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INTEGRAL INEQUALITIES OF LEVINSON'S TYPE IN TIME SCALE SETTINGS

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Abstract. A new class of functions, $\mathscr{K}_1^c(I)$, has been recently introduced by Baloch, Pečarić and Praljak. The authors proved that $\mathscr{K}_1^c(I)$ is the largest class of functions for which Levinson's inequality holds under Mercer's assumptions. We obtain Levinson's type inequalities in time scale settings by using the class $\mathscr{K}_1^c(I)$ and some known results regarding integral inequalites for convex (concave) functions on time scale sets.

1. Introduction

1.1. On Levinson's inequality

The well known Levinson's inequality, ([16]), is given in the next theorem.

THEOREM 1. *Let $f:(0,2c)\to\mathbb{R}$ satisfy $f'''\geqslant 0$ and let p_i, x_i, y_i, $i=1,2,\dots,n$, be such that $p_i>0$, $\sum_{i=1}^n p_i=1$, $0\leqslant x_i\leqslant c$, and*

$$x_1+y_1=x_2+y_2=\cdots=x_n+y_n=2c. \tag{1}$$

Then,

$$\sum_{i=1}^n p_i f(x_i)-f(\overline{x})\leqslant\sum_{i=1}^n p_i f(y_i)-f(\overline{y}), \tag{2}$$

where $\overline{x}=\sum_{i=1}^n p_i x_i$ and $\overline{y}=\sum_{i=1}^n p_i y_i$ are the weighted arithmetic means.

In the same year, 1964, Popoviciu ([20]), generalised Levinson's inequality by showing that for (2) to hold it is enough that f is 3-convex function.

In 1973, P. S. Bullen gave in [10] an alternative proof using mathematical induction. By rescaling axes, he proved that if $f:[a,b]\to\mathbb{R}$ is 3-convex and p_i,x_i,y_i, $i=1,2,\dots,n$, are such that $p_i>0$, $\sum_{i=1}^n p_i=1$, $a\leqslant x_i,y_i\leqslant b$, $x_i+y_i=c$ and

$$\max\{x_1,\dots,x_n\}\leqslant\min\{y_1,\dots,y_n\}, \tag{3}$$

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then (2) holds.

Recently, in 2010, Mercer ([18]) improved Levinson's inequality showing that for $f:[a,b]\to\mathbb{R}$ which satisfies $f'''\geqslant 0$ and p_i,x_i,y_i, $i=1,2,\dots,n$, are such that $p_i>0$, $\sum_{i=1}^n p_i=1$, $a\leqslant x_i,y_i\leqslant b$ and $\max\{x_1,\dots,x_n\}\leqslant\min\{y_1,\dots,y_n\}$, inequality (2) holds when the condition (1) is replaced by the following weaker condition

$$\sum_{i=1}^n p_i(x_i-\overline{x})^2=\sum_{i=1}^n p_i(y_i-\overline{y})^2. \tag{4}$$

In 2012 Witkowski ([21]) weakened Mercer's assumption (4) replacing equality by inequality in certain direction and, also, proved that it is enough to assume for function f to be 3 - convex.

In this paper, we will obtain Levinson's type inequality on time scale by using a new class of functions, $\mathscr{K}_1^c(I)$, that has been recently introduced by Baloch, Pečarić and Praljak in [4]. The authors proved that $\mathscr{K}_1^c(I)$ is the largest class of functions for which Levinson's inequality hold under Mercer's assumptions and it is described in the following definition.

DEFINITION 1. Let $f\colon I\to\mathbb{R}$ and $c\in I^0$, where I^0 is the interior of the interval I. We say that $f\in\mathscr{K}_1^c(I)$, $(resp. f\in\mathscr{K}_2^c(I))$, if there exists a constant α such that the function $F(x)=f(x)-\frac{\alpha}{2}x^2$ is concave (resp. convex) on $(-\infty,c]\cap I$ and convex (resp. concave) on $I\cap[c,\infty)$.

Now, for the function f which belongs to class $\mathscr{K}_1^c(I)$ we say that it is 3-convex at point c. So, the class $\mathscr{K}_1^c(I)$ generalizes 3-convex functions in the following sense: a function is 3-convex on I if and only if it is 3-convex at every $c\in I^0$.

As a simple consequence of the probabilistic version of the Levinson's inequality, Pečarić, Praljak and Witkowski, proved in [19], the following corollary where they showed that, in discrete Levinson's inequality, the number of the points of two sequences and associated weights do not need to be same.

COROLLARY 1. *If* $x_i\in I\cap(-\infty,c]$, $y_j\in I\cap[c,\infty)$, $p_i>0$ *and* $q_j>0$, *for* $i=1,\dots,n$, $j=1,\dots,m$, *are such that* $\sum_{i=1}^n p_i=\sum_{j=1}^m q_j=1$ *and*

$$\sum_{i=1}^n p_i(x_i-\overline{x})^2=\sum_{j=1}^m q_j(y_j-\overline{y})^2,$$

where $\overline{x}=\sum_{i=1}^n p_ix_i$ *and* $\overline{y}=\sum_{j=1}^m q_jy_j$, *then the inequality*

$$\sum_{i=1}^n p_if(x_i)-f(\overline{x})\leqslant\sum_{j=1}^m q_jf(y_j)-f(\overline{y}) \tag{5}$$

holds for every $f\in\mathscr{K}_1^c(I)$.

1.2. On time scale calculus

The theory of time scales was introduced by Stefan Hilger in his PhD thesis [12] in 1988 as a unification of the theory of difference equations with that of differential equations, unifying integral and differential calculus with the calculus of finite differences, extending to cases "in between" and offering a formalism for studying hybrid discrete-continuous dynamic systems. It has applications in any field that requires simultaneous modelling of discrete and continuous time. Now, we briefly introduce the time scales calculus and refer to [2, 13, 14] and the books [9, 15] for further details.

