Mathematical
Inequalities
& Applications
Volume 22, Number 4 (2019), 1451–1462 doi:10.7153/mia-2019-22-101

INEQUALITIES WITH INFINITE CONVEX COMBINATIONS IN THE SIMPLICES

ZLATKO PAVIĆ

(*Communicated by I. Perić*)

Abstract. The article deals with convex combinations containing infinite number of terms (infinite convex combinations). The inequalities for convex functions and infinite convex combinations of points from the simplices are investigated. This research relies on the extended discrete form of Jensen's inequality.

1. Introduction

Let $\mathbb{X}$ be a vector space over the field $\mathbb{R}$. A linear combination $\alpha x+\beta y$ of points $x,y\in\mathbb{X}$ and coefficients $\alpha,\beta\in\mathbb{R}$ is said to be convex if $\alpha,\beta\in[0,1]$ and $\alpha+\beta=1$. A set $C\subseteq\mathbb{X}$ is said to be convex if it contains each convex combination of each pair of its points. A function $f:C\to\mathbb{R}$ is said to be convex if the inequality

$$f(\alpha x+\beta y)\leqslant\alpha f(x)+\beta f(y) \tag{1}$$

holds for each convex combination $\alpha x+\beta y$ of each pair of points $x,y\in C$.

The convex hull $\mathrm{conv}X$ of a set $X\subseteq\mathbb{X}$ is defined as the set containing each convex combination of points from X. The set $\mathrm{conv}X$ is the smallest convex set containing X.

By using the mathematical induction, it can be demonstrated that the set convexity, function convexity and convex hull apply to n-membered convex combinations for every integer $n\geqslant 2$. In that case, formula (1) represents the discrete form of Jensen's inequality, see [2].

Let $\sum_{i=1}^{n}\lambda_i x_i$ be a convex combination of points $x_i\in\mathbb{X}$. The combination center x can be formally defined by the equation $\sum_{i=1}^{n}\lambda_i(x_i-x)=0$. Thus $x=\sum_{i=1}^{n}\lambda_i x_i$ and it belongs to the convex hull of the set $\{x_1,\dots,x_n\}$. The center x stands out in creating inequalities with convex and concave functions.

Mathematics subject classification (2010): 26B25, 26D15.
Keywords and phrases: Infinite convex combination, convergence, Jensen's inequality.

© ELEMENT, Zagreb
Paper MIA-22-101

2. Inequalities with infinite convex combinations

In this section, we briefly recall some main results obtained in [8].

DEFINITION 1. An infinite linear combination $\sum_{i=1}^{\infty}\lambda_i x_i$ of points x_i from the space $\mathbb{X}$ is said to be convex if $\lambda_i \in [0,1]$ and $\sum_{i=1}^{\infty}\lambda_i = 1$.

The next three theorems refer to infinite convex combinations of points from a bounded closed interval of real numbers.

THEOREM A. *Let $\sum_{i=1}^{\infty}\lambda_i x_i$ be an infinite convex combination of points x_i from the interval $[a,b]$.*

Then the combination $\sum_{i=1}^{\infty}\lambda_i x_i$ converges in the interval $[a,b]$.

THEOREM B. *Let X be a nonempty set, let $g : X \to \mathbb{R}$ be a function with the image in the interval $[a,b]$, and let $\sum_{i=1}^{\infty}\lambda_i g(x_i)$ be an infinite convex combination of the function values $g(x_i)$ with arguments x_i from the set X.*

Then the combination $\sum_{i=1}^{\infty}\lambda_i g(x_i)$ converges in the interval $[a,b]$.

THEOREM C. *Let $\sum_{i=1}^{\infty}\lambda_i x_i$ be an infinite convex combination of points x_i from the interval $[a,b]$, and let $\alpha a+\beta b$ be the convex combination of the endpoints a and b such that $\alpha a+\beta b=\sum_{i=1}^{\infty}\lambda_i x_i$.*

Then each convex function $f : [a,b] \to \mathbb{R}$ satisfies the double inequality

$$f(\alpha a+\beta b) \leqslant \sum_{i=1}^{\infty}\lambda_i f(x_i) \leqslant \alpha f(a)+\beta f(b). \tag{2}$$

We want to generalize the extended discrete form of Jensen's inequality in formula (2) to simplices in higher dimensions.

