Mathematical
Inequalities
& Applications
Volume 22, Number 1 (2019), 123–132 doi:10.7153/mia-2019-22-09

WINTGEN INEQUALITY FOR STATISTICAL SURFACES

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(*Communicated by S. Varošanec*)

Abstract. The Wintgen inequality (1979) is a sharp geometric inequality for surfaces in the 4-dimensional Euclidean space involving the Gauss curvature (intrinsic invariant) and the normal curvature and squared mean curvature (extrinsic invariants), respectively. In the present paper we obtain a Wintgen inequality for statistical surfaces.

1. Preliminaries

For surfaces M^2 of the Euclidean space $\mathbb{E}^3$, the Euler inequality $K \leqslant \|H\|^2$ is fulfilled, where K is the (intrinsic) Gauss curvature of M^2 and $\|H\|^2$ is the (extrinsic) squared mean curvature of M^2.

Furthermore, $K = \|H\|^2$ everywhere on M^2 if and only if M^2 is totally umbilical, or still, by a theorem of Meusnier, if and only if M^2 is (a part of) a plane $\mathbb{E}^2$ or, it is (a part of) a round sphere S^2 in $\mathbb{E}^3$.

In 1979, P. Wintgen [30] proved that the Gauss curvature K, the squared mean curvature $\|H\|^2$ and the normal curvature $K^\perp$ of any surface M^2 in $\mathbb{E}^4$ always satisfy the inequality

$$K \leqslant \|H\|^2 - |K^\perp|;$$

the equality holds if and only if the ellipse of curvature of M^2 in $\mathbb{E}^4$ is a circle.

For some explicit examples of surfaces satisfying the equality case of Wintgen's inequality, see, e.g., [7]

The Whitney 2-sphere satisfies the equality case of the Wintgen inequality identically.

A survey containing recent results on surfaces satisfying identically the equality case of Wintgen inequality can be read in [5].

Later, the Wintgen inequality was extended by B. Rouxel [26] and by I. V. Guadalupe and L. Rodriguez [13] independently, for surfaces M^2 of arbitrary codimension m in real space forms $\widetilde{M}^{2+m}(c)$; namely

$$K \leqslant \|H\|^2 - |K^\perp| + c.$$

The equality case was also investigated.

Mathematics subject classification (2010): 53C05, 53C40, 53A40.
Keywords and phrases: Wintgen inequality, statistical manifold, statistical surface, dual connections.

A corresponding inequality for totally real surfaces in n-dimensional complex space forms was obtained in [18]. The equality case was studied and a non-trivial example of a totally real surface satisfying the equality case identically was given (see also [19]).

In 1999, P. J. De Smet, F. Dillen, L. Verstraelen and L. Vrancken [8] formulated the conjecture on Wintgen inequality for submanifolds of real space forms, which is also known as the *DDVV conjecture*.

This conjecture was proven by the authors for submanifolds M^n of arbitrary dimension $n \geqslant 2$ and codimension 2 in real space forms $\tilde{M}^{n+2}(c)$ of constant sectional curvature c.

Recently, the DDVV conjecture was finally settled for the general case by Z. Lu [17] and independently by J. Ge and Z. Tang [12].

One of the present authors obtained generalized Wintgen inequalities for Lagrangian submanifolds in complex space forms [20] and Legendrian submanifolds in Sasakian space forms [21], respectively.

2. Statistical manifolds and their submanifolds

A *statistical manifold* is a Riemannian manifold $\left(\tilde{M}^{n+k}, \tilde{g}\right)$ of dimension $(n+k)$, endowed with a pair of torsion-free affine connections $\tilde{\nabla}$ and $\tilde{\nabla}^*$ satisfying

$$Z\tilde{g}(X,Y) = \tilde{g}\left(\tilde{\nabla}_Z X, Y\right) + \tilde{g}\left(X, \tilde{\nabla}^*_Z Y\right), \tag{2.1}$$

for any X,Y and $Z \in \Gamma\left(T\tilde{M}\right)$. The connections $\tilde{\nabla}$ and $\tilde{\nabla}^*$ are called *dual connections* (see [1], [23]), and it is easily shown that $\left(\tilde{\nabla}^*\right)^* = \tilde{\nabla}$. The pairing $\left(\tilde{\nabla}, \tilde{g}\right)$ is said to be a *statistical structure*. If $\left(\tilde{\nabla}, \tilde{g}\right)$ is a statistical structure on $\tilde{M}^{n+k}$, so is $\left(\tilde{\nabla}^*, \tilde{g}\right)$ [1, 29].