By a time scale $\mathbb{T}$ we mean any closed subset of $\mathbb{R}$. The two most popular examples of time scales are the real numbers $\mathbb{R}$ and the integers $\mathbb{Z}$. Since the time scale $\mathbb{T}$ may or may not be connected, we need the concept of jump operators.

For $t \in \mathbb{T}$, we define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$$

and the *backward jump operator* by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

In this definition, the convention is $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(t) = t$ if $\mathbb{T}$ has a maximum t) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(t) = t$ if $\mathbb{T}$ has a minimum t). If $\sigma(t) > t$, then we say that t is *right-scattered*, and if $\rho(t) < t$, then we say that t is *left-scattered*. Points that are right-scattered and left-scattered at the same time are called *isolated*. Also, if $\sigma(t) = t$, then t is said to be *right-dense*, and if $\rho(t) = t$, then t is said to be *left-dense*. Points that are simultaneously right-dense and left-dense are called *dense*. The mapping $\mu : \mathbb{T} \to [0, \infty)$ defined by

$$\mu(t) = \sigma(t) - t$$

is called the *graininess function*. If $\mathbb{T}$ has a left-scattered maximum M, then we define $\mathbb{T}^\kappa = \mathbb{T} \setminus \{M\}$; otherwise $\mathbb{T}^\kappa = \mathbb{T}$. If $f : \mathbb{T} \to \mathbb{R}$ is a function, then we define the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ by

$$f^\sigma(t) = f(\sigma(t)) \quad \text{for all} \quad t \in \mathbb{T}.$$

In the following considerations, $\mathbb{T}$ will denote a time scale, $I_{\mathbb{T}} = I \cap \mathbb{T}$ will denote a time scale interval (for any open or closed interval I in $\mathbb{R}$), and $[0, \infty)_{\mathbb{T}}$ will be used for the time scale interval $[0, \infty) \cap \mathbb{T}$.

DEFINITION 2. Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$\left| f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s) \right| \leqslant \varepsilon |\sigma(t) - s| \quad \text{for all} \quad s \in U.$$

We call $f^\Delta(t)$ the *delta derivative* of f at t. We say that f is *delta differentiable* on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$.

For all $t \in \mathbb{T}^\kappa$, we have the following properties:

(i) If f is delta differentiable at t, then f is continuous at t.

(ii) If f is continuous at t and t is right-scattered, then f is delta differentiable at t with $f^\Delta(t) = \frac{f(\sigma(t))-f(t)}{\mu(t)}$.

(iii) If t is right-dense, then f is delta differentiable at t iff the limit $\lim_{s\to t} \frac{f(t)-f(s)}{t-s}$ exists as a finite number. In this case, $f^\Delta(t) = \lim_{s\to t} \frac{f(t)-f(s)}{t-s}$.

(iv) If f is delta differentiable at t, then $f(\sigma(t)) = f(t) + \mu(t) f^\Delta(t)$.

DEFINITION 3. A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* if it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limits are finite at all left-dense points in $\mathbb{T}$. We denote by $\mathrm{C_{rd}}$ the set of all rd-continuous functions. We say that f is rd-continuously delta differentiable (and write $f \in \mathrm{C^1_{rd}}$) if $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$ and $f^\Delta \in \mathrm{C_{rd}}$.

DEFINITION 4. A function $F : \mathbb{T} \to \mathbb{R}$ is called a *delta antiderivative* of $f : \mathbb{T} \to \mathbb{R}$ if $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}^\kappa$. Then we define the *delta integral* by

$$\int_a^t f(s)\Delta s = F(t) - F(a).$$

The importance of rd-continuous function is revealed by the following result.

THEOREM 2. *Every rd-continuous function has a delta antiderivative.*

Now we give some properties of the delta integral.

THEOREM 3. *If* $a,b,c \in \mathbb{T}$, $\alpha \in \mathbb{R}$ *and* $f,g \in \mathrm{C_{rd}}$, *then*

(i) $\int_a^b (f(t)+g(t))\,\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t$;

(ii) $\int_a^b \alpha f(t)\Delta t = \alpha \int_a^b f(t)\Delta t$;

(iii) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t$;

(iv) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$;

(v) $\int_a^a f(t)\Delta t = 0$;

(vi) *if* $f(t) \geqslant 0$ *for all* t, *then* $\int_a^b f(t)\Delta t \geqslant 0$.

Jensen's inequality is of great interest in the theory of differential and difference equations as well as other areas of mathematics.

The Jensen inequality on time scales via the Δ-integral has been recently obtained in [2] by Agarwal, Bohner and Peterson.

THEOREM 4. *Let* $a,b \in \mathbb{T}$, $a < b$ *and suppose* $I \subset \mathbb{R}$ *is an interval. If* $F : I \to \mathbb{R}$ *is convex (resp., concave) and* $f \in \mathrm{C_{rd}}([a,b],I)$, *then*

$$F\left(\frac{\int_a^b f(t)\Delta t}{b-a}\right) \leqslant \frac{\int_a^b F(f(t))\Delta t}{b-a} \tag{6}$$

(resp., the reversed inequality is valid). Moreover, if F *is strictly convex or strictly concave, then equality in* (6) *holds if and only if* f *is constant.*

2. New results

2.1. Levinson's inequality on time scale

Following the method of Pečarić, Praljak and Witkowski, [19], in the next theorem we establish Levinson's type inequality in the settings of time scale calculus. For simplicity, in what follows, we use following notation

$$\mathscr{D}_{[a,b]}(f) = \frac{1}{b-a}\left(\int_a^b (f(t))^2\,\Delta t - \frac{1}{b-a}\left(\int_a^b f(t)\Delta t\right)^2\right).$$

