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ESTIMATES ON THE GAP IN BULLEN'S INEQUALITY

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Abstract. Some companions and estimates of the gap in Bullen's inequality are shown. The results are applied to special means and connections with the methods of computational fluid dynamics are indicated.

1. Introduction

We start from an important result related to the convex functions known as Bullen's inequality [3] (see also [13] and [21, p. 52]) which for a convex function $f:[a,b]\to\mathbb{R}$ asserts that

$$\frac{2}{b-a}\int_a^b f(x)\,\mathrm{d}x \leqslant \frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right). \tag{1.1}$$

It is strongly related and also an immediate consequence of the Hermite-Hadamard inequality ([12], [14])

$$f\left(\frac{x+y}{2}\right)\leqslant \frac{1}{y-x}\int_x^y f(t)\mathrm{d}t \leqslant \frac{f(x)+f(y)}{2}, \tag{1.2}$$

applied on the intervals $\left[a,\frac{a+b}{2}\right]$, $\left[\frac{a+b}{2},b\right]$.

REMARK 1. By another approach, Bullen's inequality can be inferred from a particular case of [20, p. 717, Theorem 1] (see also [4, Corollary 1]) as follows. It holds that

$$\begin{aligned}
&2\min\{\lambda,1-\lambda\}\left(\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right)\\
\leqslant&\lambda f(a)+(1-\lambda)f(b)-f(\lambda a+(1-\lambda)b)\\
\leqslant&2\max\{\lambda,1-\lambda\}\left(\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right),
\end{aligned}$$

for all $\lambda\in[0,1]$. One should integrate the left-hand side of this inequality with respect to λ on $[0,1]$.

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For more inequalities which imply Bullen inequality see [2]. We also refer the interested reader to classic papers from which such problems arise, such as [15], [17].

The following results will be extended in later sections.

THEOREM 1. *(Vasić and Lacković [26], and Lupaş [16]).* *Let p and q be two positive numbers. Then the inequalities*

$$f\left(\frac{pa+qb}{p+q}\right) \leqslant \frac{1}{2y}\int_{A-y}^{A+y} f(x)\,\mathrm{d}x \leqslant \frac{pf(a)+qf(b)}{p+q}$$

hold for $A=\frac{pa+qb}{p+q}$, $y>0$ and all continuous convex functions $f:[a,b]\to\mathbb{R}$ if and only if

$$y \leqslant \frac{b-a}{p+q}\min\{p,q\}.$$

See also [24, p.143].

Dragomir & Pearce [8] obtained a refinement of (1.1) in the following form.

THEOREM 2. *(Dragomir & Pearce [8, Theorem 34]) Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function for which there exist real constants m and M such that $m \leqslant f''(x) \leqslant M$ for all $x\in[a,b]$. Then*

$$m\frac{(b-a)^2}{24} \leqslant \frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right) - \frac{2}{b-a}\int_a^b f(x)\,\mathrm{d}x \leqslant M\frac{(b-a)^2}{24}. \tag{1.3}$$

See also [5], [6] and [7].

The following two reverse inequalities of Bullen's inequality were obtained in [19]:

$$\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right) - \frac{2}{b-a}\int_a^b f(x)\,\mathrm{d}x \leqslant \frac{(b-a)(f'(b)-f'(a))}{16}, \tag{1.4}$$

where f is a twice differentiable and convex function, and

$$\left|\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right) - \frac{2}{b-a}\int_a^b f(x)\,\mathrm{d}x - \frac{(b-a)(f'(b)-f'(a))}{24}\right| \leqslant \frac{(M-m)(b-a)^2}{64}, \tag{1.5}$$

under the assumptions of Theorem 2.

Acu and Gonska [1] extended Bullen's inequality for continuous functions by using the second order modulus of smoothness.

In the sequel the symbol $L^2_{[a,b]}$ stands for the space of all 2 -power integrable functions on the interval $[a,b]$.

Niezgoda [23] proved the following result.

THEOREM 3. ([23, Corollary 4.5]) *Let* $f,g,\alpha,\beta,\gamma,\delta \in L^2_{[a,b]}$ *be functions such that the following two conditions are satisfied:*

i) $\alpha(x) \leqslant f(x) \leqslant \beta(x)$ *and* $\gamma(x) \leqslant g(x) \leqslant \delta(x)$ *for all* $x \in [a,b]$, *or more generally*

$$\int_a^b (\beta(x)-f(x))(f(x)-\alpha(x))\,\mathrm{d}x \geqslant 0 \quad \text{and} \quad \int_a^b (\delta(x)-g(x))(g(x)-\gamma(x))\,\mathrm{d}x \geqslant 0;$$

ii) $\alpha+\beta$ *and* $\gamma+\delta$ *are constant functions, that is for some real constants* C_1, C_2 *it holds that* $\alpha(x)+\beta(x) = C_1$ *and* $\gamma(x)+\delta(x) = C_2$ *for* $x \in [a,b]$.