On the other hand, any torsion-free affine connection $\tilde{\nabla}$ always has a dual connection given by

$$\tilde{\nabla} + \tilde{\nabla}^* = 2\tilde{\nabla}^0, \tag{2.2}$$

where $\tilde{\nabla}^0$ is Levi-Civita connection on $\tilde{M}^{n+k}$.

In affine differential geometry the dual connections are called *conjugate connections* (see [15], [9]).

Denote by $\tilde{R}$ and $\tilde{R}^*$ the curvature tensor fields of $\tilde{\nabla}$ and $\tilde{\nabla}^*$, respectively.

A statistical structure $\left(\tilde{\nabla}, \tilde{g}\right)$ is said to be of constant curvature $c \in \mathbb{R}$ if

$$\tilde{R}(X,Y)Z = c\left\{\tilde{g}(Y,Z)X - \tilde{g}(X,Z)Y\right\}. \tag{2.3}$$

A statistical structure $\left(\tilde{\nabla}, \tilde{g}\right)$ of constant curvature 0 is called *a Hessian structure.*

The curvature tensor fields $\tilde{R}$ and $\tilde{R}^*$ of dual connections satisfy

$$\tilde{g}\left(\tilde{R}^*(X,Y)Z, W\right) = -\tilde{g}\left(Z, \tilde{R}(X,Y)W\right). \tag{2.4}$$

From (2.4) it follows immediately that if $\left(\tilde{\nabla}, \tilde{g}\right)$ is a statistical structure of constant curvature c, then $\left(\tilde{\nabla}^*, \tilde{g}\right)$ is also statistical structure of constant curvature c. In particular, if $\left(\tilde{\nabla}, \tilde{g}\right)$ is Hessian, so is $\left(\tilde{\nabla}^*, \tilde{g}\right)$ [10].

If $\left(\tilde{M}^{n+k},\tilde{g}\right)$ is a statistical manifold and M^n a submanifold of dimension n of $\tilde{M}^{n+k}$, then (M^n,g) is also a statistical manifold with the induced connection ∇ by $\tilde{\nabla}$ and induced metric g. In the case that $\left(\tilde{M}^{n+k},\tilde{g}\right)$ is a semi-Riemannian manifold, the induced metric g has to be non-degenerate. For details, see ([28, 29]).

In the geometry of Riemannian submanifolds (see [4]), the fundamental equations are the Gauss and Weingarten formulas and the equations of Gauss, Codazzi and Ricci.

Let denote the set of the sections of the normal bundle to M^n by $\Gamma\left(TM^{n\perp}\right)$.

In our case, for any $X,Y\in\Gamma\left(TM^n\right)$, according to [29], the corresponding Gauss formulas are

$$\tilde{\nabla}_X Y=\nabla_X Y+h\left(X,Y\right), \tag{2.5}$$

$$\tilde{\nabla}^*_X Y=\nabla^*_X Y+h^*\left(X,Y\right), \tag{2.6}$$

where $h,\ h^*:\Gamma(TM^n)\times\Gamma(TM^n)\to\Gamma(TM^{n\perp})$ are symmetric and bilinear, called the imbedding curvature tensor of M^n in $\tilde{M}^{n+k}$ for $\tilde{\nabla}$ and the imbedding curvature tensor of M^n in $\tilde{M}^{n+k}$ for $\tilde{\nabla}^*$, respectively.

In [29], it is also proved that (∇,g) and (∇^*,g) are dual statistical structures on M^n.