THEOREM 5. *Let* $a,b,d,e \in \mathbb{T}$, $a < b$, $d < e$ *and suppose* $I \subset \mathbb{R}$ *is an interval. Let* $f \in \mathrm{C_{rd}}([a,b],I)$, $g \in \mathrm{C_{rd}}([d,e],I)$ *and suppose there exists* $c \in I^0$ *such that*

$$\sup_{x\in[a,b]} f(x) \leqslant c \leqslant \inf_{x\in[d,e]} g(x). \tag{7}$$

If

$$\mathscr{D}_{[a,b]}(f) = \mathscr{D}_{[d,e]}(g) \tag{8}$$

then the inequality

$$\Phi\left(\frac{\int_d^e g(t)\Delta t}{e-d}\right) - \frac{1}{e-d}\int_d^e \Phi(g(t))\,\Delta t \leqslant \Phi\left(\frac{\int_a^b f(t)\Delta t}{b-a}\right) - \frac{1}{b-a}\int_a^b \Phi(f(t))\,\Delta t, \tag{9}$$

holds for every function $\Phi \in \mathscr{K}_1^c(I)$. *If the function* Φ *is contained in* $\mathscr{K}_2^c(I)$ *then the sign of inequality* (9) *is reversed.*

Proof. Following the definition 1 and using the function $\Phi \in \mathscr{K}_1^c(I)$, $c \in I^0$, we now define the function $F(x) = \Phi(x) - \frac{\alpha}{2}x^2$, where α is a constant such that the function F is concave on $(-\infty, c] \cap I$ and convex on $I \cap [c, \infty)$. Since F is concave on $(-\infty, c] \cap I$, from theorem 4, we get

$$F\left(\frac{\int_a^b f(t)\Delta t}{b-a}\right) \geqslant \frac{\int_a^b F(f(t))\Delta t}{b-a}. \tag{10}$$

Replacing $F(x)$ by $\Phi(x) - \frac{\alpha}{2}x^2$, it follows

$$\Phi\left(\frac{\int_a^b f(t)\Delta t}{b-a}\right) - \frac{\alpha}{2}\left(\frac{\int_a^b f(t)\Delta t}{b-a}\right)^2 - \frac{\int_a^b \left(\Phi(f(t)) - \frac{\alpha}{2}(f(t))^2\right)}{b-a} \geqslant 0$$

$$\Phi\left(\frac{\int_a^b f(t)\Delta t}{b-a}\right) - \frac{\alpha}{2(b-a)^2}\left(\int_a^b f(t)\Delta t\right)^2$$

$$-\frac{1}{b-a}\int_a^b \Phi(f(t))\Delta t + \frac{\alpha}{2(b-a)}\int_a^b (f(t))^2\Delta t \geqslant 0$$

$$\Phi\left(\frac{\int_a^b f(t)\Delta t}{b-a}\right) - \frac{1}{b-a}\int_a^b \Phi(f(t))\Delta t$$

$$+\frac{\alpha}{2(b-a)}\left(\int_a^b (f(t))^2\Delta t - \frac{1}{b-a}\left(\int_a^b f(t)\Delta t\right)^2\right) \geqslant 0. \tag{11}$$

Since F is convex on $I \cap [c, \infty)$ from theorem 4, we have

$$F\left(\frac{\int_d^e g(t)\Delta t}{e-d}\right) \leqslant \frac{\int_d^e F(g(t))\Delta t}{e-d}.$$

Replacing $F(x)$ by $\Phi(x) - \frac{\alpha}{2}x^2$, it follows

$$\Phi\left(\frac{\int_d^e g(t)\Delta t}{e-d}\right) - \frac{\alpha}{2}\left(\frac{\int_d^e g(t)\Delta t}{e-d}\right)^2 - \frac{\int_d^e \left(\Phi(g(t)) - \frac{\alpha}{2}(g(t))^2\right)}{e-d} \leqslant 0$$

$$\Phi\left(\frac{\int_d^e g(t)\Delta t}{e-d}\right) - \frac{\alpha}{2(e-d)^2}\left(\int_d^e g(t)\Delta t\right)^2$$

$$-\frac{1}{e-d}\int_d^e \Phi(g(t))\Delta t + \frac{\alpha}{2(e-d)}\int_d^e (g(t))^2\Delta t \leqslant 0$$

$$\Phi\left(\frac{\int_d^e g(t)\Delta t}{e-d}\right) - \frac{1}{e-d}\int_d^e \Phi(g(t))\Delta t$$

$$+\frac{\alpha}{2(e-d)}\left(\int_d^e (g(t))^2\Delta t-\frac{1}{e-d}\left(\int_d^e g(t)\Delta t\right)^2\right)\leqslant 0. \tag{12}$$

Rearranging (11), we get

$$-\frac{\alpha}{2(b-a)}\left(\int_a^b (f(t))^2\Delta t-\frac{1}{b-a}\left(\int_a^b f(t)\Delta t\right)^2\right)$$

$$\leqslant\Phi\left(\frac{\int_a^b f(t)\Delta t}{b-a}\right)-\frac{1}{b-a}\int_a^b \Phi(f(t))\Delta t. \tag{13}$$

Rearranging (12), we have

$$\frac{\alpha}{2(e-d)}\left(\int_d^e (g(t))^2\Delta t-\frac{1}{e-d}\left(\int_d^e g(t)\Delta t\right)^2\right)$$

$$\leqslant-\Phi\left(\frac{\int_d^e g(t)\Delta t}{e-d}\right)+\frac{1}{e-d}\int_d^e \Phi(g(t))\Delta t. \tag{14}$$

Adding up inequalities (13) and (14) and taking into account condition (8), we get inequality (9) which complites the proof. □

REMARK 1. It is obvious from the previus proof that the inequality (9) holds if the condition (8) is replaced by the weaker condition

$$\alpha\left(\mathscr{D}_{[d,e]}(g)-\mathscr{D}_{[a,b]}(f)\right)\geqslant 0. \tag{15}$$

Furthermore, the condition (15) can be weakened to

$$\mathscr{D}_{[d,e]}(g)-\mathscr{D}_{[a,b]}(f)\geqslant 0,$$

if, additionally, Φ is convex, since $\Phi''_-(c)\leqslant\alpha\leqslant\Phi''_+(c)$. Also, if Φ is concave, the condition (15) can be weakened to $\mathscr{D}_{[d,e]}(g)-\mathscr{D}_{[a,b]}(f)\leqslant 0$.