3. Convex combinations in the triangle

Let a, b and c be non-collinear points in the plane $\mathbb{R}^2$. Then the triangle with the vertices a, b and c can be introduced as the set

$$\triangle_{abc} = \{\alpha a+\beta b+\gamma c : \alpha,\beta,\gamma \in [0,1],\ \alpha+\beta+\gamma=1\}. \tag{3}$$

Thus the triangle $\triangle_{abc}$ is the convex hull of the vertices set $\{a,b,c\}$. Each point $x \in \triangle_{abc}$ is represented by the unique trinomial convex combination $x=\alpha a+\beta b+\gamma c$ because the vectors $a-b$ and $a-c$ are linearly independent. We can obtain the representation of x by the convex combination

$$x=\frac{\mathrm{ar}(\triangle_{xbc})}{\mathrm{ar}(\triangle_{abc})}a+\frac{\mathrm{ar}(\triangle_{xac})}{\mathrm{ar}(\triangle_{abc})}b+\frac{\mathrm{ar}(\triangle_{xab})}{\mathrm{ar}(\triangle_{abc})}c \tag{4}$$

indicating that

$$\alpha=\frac{\mathrm{ar}(\triangle_{xbc})}{\mathrm{ar}(\triangle_{abc})},\ \beta=\frac{\mathrm{ar}(\triangle_{xac})}{\mathrm{ar}(\triangle_{abc})},\ \gamma=\frac{\mathrm{ar}(\triangle_{xab})}{\mathrm{ar}(\triangle_{abc})}. \tag{5}$$

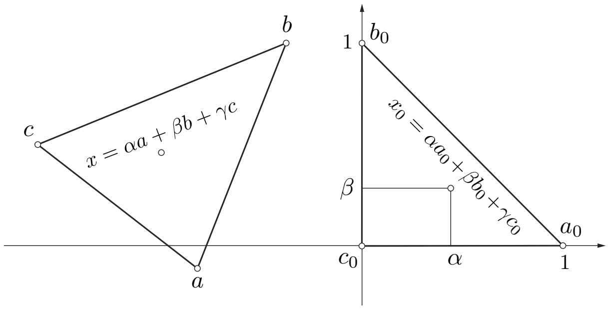

Figure 1: *The coefficients representation by using the unit triangle*

The sets $\triangle_{xbc}$, $\triangle_{xac}$ and $\triangle_{xab}$ defined as the convex hulls by formula (3) are not necessarily triangles, but at least one of them is a triangle. The graphic design of coefficients is presented in Figure 1.

The fundamental theorem for convex functions and convex combinations in the triangle states the following.

THEOREM D. *Let $\sum_{i=1}^{n}\lambda_i x_i$ be a convex combination of points x_i from the triangle $\triangle_{abc}$, and let $\alpha a+\beta b+\gamma c$ be the convex combination of the vertices a, b and c such that $\alpha a+\beta b+\gamma c=\sum_{i=1}^{n}\lambda_i x_i$.*

Then each convex function $f:\triangle_{abc}\to\mathbb{R}$ satisfies the double inequality

$$f(\alpha a+\beta b+\gamma c)\leqslant\sum_{i=1}^{n}\lambda_i f(x_i)\leqslant\alpha f(a)+\beta f(b)+\gamma f(c). \tag{6}$$

The inequality in formula (6) expresses the nature of growth of convex functions on the triangle. The convex function values, taken in the forms of convex combinations, grow from the center across the middle to the vertices. In the important formula (6), we want to replace n with infinity.

The functional approach to the inequality in formula (6) can be found in [9].

4. Main results

In this section, we investigate inequalities for convex functions and infinite convex combinations of points from the triangle.

LEMMA 4.1. *Let $\sum_{i=1}^{\infty}\lambda_i x_i$ be an infinite convex combination of points x_i from the triangle $\triangle_{abc}$.*

Then the combination $\sum_{i=1}^{\infty}\lambda_i x_i$ converges in the triangle $\triangle_{abc}$.

Proof. Each point x_i can be represented by the unique convex combination

$$x_i=\alpha_i a+\beta_i b+\gamma_i c. \tag{7}$$

Thus $\alpha_i,\beta_i,\gamma_i\in[0,1]$ and $\alpha_i+\beta_i+\gamma_i=1$. According to Theorem A, the infinite convex combinations $\sum_{i=1}^{\infty}\lambda_i\alpha_i$, $\sum_{i=1}^{\infty}\lambda_i\beta_i$ and $\sum_{i=1}^{\infty}\lambda_i\gamma_i$ of points α_i, β_i and γ_i converge in the interval $[0,1]$, and therefore we can write down

$$\sum_{i=1}^{\infty}\lambda_i\alpha_i=\alpha,\ \sum_{i=1}^{\infty}\lambda_i\beta_i=\beta,\ \sum_{i=1}^{\infty}\lambda_i\gamma_i=\gamma. \tag{8}$$