Since h and h^* are bilinear, we have the linear transformations A_ξ and A^*_ξ on TM^n defined by

$$g\left(A_\xi X,Y\right)=\tilde{g}\left(h\left(X,Y\right),\xi\right), \tag{2.7}$$

$$g\left(A^*_\xi X,Y\right)=\tilde{g}\left(h^*\left(X,Y\right),\xi\right), \tag{2.8}$$

for any $\xi\in\Gamma\left(TM^{n\perp}\right)$ and $X,Y\in\Gamma\left(TM^n\right)$. Further, see [29], the corresponding Weingarten formulas are

$$\tilde{\nabla}_X\xi=-A^*_\xi X+\nabla^\perp_X\xi, \tag{2.9}$$

$$\tilde{\nabla}^*_X\xi=-A_\xi X+\nabla^{*\perp}_X\xi, \tag{2.10}$$

for any $\xi\in\Gamma\left(TM^{n\perp}\right)$ and $X\in\Gamma\left(TM^n\right)$. The connections $\nabla^\perp_X$ and $\nabla^{*\perp}_X$ given by (2.9) and (2.10) are Riemannian dual connections with respect to induced metric on $\Gamma\left(TM^{n\perp}\right)$.

Let $\{e_1,\ldots,e_n\}$ and $\{e_{n+1},\ldots,e_{n+k}\}$ be orthonormal tangent and normal frames, respectively, on M. Then the mean curvature vector fields are defined by

$$H=\frac{1}{n}\sum_{i=1}^{n}h\left(e_i,e_i\right)=\frac{1}{n}\sum_{\alpha=1}^{k}\left(\sum_{i=1}^{n}h^\alpha_{ii}\right)e_{n+\alpha},\ h^\alpha_{ij}=\tilde{g}\left(h\left(e_i,e_j\right),e_{n+\alpha}\right) \tag{2.11}$$

and

$$H^*=\frac{1}{n}\sum_{i=1}^{n}h^*\left(e_i,e_i\right)=\frac{1}{n}\sum_{\alpha=1}^{k}\left(\sum_{i=1}^{n}h^{*\alpha}_{ii}\right)e_{n+\alpha},\ h^{*\alpha}_{ij}=\tilde{g}\left(h^*\left(e_i,e_j\right),e_{n+\alpha}\right), \tag{2.12}$$

for $1\leqslant i,j\leqslant n$ and $1\leqslant\alpha\leqslant k$ (see also [6]).

The corresponding Gauss, Codazzi and Ricci equations are given by the following result.

THEOREM 1. [29] *Let $\tilde{\nabla}$ be a dual connection on $\tilde{M}^{n+k}$ and ∇ the induced connection on M^n. Let $\tilde{R}$ and R be the Riemannian curvature tensors of $\tilde{\nabla}$ and ∇, respectively. Then,*

$$\tilde{g}\left(\tilde{R}(X,Y)Z,W\right) = g(R(X,Y)Z,W) + \tilde{g}(h(X,Z),h^*(Y,W)) - \tilde{g}(h^*(X,W),h(Y,Z)), \tag{2.13}$$

$$\begin{aligned}\left(\tilde{R}(X,Y)Z\right)^{\perp} = {} & \nabla_X^{\perp}h(Y,Z) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z) \\ & - \left\{\nabla_Y^{\perp}h(Y,Z) - h(\nabla_Y X,Z) - h(X,\nabla_Y Z)\right\},\end{aligned}$$

$$\tilde{g}\left(R^{\perp}(X,Y)\xi,\eta\right) = \tilde{g}\left(\tilde{R}(X,Y)\xi,\eta\right) + g\left(\left[A_\xi^*,A_\eta\right]X,Y\right), \tag{2.14}$$

where $R^{\perp}$ is the Riemannian curvature tensor on $TM^{n\perp}$, $\xi,\eta \in \Gamma\left(TM^{n\perp}\right)$ and $\left[A_\xi^*,A_\eta\right]$ $= A_\xi^* A_\eta - A_\eta A_\xi^*$.