In [22] authors gave the following version of theorem 4.

THEOREM 6. *Let $a,b\in\mathbb{T}$, $a<b$ and suppose $I\subset\mathbb{R}$ is an interval. Assume $h\in C_{\mathrm{rd}}([a,b],\mathbb{R})$ is nonnegative function satisfying $\int_a^b h(t)\Delta t>0$. If $F:I\to\mathbb{R}$ is convex (resp., concave) and $f\in C_{\mathrm{rd}}([a,b],I)$, then*

$$F\left(\frac{\int_a^b h(t)f(t)\Delta t}{\int_a^b h(t)\Delta t}\right)\leqslant\frac{\int_a^b h(t)F(f(t))\Delta t}{\int_a^b h(t)\Delta t} \tag{16}$$

(resp., the reversed inequality is valid).

Using this result we generalize theorem 5. Let us denote

$$\overline{\mathscr{D}}_{[a,b]}(h,f)=\frac{\int_a^b h(t)\left(f(t)\right)^2\Delta t}{\int_a^b h(t)\Delta t}-\left(\frac{\int_a^b h(t)f(t)\Delta t}{\int_a^b h(t)\Delta t}\right)^2.$$

THEOREM 7. *Let $a,b,d,e\in\mathbb{T}$, $a<b$, $d<e$ and suppose $I\subset\mathbb{R}$ is an interval. Assume $h_1\in C_{rd}([a,b],\mathbb{R})$ and $h_2\in C_{rd}([d,e],\mathbb{R})$ are nonegative functions such that $\int_a^b h_1(t)\Delta t>0$, $\int_d^e h_2(t)\Delta t>0$. Let $f\in C_{rd}([a,b],I)$, $g\in C_{rd}([d,e],I)$ and suppose there exists $c\in I^0$ such that*

$$\sup_{x\in[a,b]} f(x)\leqslant c\leqslant \inf_{x\in[d,e]} g(x). \tag{17}$$

If

$$\overline{\mathscr{D}}_{[a,b]}(h_1,f)=\overline{\mathscr{D}}_{[d,e]}(h_2,g) \tag{18}$$

then the inequality

$$\Phi\left(\frac{\int_d^e h_2(t)g(t)\Delta t}{\int_d^e h_2(t)\Delta t}\right)-\frac{\int_d^e h_2(t)\Phi\left(g(t)\right)\Delta t}{\int_d^e h_2(t)\Delta t}\leqslant\Phi\left(\frac{\int_a^b h_1(t)f(t)\Delta t}{\int_a^b h_1(t)\Delta t}\right)-\frac{\int_a^b h_1(t)\Phi\left(f(t)\right)\Delta t}{\int_a^b h_1(t)\Delta t} \tag{19}$$

holds for every function $\Phi\in\mathscr{K}_1^c(I)$. If the function Φ is contained in $\mathscr{K}_2^c(I)$ then the sign of inequality (19) is reversed.

We omit the proof since it is similar to the proof of theorem 5.

2.2. Multidimensional case

In what follows we obtain multidimensional time scale versions of some inequalities of Levinson's type.

Multiple Riemann integration and multiple Lebesgue integration on time scale was introduced in [7] and [8], respectively.

Using the fact that the time scale integral is a positive linear functional, authors M. Anwar, R. Bibi, M. Bohner and J. Pečarić proved in [3] the generalization of inequality (16) in the terms of multiple Lebesgue delta integral.

THEOREM 8. *Assume $F\in C(I,\mathbb{R})$ is convex (resp., concave), where $I\subset\mathbb{R}$ is an interval. For time scales $\mathbb{T}_1,\ldots,\mathbb{T}_n$, suppose $\mathscr{E}\subset([a_1,b_1)\cap\mathbb{T}_1)\times\cdots\times([a_n,b_n)\cap\mathbb{T}_n)\subset\mathbb{R}^n$ and f is Δ-integrable on $\mathscr{E}$ such that $f(\mathscr{E})=I$. Moreover, let $h:\mathscr{E}\to\mathbb{R}$ be nonnegative, Δ-integrable such that $\int_{\mathscr{E}} h(t)\Delta t>0$. Then*

$$F\left(\frac{\int_{\mathscr{E}} h(t)f(t)\Delta t}{\int_{\mathscr{E}} h(t)\Delta t}\right)\leqslant\frac{\int_{\mathscr{E}} h(t)F(f(t))\Delta t}{\int_{\mathscr{E}} h(t)\Delta t}.$$

Analogusly as in theorem 5 and theorem 7, we can easily obtain following multiple version of Levinson's type inequality in the settings of time scale calculus. We omit the proof since it is similar to the proof of theorem 5.

Throughout this section and following sections let

$$\mathscr{E}_i \subset ([a_{i1},b_{i1})\cap\mathbb{T}_{i1})\times\cdots\times([a_{in},b_{in})\cap\mathbb{T}_{in})\subset\mathbb{R}^n,$$

be Lebesgue Δ-measurable sets for any index $i\in\mathbb{N}$ and let $\mathbb{T}_{i1},\ldots,\mathbb{T}_{in}$ be time scales, with $a_{ij},b_{ij}\in\mathbb{T}_j$, $a_{ij}<b_{ij}$, $1\leqslant j\leqslant n$.