So, the coefficients α, β and γ are in $[0,1]$. In addition, their sum

$$\alpha+\beta+\gamma=\sum_{i=1}^{\infty}\lambda_i(\alpha_i+\beta_i+\gamma_i)=\sum_{i=1}^{\infty}\lambda_i=1.$$

By using the relations in formula (7) and formula (8), we get

$$\sum_{i=1}^{\infty}\lambda_i x_i=\sum_{i=1}^{\infty}\lambda_i(\alpha_i a+\beta_i b+\gamma_i c)=\alpha a+\beta b+\gamma c.$$

The above presentation shows that the series $\sum_{i=1}^{\infty}\lambda_i x_i$ converges to the convex combination $\alpha a+\beta b+\gamma c$ belonging to the triangle $\triangle_{abc}$. $\square$

Relying on the above lemma, we can establish the fundamental inequality for convex functions and infinite convex combinations in the triangle.

THEOREM 4.2. *Let $\sum_{i=1}^{\infty}\lambda_i x_i$ be an infinite convex combination of points x_i from the triangle $\triangle_{abc}$, and let $\alpha a+\beta b+\gamma c$ be the convex combination of the vertices a, b and c such that $\alpha a+\beta b+\gamma c=\sum_{i=1}^{\infty}\lambda_i x_i$.*

Then each convex function $f:\triangle_{abc}\to\mathbb{R}$ satisfies the double inequality

$$f(\alpha a+\beta b+\gamma c)\leqslant\sum_{i=1}^{\infty}\lambda_i f(x_i)\leqslant\alpha f(a)+\beta f(b)+\gamma f(c). \tag{9}$$

Proof. We use the points representations in formula (7), and coefficients relations in formula (8). Let $n\geqslant 2$, let

$$\varepsilon_n=1-\sum_{i=1}^{n-1}\lambda_i,$$

and let

$$\tilde{\alpha}_n=\sum_{i=1}^{n-1}\lambda_i\alpha_i+\varepsilon_n\alpha_n,\ \tilde{\beta}_n=\sum_{i=1}^{n-1}\lambda_i\beta_i+\varepsilon_n\beta_n,\ \tilde{\gamma}_n=\sum_{i=1}^{n-1}\lambda_i\gamma_i+\varepsilon_n\gamma_n. \tag{10}$$

Then $\tilde{\alpha}_n,\tilde{\beta}_n,\tilde{\gamma}_n\in[0,1]$ and $\tilde{\alpha}_n+\tilde{\beta}_n+\tilde{\gamma}_n=1$. Further, since $\lim_{n\to\infty}\varepsilon_n=0$, it follows that

$$\lim_{n\to\infty}\tilde{\alpha}_n=\alpha,\ \lim_{n\to\infty}\tilde{\beta}_n=\beta,\ \lim_{n\to\infty}\tilde{\gamma}_n=\gamma.$$

By utilizing the coefficients relations in formula (10), we get the presentation

$$\begin{aligned}\tilde{\alpha}_n a+\tilde{\beta}_n b+\tilde{\gamma}_n c &= \sum_{i=1}^{n-1}\lambda_i(\alpha_i a+\beta_i b+\gamma_i c)+\varepsilon_n(\alpha_n a+\beta_n b+\gamma_n c)\\ &= \sum_{i=1}^{n-1}\lambda_i x_i+\varepsilon_n x_n,\end{aligned}$$

and as a goal, we have the convex combinations equality

$$\tilde{\alpha}_n a+\tilde{\beta}_n b+\tilde{\gamma}_n c = \sum_{i=1}^{n-1}\lambda_i x_i+\varepsilon_n x_n. \tag{11}$$

Let

$$x=\lim_{n\to\infty}(\tilde{\alpha}_n a+\tilde{\beta}_n b+\tilde{\gamma}_n c)=\sum_{i=1}^{\infty}\lambda_i x_i=\alpha a+\beta b+\gamma c.$$

By applying formula (6) to the equality in formula (11), we obtain

$$f(\tilde{\alpha}_n a+\tilde{\beta}_n b+\tilde{\gamma}_n c)\leqslant\sum_{i=1}^{n-1}\lambda_i f(x_i)+\varepsilon_n f(x_n)\leqslant\tilde{\alpha}_n f(a)+\tilde{\beta}_n f(b)+\tilde{\gamma}_n f(c). \tag{12}$$

We want to apply the reflection moment to formula (12) by letting n tend to infinity. Then the second member approaches $\sum_{i=1}^{\infty}\lambda_i f(x_i)$, and the third member approaches $\alpha f(a)+\beta f(b)+\gamma f(c)$. As regards the limit of the first member, we discuss the following three cases.