For the equations of Gauss, Codazzi and Ricci with respect to the dual connection $\tilde{\nabla}^*$ on M^n, we have

THEOREM 2. *Let $\tilde{\nabla}^*$ be a dual connection on $\tilde{M}^{n+k}$ and ∇^* the induced connection on M^n. Let $\tilde{R}^*$ and R^* be the Riemannian curvature tensors for $\tilde{\nabla}^*$ and ∇^*, respectively. Then,*

$$\tilde{g}\left(\tilde{R}^*(X,Y)Z,W\right) = g(R^*(X,Y)Z,W) + \tilde{g}(h^*(X,Z),h(Y,W)) - \tilde{g}(h(X,W),h^*(Y,Z)), \tag{2.15}$$

$$\begin{aligned}\left(\tilde{R}^*(X,Y)Z\right)^{\perp} = {} & \nabla_X^{*\perp}h^*(Y,Z) - h^*(\nabla_X^* Y,Z) - h^*(Y,\nabla_X^* Z) \\ & - \left\{\nabla_Y^{*\perp}h^*(Y,Z) - h^*(\nabla_Y^* X,Z) - h^*(X,\nabla_Y^* Z)\right\}\end{aligned}$$

$$\tilde{g}\left(R^{*\perp}(X,Y)\xi,\eta\right) = \tilde{g}\left(\tilde{R}^*(X,Y)\xi,\eta\right) + g\left(\left[A_\xi,A_\eta^*\right]X,Y\right), \tag{2.16}$$

where $R^{*\perp}$ is the Riemannian curvature tensor for $\nabla^{\perp *}$ on $TM^{n\perp}$, $\xi,\eta \in \Gamma\left(TM^{n\perp}\right)$ and $\left[A_\xi,A_\eta^*\right] = A_\xi A_\eta^* - A_\eta^* A_\xi$.

Geometric inequalities for statistical submanifolds in statistical manifolds with constant curvature were obtained in [2].

3. Sectional curvature for statistical manifolds

Let (M^n,g) be a statistical manifold of dimension n endowed with dual connections $\tilde{\nabla}$ and $\tilde{\nabla}^*$. Unfortunately, the $(0,4)$-tensor field $g(R(X,Y)Z,W)$ is not skew-symmetric relative to Z and W. Then we cannot define a sectional curvature on M^n by the standard definition.

We shall define a skew-symmetric $(0,4)$-tensor field on M^n by

$$T(X,Y,Z,W) = \frac{1}{2}\left[g(R(X,Y)Z,W) + g(R^*(X,Y)Z,W)\right],$$

for all $X,Y,Z,W \in \Gamma(TM^n)$.

Then we are able to define a sectional curvature on M^n by the formula

$$K(X \wedge Y) = \frac{T(X,Y,X,Y)}{g(X,X)g(Y,Y) - g^2(X,Y)},$$

for any linearly independent tangent vectors X,Y at $p \in M^n$.

We want to point-out than this definition has the opposite sign that the sectional curvature defined by B. Opozda [25]. Another sectional curvature was considered in [24] (see also [27]).

In particular, for a statistical surface M^2, we can define a Gauss curvature by

$$G = K(e_1 \wedge e_2),$$

for any orthonormal frame $\{e_1,e_2\}$ on M^2.

Analogously, we shall consider a normal curvature of a statistical surface M^2 in an orientable 4-dimensional statistical manifold $\tilde{M}^4$. Let $\{e_1,e_2,e_3,e_4\}$ be a positive oriented orthonormal frame on $\tilde{M}^4$ such that e_1,e_2 are tangent to M^2. Let

$$G^\perp = \frac{1}{2}\left[g\left(R^\perp(e_1,e_2)e_3,e_4\right) + g\left(R^{*\perp}(e_1,e_2)e_3,e_4\right)\right]$$

be a normal curvature of M^2.

Remark that $\left|G^\perp\right|$ does not depend on the orientation of the statistical manifold. Then $\left|G^\perp\right|$ can be defined for any surface M^2 of any 4-dimensional statistical manifold.

We state a version of Euler inequality for surfaces in 3-dimensional statistical manifolds of constant curvature.