THEOREM 9. *Let $I\subset\mathbb{R}$ be an interval and $h_i:\mathscr{E}_i\to\mathbb{R}$, $i=1,2$, be nonnegative, Δ-integrable such that $\int_{\mathscr{E}_i}h_i(t)\Delta t>0$. Let $f:\mathscr{E}_1\to I$ and $g:\mathscr{E}_2\to I$ be Δ-integrable functions on $\mathscr{E}_i$, respectively, and suppose there exists $c\in I^0$ such that*

$$\sup_{T\in\mathscr{E}_1} f(T)\leqslant c\leqslant \inf_{T\in\mathscr{E}_2} g(T). \tag{20}$$

If

$$\overline{\mathscr{D}}_{\mathscr{E}_1}(h_1,f)=\overline{\mathscr{D}}_{\mathscr{E}_2}(h_2,g) \tag{21}$$

then the inequality

$$\Phi\left(\frac{\int_{\mathscr{E}_2}h_2(t)g(t)\Delta t}{\int_{\mathscr{E}_2}h_2(t)\Delta t}\right)-\frac{\int\limits_{\mathscr{E}_2}h_2(t)\Phi(g(t))\Delta t}{\int_{\mathscr{E}_2}h_2(t)\Delta t}\leqslant\Phi\left(\frac{\int_{\mathscr{E}_1}h_1(t)f(t)\Delta t}{\int_{\mathscr{E}_1}h_1(t)\Delta t}\right)-\frac{\int\limits_{\mathscr{E}_1}h_1(t)\Phi(f(t))\Delta t}{\int_a^b h_1(t)\Delta t}, \tag{22}$$

holds for every function $\Phi\in\mathscr{K}_1^c(I)$.

2.3. Converse inequalities

Following converse of delta integral Jensen's inequality for convex functions is obtained in [3].

THEOREM 10. *Assume $F\in C(I,\mathbb{R})$ is convex (resp., concave), where $I=[m,M]\subset\mathbb{R}$, with $m<M$. For time scales $\mathbb{T}_1,\ldots,\mathbb{T}_n$, suppose $\mathscr{E}\subset([a_1,b_1)\cap\mathbb{T}_1)\times\cdots\times([a_n,b_n)\cap\mathbb{T}_n)\subset\mathbb{R}^n$ and f is Δ-integrable on $\mathscr{E}$ such that $f(\mathscr{E})=I$. Moreover, let $h:\mathscr{E}\to\mathbb{R}$ be nonnegative, Δ-integrable such that $\int\limits_{\mathscr{E}}h(t)\Delta t>0$. Then*

$$\frac{\int\limits_{\mathscr{E}}h(t)F(f(t))\Delta t}{\int\limits_{\mathscr{E}}h(t)\Delta t}\leqslant\frac{M-\frac{\int\limits_{\mathscr{E}}h(t)f(t)\Delta t}{\int\limits_{\mathscr{E}}h(t)\Delta t}}{M-m}F(m)+\frac{\frac{\int\limits_{\mathscr{E}}h(t)f(t)\Delta t}{\int\limits_{\mathscr{E}}h(t)\Delta t}-m}{M-m}F(M), \tag{23}$$

(resp., the reversed inequality is valid).

Now, we state new converses of Levinson's type inequality on time scale.

In order to simplify the notation, we denote:

$$\overline{f}=\frac{\int_{\mathscr{E}_1}h_1(t)f(t)\Delta t}{\int_{\mathscr{E}_1}h_1(t)\Delta t}\quad\text{and}\quad \overline{g}=\frac{\int_{\mathscr{E}_2}h_2(t)g(t)\Delta t}{\int_{\mathscr{E}_2}h_2(t)\Delta t}.$$

THEOREM 11. *Let $h_i:\mathscr{E}_i\to\mathbb{R}$, $i=1,2$, be nonnegative, Δ-integrable such that $\int_{\mathscr{E}_i}h_i(t)\Delta t>0$. Let $f:\mathscr{E}_1\to[m,M]\subset I$ and $g:\mathscr{E}_2\to[n,N]\subset I$, $m<M$, $n<N$, be Δ-integrable functions on $\mathscr{E}_i$, respectively, and suppose there exists $c\in I^0$ such that*

$$\sup_{T\in\mathscr{E}_1}f(T)\leqslant c\leqslant\inf_{T\in\mathscr{E}_2}g(T). \tag{24}$$

If

$$\frac{M-\overline{f}}{M-m}m^2+\frac{\overline{f}-m}{M-m}M^2-\frac{\int_{\mathscr{E}_1}h_1(t)f^2(t)\Delta t}{\int_{\mathscr{E}_1}h_1(t)\Delta t}=\frac{N-\overline{g}}{N-n}n^2+\frac{\overline{g}-n}{N-n}N^2-\frac{\int_{\mathscr{E}_2}h_2(t)g^2(t)\Delta t}{\int_{\mathscr{E}_2}h_2(t)\Delta t} \tag{25}$$

then inequality

$$\begin{aligned}&\frac{M-\overline{f}}{M-m}\Phi(m)+\frac{\overline{f}-m}{M-m}\Phi(M)-\frac{\int_{\mathscr{E}_1}h_1(t)\Phi(f(t))\Delta t}{\int_{\mathscr{E}_1}h_1(t)\Delta t}\\ \leqslant&\frac{N-\overline{g}}{N-n}\Phi(n)+\frac{\overline{g}-n}{N-n}\Phi(N)-\frac{\int_{\mathscr{E}_2}h_2(t)\Phi(g(t))\Delta t}{\int_{\mathscr{E}_2}h_2(t)\Delta t}\end{aligned} \tag{26}$$

holds for every function $\Phi\in\mathscr{K}_1^c(I)$.