If $\alpha,\beta,\gamma>0$, then the point $x=\alpha a+\beta b+\gamma c$ belongs to the interior $\triangle_{abc}^{o}$ of the triangle $\triangle_{abc}$. There is a closed ball $\mathscr{B}\subset\triangle_{abc}^{o}$ containing the point x and almost all points $\tilde{\alpha}_n a+\tilde{\beta}_n b+\tilde{\gamma}_n c$. The convex function f is continuous on $\triangle_{abc}^{o}$, and so f is continuous on $\mathscr{B}$. Then it follows that

$$\lim_{n\to\infty}f(\tilde{\alpha}_n a+\tilde{\beta}_n b+\tilde{\gamma}_n c)=f\big(\lim_{n\to\infty}(\tilde{\alpha}_n a+\tilde{\beta}_n b+\tilde{\gamma}_n c)\big)=f(\alpha a+\beta b+\gamma c).$$

If $\alpha,\beta>0$ and $\gamma=0$, then $x=\alpha a+\beta b$, and we have $\sum_{i=1}^{\infty}\lambda_i x_i=\alpha a+\beta b$. The fact is that $\lambda_i>0$ implies $x_i=\alpha_i a+\beta_i b$. So, we can assume that each x_i belongs to the line segment $\triangle_{ab}$. Since the point x belongs to the relative interior $\triangle_{ab}^{o}$ of $\triangle_{ab}$, we can apply the previous case to $\triangle_{ab}$.

If $\alpha=1$ and $\beta=\gamma=0$, then $x=a$, and we have $\sum_{i=1}^{\infty}\lambda_i x_i=a$. The fact is that $\lambda_i>0$ implies $x_i=a$. So, we can assume that each x_i is equal to a. Since

$$\sum_{i=1}^{\infty}\lambda_i f(x_i)=\sum_{i=1}^{\infty}\lambda_i f(a)=f(a),$$

the trivial double inequality $f(a)\leqslant f(a)\leqslant f(a)$ represents formula (9).

To conclude, formula (12) approaches formula (9) as n approaches infinity in each of the above three cases. $\square$

Let $(x_n)_{n=1}^{\infty}$ be a convergent sequence of points $x_n \in \triangle_{abc}$, and let $f : \triangle_{abc} \to \mathbb{R}$ be a convex function. By using the convex combinations $x_n = \alpha_n a + \beta_n b + \gamma_n c$, utilizing the convexity of f, and exploiting the continuity of f on the triangle interior and sides relative interiors, we can prove the inequality

$$\lim_{n\to\infty} f(x_n) \leqslant f\big(\lim_{n\to\infty} x_n\big). \tag{13}$$

To verify the above inequality, we point out the limit $x = \lim_{n\to\infty} x_n$ and its convex combination $x = \alpha a + \beta b + \gamma c$. Then we have $\alpha = \lim_{n\to\infty} \alpha_n$, $\beta = \lim_{n\to\infty} \beta_n$ and $\gamma = \lim_{n\to\infty} \gamma_n$. We consider the next three positions of the point x.

If $x \in \triangle_{abc}^{o}$, then almost all members x_n are in $\triangle_{abc}^{o}$. Since the function f is continuous on $\triangle_{abc}^{o}$, herein the equality holds in formula (13).

If $x \in \triangle_{ab}^{o}$, then the convex combination $x = \alpha a + \beta b$ with positive coefficients α and β is effective. So, here is $\gamma = \lim_{n\to\infty} \gamma_n = 0$. By applying the convexity of f to the representations (in the form of binomial convex combinations)

$$x_n = \alpha_n a + \beta_n b + \gamma_n c = (1-\gamma_n)\left(\frac{\alpha_n}{1-\gamma_n}a + \frac{\beta_n}{1-\gamma_n}b\right) + \gamma_n c,$$

we get

$$f(x_n) \leqslant (1-\gamma_n) f\left(\frac{\alpha_n}{1-\gamma_n}a + \frac{\beta_n}{1-\gamma_n}b\right) + \gamma_n f(c),$$

and calculating the limit taking into account the continuity of f on $\triangle_{ab}^{o}$, we obtain

$$\begin{aligned}\lim_{n\to\infty} f(x_n) &\leqslant \lim_{n\to\infty} f\left(\frac{\alpha_n}{1-\gamma_n}a + \frac{\beta_n}{1-\gamma_n}b\right) \;=\; f\left(\lim_{n\to\infty}\left(\frac{\alpha_n}{1-\gamma_n}a + \frac{\beta_n}{1-\gamma_n}b\right)\right)\\ &= f(\alpha a + \beta b) \;=\; f(x) \;=\; f\big(\lim_{n\to\infty} x_n\big).\end{aligned}$$