Then we have the inequality

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x - \frac{1}{(b-a)^2}\int_a^b f(x)\,\mathrm{d}x \cdot \int_a^b g(x)\,\mathrm{d}x\right| \leqslant \frac{1}{4(b-a)}\left(\int_a^b (\beta(x)-\alpha(x))^2\,\mathrm{d}x\right)^{1/2}\left(\int_a^b (\delta(x)-\gamma(x))^2\,\mathrm{d}x\right)^{1/2}. \tag{1.6}$$

See also [22]. Moreover, the above theorem contains the classic Grüss inequality [11] as a special case.

COROLLARY 1. *Let* $f,g \in L^2_{[a,b]}$ *be functions and* $\alpha_0, \beta_0, \gamma_0, \delta_0$ *be real constants such that* $\alpha_0 \leqslant f(x) \leqslant \beta_0$ *and* $\gamma_0 \leqslant g(x) \leqslant \delta_0$ *for all* $x \in [a,b]$. *Then*

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x - \frac{1}{(b-a)^2}\int_a^b f(x)\,\mathrm{d}x \cdot \int_a^b g(x)\,\mathrm{d}x\right| \leqslant \frac{1}{4}(\beta_0-\alpha_0)(\delta_0-\gamma_0). \tag{1.7}$$

Notice that in some situations such inequalities with functions $\alpha(x)$ and $\beta(x)$ can provide tighter estimates than the original ones with constants α_0 and β_0 (see [23, Example 4.6]).

2. Main results

2.1. Bounds for the generalized Bullen functional

Motivated by the above results, in this section we estimate the *generalized Bullen functional*

$$\mathscr{B}(f;a,b,c) = (b-a)\left(f(c) + \frac{c-a}{b-a}f(a) + \frac{b-c}{b-a}f(b) - \frac{2}{b-a}\int_a^b f(x)\,\mathrm{d}x\right),$$

$c \in [a,b]$. We see that $\mathscr{B}(f;a,b,c)$ takes positive values for convex functions f, as a consequence of the Hermite-Hadamard inequality applied on the intervals $[a,c]$, $[c,b]$.

The standard case $\mathscr{B}(f;a,b)$ of this functional is obtained for $c = \frac{a+b}{2}$, as follows

$$\mathscr{B}(f;a,b) = (b-a)\left(f\left(\frac{a+b}{2}\right) + \frac{1}{2}f(a) + \frac{1}{2}f(b) - \frac{2}{b-a}\int_a^b f(x)\,\mathrm{d}x\right)$$

(see [1], [11]).

The following result corresponds to the case $n=2$ of the identity (8) in [10], so we omit its proof.

LEMMA 1. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function such that f'' is continuous. Then*

$$\int_a^b (x-c)q_c(x)f''(x)\,\mathrm{d}x = \mathscr{B}(f;a,b,c), \tag{2.1}$$

where $a<c<b$ and

$$q_c(x) := \begin{cases} a-x, & x\in[a,c), \\ b-x, & x\in[c,b] \end{cases}.$$

As a particular case of (2.1), for $f(x) = \frac{1}{2}x^2$ one has

$$\int_a^b (x-c)q_c(x)\,\mathrm{d}x = \frac{b-a}{6}(a^2+b^2+3c^2+ab-3bc-3ac). \tag{2.2}$$

THEOREM 4. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function such that f'' is continuous and there exist real constants m and M, $m\leqslant f''(x)\leqslant M$ for all $x\in[a,b]$. Then*

$$m\frac{b-a}{6}(a^2+b^2+3c^2+ab-3bc-3ac) \leqslant \mathscr{B}(f;a,b,c) \leqslant M\frac{b-a}{6}(a^2+b^2+3c^2+ab-3bc-3ac). \tag{2.3}$$

Proof. Clearly, $(x-c)q_c(x)\geqslant 0$ for $x\in[a,b]$. Consequently, since $m\leqslant f''(x)\leqslant M$ for all $x\in[a,b]$, we have

$$m(x-c)q_c(x) \leqslant (x-c)q_c(x)f''(x) \leqslant M(x-c)q_c(x). \tag{2.4}$$

Integrating from a to b and using (2.2) and Lemma 1, we obtain (2.3). □

The particular case $c=\frac{a+b}{2}$ gives the inequalities (1.3).

THEOREM 5. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable and convex function such that f'' is continuous. Then the following inequality holds:*

$$0\leqslant \mathscr{B}(f;a,b,c)\leqslant \frac{1}{4}\max\{(a-c)^2,(b-c)^2\}[f'(b)-f'(a)], \tag{2.5}$$

where $a<c<b$.