Proof. Let us define the function $F(x)=\Phi(x)-\frac{\alpha}{2}x^2$, where α is a constant from definition 1. Since F is concave on $[m,M]\cap I$, from theorem 10, we have

$$\begin{aligned}&\frac{\int_{\mathscr{E}_1}h_1(t)F(f(t))\Delta t}{\int_{\mathscr{E}_1}h_1(t)\Delta t}\\ \geqslant&\frac{M-\overline{f}}{M-m}F(m)+\frac{\overline{f}-m}{M-m}F(M)\frac{\int_{\mathscr{E}_1}h_1(t)\left(\Phi(f(t))-\frac{\alpha}{2}f^2(t)\right)\Delta t}{\int_{\mathscr{E}_1}h_1(t)\Delta t}\\ \geqslant&\frac{M-\overline{f}}{M-m}\left(\Phi(m)-\frac{\alpha}{2}m^2\right)+\frac{\overline{f}-m}{M-m}\left(\Phi(M)-\frac{\alpha}{2}M^2\right)\frac{\int_{\mathscr{E}_1}h_1(t)\Phi(f(t))\Delta t}{\int_{\mathscr{E}_1}h_1(t)\Delta t}\\ &-\frac{\alpha}{2}\frac{\int_{\mathscr{E}_1}h_1(t)f^2(t)\Delta t}{\int_{\mathscr{E}_1}h_1(t)\Delta t}\\ \geqslant&\frac{M-\overline{f}}{M-m}\Phi(m)-\frac{\alpha}{2}m^2\cdot\frac{M-\overline{f}}{M-m}+\frac{\overline{f}-m}{M-m}\Phi(M)-\frac{\alpha}{2}M^2\cdot\frac{\overline{f}-m}{M-m}\frac{M-\overline{f}}{M-m}\Phi(m)\\ &+\frac{\overline{f}-m}{M-m}\Phi(M)-\frac{\int_{\mathscr{E}_1}h_1(t)\Phi(f(t))\Delta t}{\int_{\mathscr{E}_1}h_1(t)\Delta t}\end{aligned}$$

$$\leqslant \frac{\alpha}{2} \cdot \left(\frac{M - \overline{f}}{M - m} m^2 + \frac{\overline{f} - m}{M - m} M^2 - \frac{\int_{\mathscr{E}_1} h_1(t) f^2(t) \Delta t}{\int_{\mathscr{E}_1} h_1(t) \Delta t} \right). \tag{27}$$

Since the function F is convex on $I \cap [n, N]$, from theorem 10, it follows

$$\begin{aligned}
&\frac{\int_{\mathscr{E}_2} h_2(t) F(g(t)) \Delta t}{\int_{\mathscr{E}_2} h_2(t) \Delta t} \\
&\leqslant \frac{N - \overline{g}}{N - n} F(n) + \frac{\overline{g} - n}{N - n} F(N) \frac{\int_{\mathscr{E}_2} h_2(t) \left(\Phi(g(t)) - \frac{\alpha}{2} g^2(t) \right) \Delta t}{\int_{\mathscr{E}_2} h_2(t) \Delta t} \\
&\leqslant \frac{N - \overline{g}}{N - n} \left(\Phi(n) - \frac{\alpha}{2} n^2 \right) + \frac{\overline{g} - n}{N - n} \left(\Phi(N) - \frac{\alpha}{2} N^2 \right) \frac{\int_{\mathscr{E}_2} h_2(t) \Phi(g(t)) \Delta t}{\int_{\mathscr{E}_2} h_2(t) \Delta t} \\
&\quad - \frac{\alpha}{2} \frac{\int_{\mathscr{E}_2} h_2(t) g^2(t) \Delta t}{\int_{\mathscr{E}_2} h_2(t) \Delta t} \\
&\leqslant \frac{N - \overline{g}}{N - n} \Phi(n) - \frac{\alpha}{2} n^2 \cdot \frac{N - \overline{g}}{N - n} + \frac{\overline{g} - n}{N - n} \Phi(N) - \frac{\alpha}{2} N^2 \cdot \frac{\overline{g} - n}{N - n} \cdot \frac{\alpha}{2} \\
&\quad \times \left(\frac{N - \overline{g}}{N - n} n^2 + \frac{\overline{g} - n}{N - n} N^2 - \frac{\int_{\mathscr{E}_2} h_2(t) g^2(t) \Delta t}{\int_{\mathscr{E}_2} h_2(t) \Delta t} \right) \\
&\leqslant \frac{N - \overline{g}}{N - n} \Phi(n) + \frac{\overline{g} - n}{N - n} \Phi(N) - \frac{\int_{\mathscr{E}_2} h_2(t) \Phi(g(t)) \Delta t}{\int_{\mathscr{E}_2} h_2(t) \Delta t}.
\end{aligned} \tag{28}$$

Adding up inequalities (27) and (28) and taking into account condition (25), we get inequality (26) which completes the proof. □

Using inequality (26) and converse of Jessen's inequality on time scale proved in [5, theorem 2.1], in the next theorem, we obtain another converse of Levinson's type inequality on time scale.