If $x = a$, then $\alpha = \lim_{n\to\infty} \alpha_n = 1$, $\beta = \lim_{n\to\infty} \beta_n = 0$ and $\gamma = \lim_{n\to\infty} \gamma_n = 0$. By applying Jensen's inequality to the convex combinations $x_n = \alpha_n a + \beta_n b + \gamma_n c$, we get

$$f(x_n) \leqslant \alpha_n f(a) + \beta_n f(b) + \gamma_n f(c),$$

and calculating the limit, we obtain

$$\lim_{n\to\infty} f(x_n) \leqslant f(a) = f(x) = f\big(\lim_{n\to\infty} x_n\big).$$

In case the function f is not continuous, the inequality in formula (13) may be strict. This is exactly the following example.

EXAMPLE 4.3. Let $f : \triangle_{abc} \to \mathbb{R}$ be a convex function having a discontinuity at the vertex a, and let $(x_n)_{n=1}^{\infty}$ be the sequence of the convex combinations of the vertices a, b and c defined by

$$x_n = \frac{n}{n+2}a + \frac{1}{n+2}b + \frac{1}{n+2}c.$$

Then the points x_n belong to the interior $\triangle^o_{abc}$, $\lim_{n\to\infty} x_n = a$, and

$$\lim_{n\to\infty} f(x_n) < f\big(\lim_{n\to\infty} x_n\big) = f(a).$$

For example, using the function

$$f(x) = \begin{cases} 1 \text{ if } x \in \triangle^o_{abc} \\ 2 \text{ if } x \notin \triangle^o_{abc} \end{cases},$$

we have

$$\lim_{n\to\infty} f(x_n) = 1 < 2 = f\big(\lim_{n\to\infty} x_n\big) = f(a).$$

The inequality of the first and second members in formula (9) can be expressed in the form

$$f\left(\sum_{i=1}^{\infty} \lambda_i x_i\right) \leqslant \sum_{i=1}^{\infty} \lambda_i f(x_i), \tag{14}$$

representing Jensen's inequality for infinite convex combinations in the triangle. New results on Jensen's inequality have been achieved in [3] and [5].

The inequality in formula (9) can be adapted to other inequalities. The following are two versions of the extended form of the Jensen-Mercer inequality (see [6]) for infinite convex combinations in the triangle.

COROLLARY 4.4. *Let $\sum_{i=1}^{\infty} \lambda_i x_i$ be an infinite convex combination of points x_i from the triangle $\triangle_{abc}$, and let $\alpha a + \beta b + \gamma c$ be the convex combination of the vertices a, b and c such that $\alpha a + \beta b + \gamma c = \sum_{i=1}^{\infty} \lambda_i x_i$.*

Then each convex function $f : \triangle_{abc} \to \mathbb{R}$ satisfies the double inequalities

$$\begin{aligned} f\left(\frac{1-\alpha}{2}a + \frac{1-\beta}{2}b + \frac{1-\gamma}{2}c\right) &\leqslant \sum_{i=1}^{\infty} \lambda_i f\left(\frac{a+b+c-x_i}{2}\right) \\ &\leqslant \frac{1-\alpha}{2} f(a) + \frac{1-\beta}{2} f(b) + \frac{1-\gamma}{2} f(c) \end{aligned} \tag{15}$$

and

$$\begin{aligned} f\left(\frac{a+b+c-\sum_{i=1}^{\infty} \lambda_i x_i}{2}\right) &\leqslant \frac{1-\alpha}{2} f(a) + \frac{1-\beta}{2} f(b) + \frac{1-\gamma}{2} f(c) \\ &\leqslant \frac{f(a)+f(b)+f(c)-\sum_{i=1}^{\infty} \lambda_i f(x_i)}{2}. \end{aligned} \tag{16}$$

Proof. By using formula (7), we get the representation

$$\frac{a+b+c-x_i}{2} = \frac{1-\alpha_i}{2}a + \frac{1-\beta_i}{2}b + \frac{1-\gamma_i}{2}c,$$

showing that the left side member belongs to $\triangle_{abc}$ as the convex combination of the vertices a, b and c. Further, by utilizing the above representation, we can derive the equalities in formula (17) and formula (18).

If we apply formula (9) to the convex combinations equality

$$\frac{1-\alpha}{2}a+\frac{1-\beta}{2}b+\frac{1-\gamma}{2}c=\sum_{i=1}^{\infty}\lambda_i\frac{a+b+c-x_i}{2}, \tag{17}$$

then we obtain formula (15).