THEOREM 3. *Let M^2 be surface in a 3-dimensional statistical manifold of constant curvature c. Then its Gauss curvature satisfies:*

$$G \leqslant 2||H|| \cdot ||H^*|| - c.$$

Proof. Let $p \in M^2$ and e_3 be a unit normal vector to M^2 at p. We can choose an orthonormal basis $\{e_1,e_2\}$ of T_pM^2 such that $h^0(e_1,e_2) = 0$, where h^0 is the second fundamental form of M^2 (with respect to the Levi-Civita connection). Then $h^3_{12} + h^{*3}_{12} = 0$. The Gauss equations for ∇ and ∇^* imply

$$G = -c - \frac{1}{2}(h^3_{11}h^{*3}_{22} + h^{*3}_{11}h^3_{22}) + h^3_{12}h^{*3}_{12}.$$

Applying the Cauchy-Schwarz inequality, it follows that

$$G \leqslant -c + \frac{1}{2}\sqrt{(h^3_{11}+h^3_{22})^2(h^{*3}_{11}+h^{*3}_{22})^2} - (h^3_{12})^2.$$

But in our case $4||H||^2 = (h^3_{11}+h^3_{22})^2$ and $4||H^*||^2 = (h^{*3}_{11}+h^{*3}_{22})^2$. Therefore

$$G \leqslant -c + 2||H|| \cdot ||H^*||. \quad \square$$

EXAMPLE 1. (A trivial example) Recall Lemma 5.3 of Furuhata [10]. Let $\left(\mathbb{H},\tilde{\nabla},\tilde{g}\right)$ be a Hessian manifold of constant Hessian curvature $\tilde{c}\neq 0$, (M,∇,g) a trivial Hessian manifold and $f:M\longrightarrow\mathbb{H}$ a statistical immmersion of codimension one. Then one has:

$$A^*=0,\quad h^*=0,\quad \|H^*\|=0.$$

Thus, if dim $M=2$, the immersion f of codimension one satisfies the equality case of the statistical version of Euler inequality given by Theorem 3.

EXAMPLE 2. Let $\left(\mathbb{H}^3,\tilde{g}\right)$ be the upper half space of constant sectional curvature -1, i.e.,

$$\mathbb{H}^3=\left\{y=\left(y^1,y^2,y^3\right)\in\mathbb{R}^3:y^3>0\right\},\quad \tilde{g}=\left(y^3\right)^{-2}\sum_{k=1}^{3}dy^kdy^k.$$

An affine connection $\tilde{\nabla}$ on $\mathbb{H}$ is given by

$$\tilde{\nabla}_{\frac{\partial}{\partial y^3}}\frac{\partial}{\partial y^3}=\left(y^3\right)^{-1}\frac{\partial}{\partial y^3},\quad \tilde{\nabla}_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j}=2\delta_{ij}\left(y^3\right)^{-1}\frac{\partial}{\partial y^3},\quad \tilde{\nabla}_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^3}=\tilde{\nabla}_{\frac{\partial}{\partial y^3}}\frac{\partial}{\partial y^j}=0,$$

where $i,j=1,2$. The curvature tensor field $\tilde{R}$ of $\tilde{\nabla}$ is identically zero, i.e., $c=0$. Thus $\left(\mathbb{H}^3,\tilde{\nabla},\tilde{g}\right)$ is a Hessian manifold of constant Hessian curvature 4. Now let consider a horosphere M^2 in $\mathbb{H}^3$ having null Gauss curvature, i.e., $G\equiv 0$. (For details, see [16]). If $f:M^2\longrightarrow\mathbb{H}^3$ is a statistical immersion of codimension one, then, by using Lemma 4.1 of [22], we deduce $A^*=0$, and then $H^*=0$. This implies that the horosphere M^2 satisfies the equality case of the statistical version of Euler inequality given by Theorem 3.

4. Wintgen inequality for statistical surfaces in a 4-dimensional statistical manifold of constant curvature

Let $\left(\tilde{M}^4,c\right)$ be a statistical manifold of constant curvature c and M^2 a statistical surface in $\left(\tilde{M}^4,c\right)$.