THEOREM 12. *Let $h_i : \mathscr{E}_i \to \mathbb{R}$, $i = 1, 2$, be nonnegative, Δ-integrable such that $\int_{\mathscr{E}_i} h_i(t) \Delta t > 0$. Let $f : \mathscr{E}_1 \to [m, M] \subset I$ and $g : \mathscr{E}_2 \to [n, N] \subset I$, $m < M$, $n < N$, be Δ-integrable functions on $\mathscr{E}_i$, respectively, and suppose there exists $c \in I^0$ such that*

$$\sup_{T \in \mathscr{E}_1} f(T) \leqslant c \leqslant \inf_{T \in \mathscr{E}_2} g(T). \tag{29}$$

If

$$(M - m)^2 + 2 \frac{\int_{\mathscr{E}_2} h_2(t) g^2(t) \Delta t}{\int_{\mathscr{E}_2} h_2(t) \Delta t} = 2 \frac{\int_{\mathscr{E}_1} h_1(t) f^2(t) \Delta t}{\int_{\mathscr{E}_1} h_1(t) \Delta t} + (N - n)^2 \tag{30}$$

then inequality

$$\frac{1}{4} (M - m) \left(\Phi'_-(M) - \Phi'_+(m) \right) - \frac{\int_{\mathscr{E}_1} h_1(t) \Phi(f(t)) \Delta t}{\int_{\mathscr{E}_1} h_1(t) \Delta t} \tag{31}$$

$$\leqslant \frac{1}{4}(N-n)\left(\Phi'_-(N)-\Phi'_+(n)\right)-\frac{\int_{\mathscr{E}_2} h_2(t)\Phi(g(t))\Delta t}{\int_{\mathscr{E}_2} h_2(t)\Delta t}$$

holds for every function $\Phi \in \mathscr{K}_1^c(I)$, *where* $\Phi'_-(M)=\lim_{x\to M^-}\frac{\Phi(x)-\Phi(M)}{x-M}$ *is a left hand derivative of* Φ *at* M, *and* $\Phi'_+(m)=\lim_{x\to m^+}\frac{\Phi(x)-\phi(m)}{x-m}$ *is a right hand derivative of* Φ *at* m, $x\in I$.

Proof. Let us define the function $F(x)=\Phi(x)-\frac{\alpha}{2}x^2$, where α is a constant from definition 1. Since F is concave on $[m,M]\cap I$, from [5, theorem 2.1], we get

$$\frac{\int_{\mathscr{E}_1} h_1(t)F(f(t))\Delta t}{\int_{\mathscr{E}_1} h_1(t)\Delta t}\geqslant \left(M-\overline{f}\right)\left(\overline{f}-m\right)\cdot\frac{F'_-(M)-F'_+(m)}{M-m}.$$

$$\begin{aligned}
&\frac{\int_{\mathscr{E}_1} h_1(t)\left(\Phi(f(t))-\frac{\alpha}{2}f^2(t)\right)\Delta t}{\int_{\mathscr{E}_1} h_1(t)\Delta t}\\
&\geqslant \left(M-\overline{f}\right)\left(\overline{f}-m\right)\cdot\frac{\left(\Phi'_-(M)-\alpha M\right)-\left(\Phi'_+(m)-\alpha m\right)}{M-m}\frac{\int_{\mathscr{E}_1} h_1(t)\Phi(f(t))\Delta t}{\int_{\mathscr{E}_1} h_1(t)\Delta t}\\
&\quad-\frac{\alpha}{2}\frac{\int_{\mathscr{E}_1} h_1(t)f^2(t)\Delta t}{\int_{\mathscr{E}_1} h_1(t)\Delta t}\\
&\geqslant \frac{1}{4}(M-m)\left(\Phi'_-(M)-\Phi'_+(m)-\alpha(M-m)\right).
\end{aligned}\tag{32}$$

Since F is convex on $I\cap[n,N]$, from [5, theorem 2.1] we obtain

$$\begin{aligned}
&\frac{\int_{\mathscr{E}_2} h_2(t)F(g(t))\Delta t}{\int_{\mathscr{E}_1} h_1(t)\Delta t}\\
&\leqslant (N-\overline{g})(\overline{g}-n)\cdot\frac{F'_-(N)-F'_+(n)}{N-n}\frac{\int_{\mathscr{E}_2} h_2(t)\left(\Phi(g(t))-\frac{\alpha}{2}g^2(t)\right)\Delta t}{\int_{\mathscr{E}_2} h_2(t)\Delta t}\\
&\leqslant (N-\overline{g})(\overline{g}-n)\cdot\frac{\left(\Phi'_-(N)-\alpha N\right)-\left(\Phi'_+(n)-\alpha n\right)}{N-n}\frac{\int_{\mathscr{E}_2} h_2(t)\Phi(g(t))\Delta t}{\int_{\mathscr{E}_2} h_2(t)\Delta t}\\
&\quad-\frac{\alpha}{2}\frac{\int_{\mathscr{E}_2} h_2(t)g^2(t)\Delta t}{\int_{\mathscr{E}_2} h_2(t)\Delta t}\\
&\leqslant \frac{1}{4}(N-n)\left(\Phi'_-(N)-\Phi'_+(n)-\alpha(N-n)\right).
\end{aligned}\tag{33}$$

Adding up inequalities (32) and (33) and taking into account condition (30), we get inequality (31) which completes the proof. □

2.4. Jensen-Mercer's type inequality

In 2003, A. McD. Mercer in [17] gave a variant of Jensen's inequality, called the Jensen–Mercer's inequality. Later, W. S. Cheung et al. generalized the Jensen–Mercer's inequality for isotonic linear functionals, called Jensen–Mercer's inequality (see [11]). Further in [1], S. Abramovich et al. gave the refinement of the Jensen–Mercer's inequality for superquadratic functions. In the following theorem we state Jensen-Mercer's inequality on time scale proved in [6].