If we apply Jensen's inequality to the convex combinations equality

$$\frac{a+b+c-\sum_{i=1}^{\infty}\lambda_i x_i}{2}=\frac{1-\alpha}{2}a+\frac{1-\beta}{2}b+\frac{1-\gamma}{2}c, \tag{18}$$

then we obtain the inequality of the first and second members in formula (16). If we utilize the inequality

$$-\alpha f(a)-\beta f(b)-\gamma f(c)\leqslant-\sum_{i=1}^{\infty}\lambda_i f(x_i)$$

derived from the inequality of the second and third members in formula (9), then we obtain the inequality of the second and third members in formula (16). □

The consequences of Theorem 4.2 and Corollary 4.4 are the following versions of the extended form of Jensen's inequality.

COROLLARY 4.5. *Let $\sum_{i=1}^{\infty}\lambda_i x_i$ be an infinite convex combination of points x_i from the triangle $\triangle_{abc}$, and let $x=\sum_{i=1}^{\infty}\lambda_i x_i$ be the combination center.*

Then each convex function $f:\triangle_{abc}\to\mathbb{R}$ satisfies the inequalities

$$\begin{aligned} f\left(\sum_{i=1}^{\infty}\lambda_i x_i\right) &\leqslant \sum_{i=1}^{\infty}\lambda_i f(x_i)\\ &\leqslant \frac{\operatorname{ar}(\triangle_{xbc})}{\operatorname{ar}(\triangle_{abc})}f(a)+\frac{\operatorname{ar}(\triangle_{xac})}{\operatorname{ar}(\triangle_{abc})}f(b)+\frac{\operatorname{ar}(\triangle_{xab})}{\operatorname{ar}(\triangle_{abc})}f(c) \end{aligned} \tag{19}$$

and

$$\begin{aligned} f\left(\frac{a+b+c-\sum_{i=1}^{\infty}\lambda_i x_i}{2}\right) &\leqslant \sum_{i=1}^{\infty}\lambda_i f\left(\frac{a+b+c-x_i}{2}\right)\\ &\leqslant \frac{\operatorname{ar}(\triangle_{abc})-\operatorname{ar}(\triangle_{xbc})}{2\operatorname{ar}(\triangle_{abc})}f(a)\\ &\quad+\frac{\operatorname{ar}(\triangle_{abc})-\operatorname{ar}(\triangle_{xac})}{2\operatorname{ar}(\triangle_{abc})}f(b)\\ &\quad+\frac{\operatorname{ar}(\triangle_{abc})-\operatorname{ar}(\triangle_{xab})}{2\operatorname{ar}(\triangle_{abc})}f(c)\\ &\leqslant \frac{f(a)+f(b)+f(c)-\sum_{i=1}^{\infty}\lambda_i f(x_i)}{2}. \end{aligned} \tag{20}$$

Proof. The double inequality in formula (19) is the version of formula (9) with coefficients presented in formula (5).

The multiple inequality in formula (20) can be achieved by combining formula (15) and formula (16), and using coefficients presented in formula (5). □

More details on convex sets, convex functions and their inequalities can be found in the books [10] and [11].

5. Generalizations to higher dimensions

The results obtained in the previous section can be generalized to the m-simplex in the space $\mathbb{R}^m$.

If $a_1,\dots,a_{m+1}\in\mathbb{R}^m$ are points such that differences $a_1-a_{m+1},\dots,a_m-a_{m+1}$ are linearly independent, then the m-simplex with vertices $a_1,\dots,a_{m+1}$ can be introduced as the set

$$\triangle_{a_1\dots a_{m+1}}=\Big\{\textstyle\sum_{j=1}^{m+1}\alpha_j a_j : \alpha_j\in[0,1],\ \sum_{j=1}^{m+1}\alpha_j=1\Big\}. \tag{21}$$

Thus we have $\triangle_{a_1\dots a_{m+1}}=\operatorname{conv}\{a_1,\dots,a_{m+1}\}$. Each point $x\in\triangle_{a_1\dots a_{m+1}}$ is represented by the unique $(m+1)$-membered convex combination $x=\sum_{j=1}^{m+1}\alpha_j a_j$ because the points $a_1-a_{m+1},\dots,a_m-a_{m+1}$ are linearly independent. By using the sets $\triangle_{a_j=x}=\operatorname{conv}\{a_1,\dots,a_{j-1},x,a_{j+1},\dots,a_{m+1}\}$ and the denotation vol_m for the volume in the space $\mathbb{R}^m$, we can obtain the representation of x by the convex combination

$$x=\sum_{j=1}^{m+1}\frac{\mathrm{vol}_m(\triangle_{a_j=x})}{\mathrm{vol}_m(\triangle_{a_1\dots a_{m+1}})}a_j \tag{22}$$

indicating that

$$\alpha_j=\frac{\mathrm{vol}_m(\triangle_{a_j=x})}{\mathrm{vol}_m(\triangle_{a_1\dots a_{m+1}})}. \tag{23}$$

Depending on the location of the point x, the set $\triangle_{a_j=x}$ appears as one of the following two volumetric shapes, it is the m-simplex in which case $\alpha_j>0$, or the facet $\triangle_{a_1\dots a_{j-1}a_{j+1}\dots a_{m+1}}$ in which case $\alpha_j=0$.