We shall prove a Wintgen inequality for the surfaces M^2 in $\left(\tilde{M}^4,c\right)$. The Gauss curvature G of M^2 is given by

$$G=\frac{1}{2}\left[g\left(R\left(e_1,e_2\right)e_1,e_2\right)+g\left(R^*\left(e_1,e_2\right)e_1,e_2\right)\right].$$

By the Gauss equation we have

$$g\left(R\left(e_1,e_2\right)e_1,e_2\right)=g\left(\tilde{R}\left(e_1,e_2\right)e_1,e_2\right)-g\left(h\left(e_1,e_1\right),h^*\left(e_2,e_2\right)\right)+g\left(h^*\left(e_1,e_2\right),h\left(e_1,e_2\right)\right),$$

or equivalently,

$$g\left(R\left(e_1,e_2\right)e_1,e_2\right)=-c-h_{11}^3h_{22}^{*3}-h_{11}^4h_{22}^{*4}+h_{12}^{*3}h_{12}^3+h_{12}^{*4}h_{12}^4.$$

Analogously

$$g\left(R^*\left(e_1,e_2\right)e_1,e_2\right)=-c-h^{*3}_{11}h^3_{22}-h^{*4}_{11}h^4_{22}+h^3_{12}h^{*3}_{12}+h^4_{12}h^{*4}_{12}.$$

It follows that

$$G=-c-\frac{1}{2}\left[h^3_{11}h^{*3}_{22}+h^{*3}_{11}h^3_{22}+h^4_{11}h^{*4}_{22}+h^{*4}_{11}h^4_{22}\right]+h^3_{12}h^{*3}_{12}+h^4_{12}h^{*4}_{12}.$$

The normal curvature $G^\perp$ of M^2 is given by

$$2\left|G^\perp\right|=\frac{1}{2}\left|g\left(R^\perp\left(e_1,e_2\right)e_3,e_4\right)+g\left(R^{*\perp}\left(e_1,e_2\right)e_3,e_4\right)\right|.$$

By the Ricci equations with respect to $\tilde{\nabla}$ and $\tilde{\nabla}^*$, respectively, we get

$$\begin{aligned}2\left|G^\perp\right| &= \left|g([A^*_{e_3},A_{e_4}]e_1,e_2)+g([A_{e_3},A^*_{e_4}]e_1,e_2)\right|\\ &= \left|h^{*3}_{12}\left(h^4_{11}-h^4_{22}\right)-h^{*4}_{12}\left(h^3_{11}-h^3_{22}\right)+h^3_{12}\left(h^{*4}_{11}-h^{*4}_{22}\right)-h^4_{12}\left(h^{*3}_{11}-h^{*3}_{22}\right)\right|.\end{aligned}$$

In order to estimate $\left|G^\perp\right|$, we shall use the inequalities

$$\pm 4ab\leqslant a^2+4b^2,\ a,b\in\mathbb{R}.$$

Then we have

$$\begin{aligned}2\left|G^\perp\right| &\leqslant \frac{1}{4}[(h^3_{11}-h^3_{22})^2+(h^4_{11}-h^4_{22})^2+(h^{*3}_{11}-h^{*3}_{22})^2+(h^{*4}_{11}-h^{*4}_{22})^2]\\ &\quad +(h^3_{12})^2+(h^4_{12})^2+(h^{*3}_{12})^2+(h^{*4}_{12})^2\\ &= \frac{1}{4}\left[\|h_{11}-h_{22}\|^2+\|h^*_{11}-h^*_{22}\|^2\right]+\|h_{12}\|^2+\|h^*_{12}\|^2,\end{aligned}$$

which yields that

$$\begin{aligned}2\left|G^\perp\right| &\leqslant \frac{1}{4}\left[\|h_{11}+h_{22}\|^2+\|h^*_{11}+h^*_{22}\|^2\right]-g\left(h_{11},h_{22}\right)-g\left(h^*_{11},h^*_{22}\right)+\|h_{12}\|^2+\|h^*_{12}\|^2\\ &= \|H\|^2+\|H^*\|^2-\left(h^3_{11}+h^{*3}_{11}\right)\left(h^3_{22}+h^{*3}_{22}\right)-\left(h^4_{11}+h^{*4}_{11}\right)\left(h^4_{22}+h^{*4}_{22}\right)\\ &\quad +h^3_{11}h^{*3}_{22}+h^{*3}_{11}h^3_{22}+h^4_{11}h^{*4}_{22}+h^{*4}_{11}h^4_{22}+\left(h^3_{12}\right)^2+\left(h^4_{12}\right)^2+\left(h^{*3}_{12}\right)^2+\left(h^{*4}_{12}\right)^2.\end{aligned}$$

It is known that $2h^0=h+h^*$, where h^0 denotes the second fundamental form of M^2 with respect to the Levi-Civita connection $\tilde{\nabla}^0$ on $\left(\tilde{M}^4,c\right)$.