THEOREM 13. *Suppose f is a Δ-integrable function on $\mathscr{E}$ such that $f(\mathscr{E}) \subseteq [m,M]$ and $h\colon \mathscr{E} \to \mathbb{R}$ is nonnegative and Δ-integrable such that $\int_{\mathscr{E}} h(t)\Delta t > 0$. Let $F \in C([m,M],\mathbb{R})$. If F is convex, then*

$$F\left(m+M-\frac{\int_{\mathscr{E}} h(t)f(t)\Delta t}{\int_{\mathscr{E}} h(t)\Delta t}\right) \leqslant \frac{\int_{\mathscr{E}} h(t)F(m+M-f)\Delta t}{\int_{\mathscr{E}} h(t)\Delta t}$$
$$\leqslant \frac{M-\frac{\int_{\mathscr{E}} h(t)f(t)\Delta t}{\int_{\mathscr{E}} h(t)\Delta t}}{M-m}F(M)+\frac{\frac{\int_{\mathscr{E}} h(t)f(t)\Delta t}{\int_{\mathscr{E}} h(t)\Delta t}-m}{M-m}F(m)$$
$$\leqslant F(m)+F(M)-\frac{\int_{\mathscr{E}} h(t)F(f(t))\Delta t}{\int_{\mathscr{E}} h(t)\Delta t}.$$

Moreover, if F is concave, then the above inequalities hold in reverse order.

Here we obtain Levinson's type generalization of Jensen-Mercer's inequality on time scale.

THEOREM 14. *Let $h_i : \mathscr{E}_i \to \mathbb{R}$, $i = 1,2$, be nonnegative, Δ-integrable such that $\int_{\mathscr{E}_i} h_i(t)\Delta t > 0$. Let $f : \mathscr{E}_1 \to [m,M] \subset I$ and $g : \mathscr{E}_2 \to [n,N] \subset I$, $m < M$, $n < N$, be Δ-integrable functions on $\mathscr{E}_i$, respectively, and suppose there exists $c \in I^0$ such that*

$$\sup_{T\in\mathscr{E}_1} f(T) \leqslant c \leqslant \inf_{T\in\mathscr{E}_2} g(T). \tag{34}$$

If

$$m+M-\overline{f}-\left(m^2+M^2\right)+\frac{\int_{\mathscr{E}_1} h_1(t)f^2(t)\Delta t}{\int_{\mathscr{E}_1} h_1(t)\Delta t} = n+N-\overline{g}-\left(n^2+N^2\right)+\frac{\int_{\mathscr{E}_2} h_2(t)g^2(t)\Delta t}{\int_{\mathscr{E}_2} h_2(t)\Delta t} \tag{35}$$

then inequality

$$\Phi(m)+\Phi(M)-\Phi\left(m+M-\overline{f}\right)-\frac{\int_{\mathscr{E}_1} h_1(t)\Phi(f(t))\Delta t}{\int_{\mathscr{E}_1} h_1(t)\Delta t} \tag{36}$$

$$\leqslant \Phi(n)+\Phi(N)-\Phi(n+N-\overline{g})-\frac{\int_{\mathscr{E}_2} h_2(t)\Phi(g(t))\Delta t}{\int_{\mathscr{E}_2} h_2(t)\Delta t} \tag{37}$$

holds for every function $\Phi \in \mathscr{K}_1^c(I)$.

Proof. If we define the function F as follows: $F(x) = \Phi(x) - \frac{\alpha}{2}x^2$, where α is a constant from definition 1, then, since F is concave on $[m,M]\cap I$, from theorem 13, we get

$$\begin{aligned}
&\Phi\left(m+M-\overline{f}\right) - \frac{\alpha}{2}\left(m+M-\overline{f}\right)^2 \\
\geqslant &\Phi(m) - \frac{\alpha}{2}m^2 + \Phi(M) - \frac{\alpha}{2}M^2 - \frac{\int_{\mathscr{E}_1} h_1(t)\left(\Phi(f(t)) - \frac{\alpha}{2}f^2(t)\right)\Delta t}{\int_{\mathscr{E}_1} h_1(t)\Delta t}\Phi\left(m+M-\overline{f}\right) \\
&- \Phi(m) - \Phi(M) + \frac{\int_{\mathscr{E}_1} h_1(t)\Phi(f(t))\Delta t}{\int_{\mathscr{E}_1} h_1(t)\Delta t} \\
\geqslant &\frac{\alpha}{2}\left(m+M-\overline{f}\right)^2 - \frac{\alpha}{2}\left(m^2+M^2\right) + \frac{\alpha}{2}\frac{\int_{\mathscr{E}_1} h_1(t)f^2(t)\Delta t}{\int_{\mathscr{E}_1} h_1(t)\Delta t}.
\end{aligned} \tag{38}$$

Since F is convex on $I\cap[n,N]$, from theorem 13, we obtain

$$\begin{aligned}
&\Phi\left(n+N-\overline{g}\right) - \frac{\alpha}{2}\left(n+N-\overline{g}\right)^2 \\
\leqslant &\Phi(n) - \frac{\alpha}{2}n^2 + \Phi(N) - \frac{\alpha}{2}N^2 - \frac{\int_{\mathscr{E}_2} h_2(t)\left(\Phi(g(t)) - \frac{\alpha}{2}g^2(t)\right)\Delta t}{\int_{\mathscr{E}_2} h_2(t)\Delta t}\Phi\left(n+N-\overline{g}\right) \\
&- \Phi(n) - \Phi(N) + \frac{\int_{\mathscr{E}_2} h_2(t)\Phi(g(t))\Delta t}{\int_{\mathscr{E}_2} h_2(t)\Delta t} \\
\leqslant &\frac{\alpha}{2}\left(n+N-\overline{g}\right)^2 - \frac{\alpha}{2}\left(n^2+N^2\right) + \frac{\alpha}{2}\frac{\int_{\mathscr{E}_2} h_2(t)g^2(t)\Delta t}{\int_{\mathscr{E}_2} h_2(t)\Delta t}.
\end{aligned} \tag{39}$$

Adding up inequalities (38) and (39) and taking into account condition (35), we get inequality (36) which completes the proof. □

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