The following is the initial lemma on infinite convex combinations of points from the m-simplex in the space $\mathbb{R}^m$.

LEMMA 5.1. *Let $\sum_{i=1}^{\infty}\lambda_i x_i$ be an infinite convex combination of points x_i from the m-simplex $\triangle_{a_1\dots a_{m+1}}$.*

Then the combination $\sum_{i=1}^{\infty}\lambda_i x_i$ converges in the m-simplex $\triangle_{a_1\dots a_{m+1}}$.

Proof. By using the unique convex combination presentation

$$x_i=\sum_{j=1}^{m+1}\alpha_{ij}a_j$$

for $i \in \mathbb{N} = \{1,2,3,\ldots\}$, and consequently the convergent series

$$\sum_{i=1}^{\infty} \lambda_i \alpha_{ij} = \alpha_j$$

for $j \in \mathbb{N}_{m+1} = \{1,\ldots,m+1\}$, we obtain the relation

$$\sum_{i=1}^{\infty} \lambda_i x_i = \sum_{j=1}^{m+1} \alpha_j a_j,$$

where the infinite convex combination on the left side coincides with the convex combination of the m-simplex vertices a_j on the right side. $\square$

By relying on the analogue of Theorem D in higher dimensions (for example, see [7, Corollary 2.2]), the fundamental inequality for convex functions and infinite convex combinations in the m-simplex is as follows.

THEOREM 5.2. *Let $\sum_{i=1}^{\infty} \lambda_i x_i$ be an infinite convex combination of points x_i from the m-simplex $\triangle_{a_1\ldots a_{m+1}}$, and let $\sum_{j=1}^{m+1} \alpha_j a_j$ be the convex combination of the vertices a_j such that $\sum_{j=1}^{m+1} \alpha_j a_j = \sum_{i=1}^{\infty} \lambda_i x_i$.*

Then each convex function $f : \triangle_{a_1\ldots a_{m+1}} \to \mathbb{R}$ satisfies the double inequality

$$f\left(\sum_{j=1}^{m+1} \alpha_j a_j\right) \leqslant \sum_{i=1}^{\infty} \lambda_i f(x_i) \leqslant \sum_{j=1}^{m+1} \alpha_j f(a_j). \tag{24}$$

Proof. To obtain the inequality in formula (24), we have to apply the reflection moment (by letting n tend to infinity) to the inequality

$$f\left(\sum_{j=1}^{m+1} \tilde{\alpha}_{nj} a_j\right) \leqslant \sum_{i=1}^{n-1} \lambda_i f(x_i) + \varepsilon_n f(x_n) \leqslant \sum_{j=1}^{m+1} \tilde{\alpha}_{nj} f(a_j)$$

with the coefficients convex combinations

$$\tilde{\alpha}_{nj} = \sum_{i=1}^{n-1} \lambda_i \alpha_{ij} + \varepsilon_n \alpha_{nj}$$

for every $j \in \mathbb{N}_{m+1}$, and the coefficient $\varepsilon_n = 1 - \sum_{i=1}^{n-1} \lambda_i$. $\square$

The variants of the Jensen-Mercer inequality for infinite convex combinations in the m-simplex are as follows.

COROLLARY 5.3. *Let $\sum_{i=1}^{\infty} \lambda_i x_i$ be an infinite convex combination of points x_i from the m-simplex $\triangle_{a_1\ldots a_{m+1}}$, and let $\sum_{j=1}^{m+1} \alpha_j a_j$ be the convex combination of the vertices a_j such that $\sum_{j=1}^{m+1} \alpha_j a_j = \sum_{i=1}^{\infty} \lambda_i x_i$.*

Then each convex function $f : \triangle_{a_1 \dots a_{m+1}} \to \mathbb{R}$ satisfies the double inequalities

$$
\begin{aligned}
f\left(\sum_{j=1}^{m+1} \frac{1-\alpha_j}{m} a_j\right) &\leqslant \sum_{i=1}^{\infty} \lambda_i f\left(\frac{\sum_{j=1}^{m+1} a_j - x_i}{m}\right) \\
&\leqslant \sum_{j=1}^{m+1} \frac{1-\alpha_j}{m} f(a_j)
\end{aligned} \tag{25}
$$

and

$$
\begin{aligned}
f\left(\frac{\sum_{j=1}^{m+1} a_j - \sum_{i=1}^{\infty} \lambda_i x_i}{m}\right) &\leqslant \sum_{j=1}^{m+1} \frac{1-\alpha_j}{m} f(a_j) \\
&\leqslant \frac{\sum_{j=1}^{m+1} f(a_j) - \sum_{i=1}^{\infty} \lambda_i f(x_i)}{m}.
\end{aligned} \tag{26}
$$

By making some modifications, combining inequalities, and using the coefficients in formula (23), we get the following.