Then we can write

$$\begin{aligned}2\left|G^\perp\right| &\leqslant \|H\|^2+\|H^*\|^2-4\left(h^{03}_{11}h^{03}_{22}+h^{04}_{11}h^{04}_{22}\right)-2G-2c\\ &\quad +2h^3_{12}h^{*3}_{12}+2h^4_{12}h^{*4}_{12}+\left(h^3_{12}\right)^2+\left(h^{*3}_{12}\right)^2+\left(h^4_{12}\right)^2+\left(h^{*4}_{12}\right)^2,\end{aligned}$$

where h^{03}_{ij} and h^{04}_{ij} are the components of the second fundamental form h^0 with respect to the Levi-Civita connection.

Recall the Gauss equation for the Levi-Civita connection

$$\tilde{K}^0(e_1 \wedge e_2) = G^0 - h_{11}^{03}h_{22}^{03} - h_{11}^{04}h_{22}^{04} + \left(h_{12}^{03}\right)^2 + \left(h_{12}^{04}\right)^2,$$

where $\tilde{K}^0(e_1 \wedge e_2)$ is the sectional curvature of M^2 in $\left(\tilde{M}^4, \tilde{\nabla}^0\right)$ and G^0 its Gaussian curvature with respect to the Levi-Civita connection.

Consequently we have

$$\begin{aligned} 2\left|G^\perp\right| &\leqslant \|H\|^2 + \|H^*\|^2 - 4G^0 + 4\tilde{K}^0(e_1 \wedge e_2) - 2G - 2c \\ &\quad - \left(h_{12}^3 + h_{12}^{*3}\right)^2 - \left(h_{12}^4 + h_{12}^{*4}\right)^2 + 2h_{12}^3h_{12}^{*3} + 2h_{12}^4h_{12}^{*4} \\ &\quad + \left(h_{12}^3\right)^2 + \left(h_{12}^{*3}\right)^2 + \left(h_{12}^4\right)^2 + \left(h_{12}^{*4}\right)^2 \\ &= \|H\|^2 + \|H^*\|^2 - 4G^0 + 4\tilde{K}^0(e_1 \wedge e_2) - 2G - 2c. \end{aligned}$$

Summing up, we state the following.

THEOREM 4. *Let M^2 be a statistical surface in a 4-dimensional statistical manifold $\left(\tilde{M}^4, c\right)$ of constant curvature c. Then*

$$G + \left|G^\perp\right| + 2G^0 \leqslant \frac{1}{2}\left(\|H\|^2 + \|H^*\|^2\right) - c + 2\tilde{K}^0(e_1 \wedge e_2).$$

In particular, for $c = 0$ we derive the following.

COROLLARY 1. *Let M^2 be a statistical surface of a 4-dimensional Hessian manifold $\tilde{M}^4$ of Hessian curvature 0. Then:*

$$G + \left|G^\perp\right| + 2G^0 \leqslant \frac{1}{2}\left(\|H\|^2 + \|H^*\|^2\right).$$

REMARK 1. In [3] we proved a generalized Wintgen inequality for statistical submanifolds in statistical manifolds of constant curvature by using different techniques than in the proof of Theorem 4. Moreover we want to point-out that the inequality from Theorem 4 is stronger than the particular case $n = 2$ which derives from the inequality obtained in [3].

Acknowledgements. This paper was written during the visit of the second author to Firat University, Turkey, in April-May 2015. The second author was supported by the Scientific and Technical Research Council of Turkey (TUBITAK) for Advanced Fellowships Programme.

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(Received April 26, 2018)

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