\sigma-2)}\int_a^b (b-x)(x-a)\xi(x)dx.\end{aligned}$$

This completes the proof. □

REMARK 4. When $c=0$, then the first pair of inequality of the above theorem reduces to Theorem 2 and the second pair of inequality of the above theorem reduces to Theorem 3.

Now, we discuss a new variant of Hermite-Hadamard inequality for differentiable strongly η-convex of order σ.

THEOREM 7. *Let $\psi : I^0 \subset \mathbb{R} \longrightarrow \mathbb{R}$ be n-times differentiable strongly η-convex function of order σ on I^0 where $a,b \in I^0$ with $a<b$ and $\psi' \in L_1[a,b]$. If $|\psi^{(n)}|^p$ is strongly η-convex function of order σ with $c \geqslant 1$, then for $n \geqslant 2$ and $p \geqslant 1$, we have*

$$\begin{aligned}\left|\frac{\psi(a)+\psi(b)}{2} - \frac{1}{b-a}\int_a^b \psi(x)dx - \sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}\psi^{(k)}(a)\right| \\ \leqslant \frac{(b-a)^n}{2n!}\alpha^{(1-\frac{1}{p})}(n)[\alpha(n)|\psi^{(n)}(b)|^p + \beta(n)\eta(|\psi^{(n)}(a)|^p,|\psi^{(n)}(b)|^p) \\ - c\gamma(n)\|a-b\|^{\sigma}]^{\frac{1}{p}},\end{aligned}$$

where

$$\alpha(n) := \frac{n-1}{n+1},$$
$$\beta(n) := \frac{n^2-2}{(n+1)(n+2)},$$
$$\gamma(n) := \frac{n-1}{(n+1)(n+3)}.$$

Proof. Recall Lemma 1;

$$\frac{\psi(a)+\psi(b)}{2} - \frac{1}{b-a}\int_a^b \psi(x)dx - \sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}\psi^{(k)}(a)$$
$$= \frac{(b-a)^n}{2n!}\int_0^1 t^{n-1}(n-2t)\psi^{(n)}(ta+(1-t)b)dt.$$

Case 1. When $p=1$, using the definition of strong η-convexity of order σ, we have

$$\left|\frac{\psi(a)+\psi(b)}{2} - \frac{1}{b-a}\int_a^b \psi(x)dx - \sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}\psi^{(k)}(a)\right|$$
$$\leqslant \frac{(b-a)^n}{2n!}\left[\int_0^1 t^{n-1}(n-2t)|\psi^{(n)}(ta+(1-t)b)|dt\right]$$
$$\leqslant \frac{(b-a)^n}{2n!}\left[|\psi^{(n)}(b)|\int_0^1 t^{n-1}(n-2t)dt\right.$$
$$+\eta(|\psi^{(n)}(a)|,|\psi^{(n)}(b)|)\int_0^1 t^n(n-2t)dt$$
$$\left.-c\|a-b\|^\sigma\int_0^1 t^n(1-t)(n-2t)dt\right]$$
$$= \frac{(b-a)^n}{2n!}\left[\frac{n-1}{n+1}|\psi^{(n)}(b)| + \frac{n^2-2}{(n+1)(n+2)}\eta(|\psi^{(n)}(a)|,|\psi^{(n)}(b)|)\right.$$
$$\left.-\frac{c(n-1)}{(n+1)(n+3)}\|a-b\|^\sigma\right].$$

Case 2. When $p>1$,

$$\left|\frac{\psi(a)+\psi(b)}{2} - \frac{1}{b-a}\int_a^b \psi(x)dx - \sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}\psi^{(k)}(a)\right|$$
$$\leqslant \frac{(b-a)^n}{2n!}\left[\int_0^1 (t^{n-1}(n-2t))^{1-\frac{1}{p}}(t^{n-1}(n-2t))^{\frac{1}{p}}|\psi^{(n)}(ta+(1-t)b)|dt\right].$$

Using Hölder's inequality, we have

$$\begin{aligned}&\left|\frac{\psi(a)+\psi(b)}{2}-\frac{1}{b-a}\int_a^b\psi(x)dx-\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}\psi^{(k)}(a)\right|\\&\leqslant\frac{(b-a)^n}{2n!}\left[\int_0^1 t^{n-1}(n-2t)dt\right]^{1-\frac{1}{p}}\left[\int_0^1 t^{n-1}(n-2t)|\psi^{(n)}(ta+(1-t)b)|^p dt\right]^{\frac{1}{p}}\\&\leqslant\frac{(b-a)^n}{2n!}\left(\frac{n-1}{n+1}\right)^{1-\frac{1}{p}}\left[|\psi^{(n)}(b)|^p\int_0^1 t^{n-1}(n-2t)dt\right.\\&\quad\left.+\eta(|\psi^{(n)}(a)|^p,|\psi^{(n)}(b)|^p)\int_0^1 t^n(n-2t)dt-c\|a-b\|^\sigma\int_0^1 t^n(1-t)(n-2t)dt\right]^{\frac{1}{p}}.\end{aligned}$$

This implies

$$\begin{aligned}&\left|\frac{\psi(a)+\psi(b)}{2}-\frac{1}{b-a}\int_a^b\psi(x)dx-\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}\psi^{(k)}(a)\right|\\&\leqslant\frac{(b-a)^n}{2n!}\left(\frac{n-1}{n+1}\right)^{1-\frac{1}{p}}\left[\frac{n-1}{n+1}|\psi^{(n)}(b)|^p+\frac{n^2-2}{(n+1)(n+2)}\eta(|\psi^{(n)}(a)|^p,|\psi^{(n)}(b)|^p)\right.\\&\quad\left.-\frac{c(n-1)}{(n+1)(n+3)}\|a-b\|^\sigma\right]^{\frac{1}{p}}.\end{aligned}$$

This completes the proof. □

4. Conclusion

The strong convexity of a function is the basis for many inequalities in mathematics. We introduced the concept of strongly η-convex functions of higher order, as a generalization of the strongly η-convex functions. Some new counterparts of Hermite-Hadamard and Hermite-Hadamard-Féjer type inequalities for strongly η-convex functions of higher order are obtained.

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