COROLLARY 5.4. *Let $\sum_{i=1}^{\infty} \lambda_i x_i$ be an infinite convex combination of points x_i from the m-simplex $\triangle_{a_1 \dots a_{m+1}}$, and let $x = \sum_{i=1}^{\infty} \lambda_i x_i$ be the combination center.*

Then each convex function $f : \triangle_{a_1 \dots a_{m+1}} \to \mathbb{R}$ satisfies the inequalities

$$
\begin{aligned}
f\left(\sum_{i=1}^{\infty} \lambda_i x_i\right) &\leqslant \sum_{i=1}^{\infty} \lambda_i f(x_i) \\
&\leqslant \sum_{j=1}^{m+1} \frac{\operatorname{vol}_m(\triangle_{a_j = x})}{\operatorname{vol}_m(\triangle_{a_1 \dots a_{m+1}})} f(a_j)
\end{aligned} \tag{27}
$$

and

$$
\begin{aligned}
f\left(\frac{\sum_{j=1}^{m+1} a_j - \sum_{i=1}^{\infty} \lambda_i x_i}{m}\right) &\leqslant \sum_{i=1}^{\infty} \lambda_i f\left(\frac{\sum_{j=1}^{m+1} a_j - x_i}{m}\right) \\
&\leqslant \sum_{j=1}^{m+1} \frac{\operatorname{vol}_m(\triangle_{a_1 \dots a_{m+1}}) - \operatorname{vol}_m(\triangle_{a_j = x})}{m \operatorname{vol}_m(\triangle_{a_1 \dots a_{m+1}})} f(a_j) \\
&\leqslant \frac{\sum_{j=1}^{m+1} f(a_j) - \sum_{i=1}^{\infty} \lambda_i f(x_i)}{m}.
\end{aligned} \tag{28}
$$

Very general variant of the Jensen-Mercer inequality for convex functions on the convex hulls in higher dimensions was obtained in [4]. The inclusion of operators into the Jensen-Mercer inequality can be seen in [1].

REFERENCES

[1] S. IVELIĆ, A. MATKOVIĆ AND J. PEČARIĆ, *On a Jensen-Mercer operator inequality*, Banach J. Math. Anal. **5** (2011), 19–28.
[2] J. L. W. V. JENSEN, *Om konvekse Funktioner og Uligheder mellem Middelværdier*, Nyt Tidsskr. Math. B, **16** (1905), 49–68.
[3] G. LU, *New refinements of Jensen's inequality and entropy upper bounds*, J. Math. Inequal. **12** (2018), 403–421.
[4] A. MATKOVIĆ AND J. PEČARIĆ, *A variant of Jensen's inequality for convex functions of several variables*, J. Math. Inequal. **1** (2007), 45–51.
[5] N. MEHMOOD, S. IHSAN BUTT, L. HORVÁTH AND J. PEČARIĆ, *Generalization of cyclic refinements of Jensen's inequality by Fink's identity*, J. Inequal. Appl. **2018** (2018), Article 51.
[6] A. MCD. MERCER, *A variant of Jensen's inequality*, JIPAM **4** (2003), Article 73.
[7] Z. PAVIĆ, Improvements of the Hermite-Hadamard inequality for the simplex, J. Inequal. Appl. **2017** (2017), Article 3.
[8] Z. PAVIĆ, *Inequalities with infinite convex combinations*, submitted to FILOMAT.
[9] Z. PAVIĆ, *The Jensen and Hermite-Hadamard inequality on the triangle*, J. Math. Inequal. **11** (2017), 1099–1112.
[10] A. W. ROBERTS AND D. E. VARBERG, *Convex Functions*, Academic Press, New York and London, 1973.
[11] B. SIMON, *Convexity: An Analytic Viewpoint*, Cambridge University Press, New York, 2011.

(Received November 1, 2018)

Zlatko Pavić
Department of Mathematics
Mechanical Engineering Faculty in Slavonski Brod
University of Osijek, 35000 Slavonski Brod, Croatia
e-mail: Zlatko.Pavic@sfsb.hr