Proof. Since f is convex, $f''(x)\geqslant 0$ for $x\in[a,b]$. Because

$$0\leqslant (x-c)\,q_c(x)\leqslant \frac{1}{4}\max\{(a-c)^2,(b-c)^2\},$$

we obtain

$$0\leqslant (x-c)\,q_c(x)f''(x)\leqslant \frac{1}{4}\max\{(a-c)^2,(b-c)^2\}f''(x),$$

for $x\in[a,b]$. Hence, by integrating the last inequality from a to b we get

$$0\leqslant \int_a^b (x-c)\,q_c(x)f''(x)\,dx\leqslant \frac{1}{4}\max\{(a-c)^2,(b-c)^2\}[f'(b)-f'(a)].$$

Indeed, the equality (2.1) together with the previous inequality leads to the conclusion. □

2.2. Bullen-Grüss type inequalities

Using this technique, in what follows we estimate the Bullen functional.

THEOREM 6. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function such that f'' is continuous and there exist real functions $\alpha(x)$ and $\beta(x)$ in $L^2_{[a,b]}$, satisfying:*

i) $\alpha(x)\leqslant f''(x)\leqslant\beta(x)$ *for all $x\in[a,b]$ and*

ii) $\alpha(x)+\beta(x)=C$ *with $x\in[a,b]$ for some real constant C.*

Then

$$\left|\mathscr{B}(f;a,b,c)-\frac{1}{6}(a^2+b^2+3c^2+ab-3bc-3ac)\frac{f'(b)-f'(a)}{b-a}\right|$$
$$\leqslant\frac{1}{16(b-a)^{1/2}}\left(\int_a^b(\beta(x)-\alpha(x))^2\,dx\right)^{1/2}\max\{(a-c)^2,(b-c)^2\},$$

where $a<c<b$.

Proof. Obviously we have

$$0 \leqslant (x-c)q_c(x) \leqslant \frac{1}{4}\max\{(a-c)^2,(b-c)^2\}$$

and $\alpha(x) \leqslant f''(x) \leqslant \beta(x)$ for $x \in [a,b]$. Thus, applying Theorem 3 we get the estimate

$$\left|\frac{1}{b-a}\int_a^b (x-c)q_c(x)f''(x)\,\mathrm{d}x - \frac{1}{(b-a)^2}\int_a^b (x-c)q_c(x)\,\mathrm{d}x \cdot \int_a^b f''(x)\,\mathrm{d}x\right|$$

$$\leqslant \frac{1}{16}\left(\frac{1}{b-a}\int_a^b (\beta(x)-\alpha(x))^2\,\mathrm{d}x\right)^{1/2} \max\{(a-c)^2,(b-c)^2\}.$$

The result now follows by (2.1) and (2.2). □

COROLLARY 2. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function such that f'' is continuous and there exist real constants m and M, $m \leqslant f''(x) \leqslant M$ for all $x \in [a,b]$. Then*

$$\left|\mathscr{B}(f;a,b,c) - \frac{1}{6}(a^2+b^2+3c^2+ab-3bc-3ac)\frac{f'(b)-f'(a)}{b-a}\right| \leqslant \frac{M-m}{16}\max\{(a-c)^2,(b-c)^2\}, \tag{2.6}$$

where $a<c<b$.

Proof. The previous theorem is applied for the constant functions $\alpha(x)=m$ and $\beta(x)=M$ for all $x\in[a,b]$. □

REMARK 2. (a) By writing $c=\lambda a+(1-\lambda)b$ with $\lambda\in[0,1]$, inequalities (2.5) and (2.6) can be rewritten as

$$0 \leqslant (b-a)f(\lambda a+(1-\lambda)b)+(b-a)[(1-\lambda)f(a)+\lambda f(b)]-2\int_a^b f(x)\,\mathrm{d}x \leqslant \frac{(b-a)^2}{4}\max\{(1-\lambda)^2,\lambda^2\}[f'(b)-f'(a)], \tag{2.7}$$

and

$$\left|(b-a)f(\lambda a+(1-\lambda)b)+(b-a)[(1-\lambda)f(a)+\lambda f(b)]-2\int_a^b f(x)\,\mathrm{d}x - \frac{(1-3\lambda(1-\lambda))(b-a)(f'(b)-f'(a))}{6}\right| \leqslant \frac{(b-a)^2(M-m)}{16}\max\{(1-\lambda)^2,\lambda^2\}.$$

(b) Letting $\lambda = \frac{1}{2}$ in the inequalities from (a) we deduce the inequalities (1.4)-(1.5).

2.3. Applications

In the remainder, we briefly focus on three particular cases of interest of the inequality (2.7):

(a) For $f(x) = x^p$ with $p > 1$ one has

$$\frac{(\lambda a+(1-\lambda)b)^p+(1-\lambda)a^p+\lambda b^p}{2} - \frac{p(b-a)(b^{p-1}-a^{p-1})}{8}\max\{(1-\lambda)^2,\lambda^2\}$$
$$\leqslant [L_{p+1}(a,b)]^p \leqslant \frac{(\lambda a+(1-\lambda)b)^p+(1-\lambda)a^p+\lambda b^p}{2},$$

where $L_p(a,b) = \left[\frac{a^p-b^p}{p(a-b)}\right]^{\frac{1}{p-1}}$ is Stolarsky's mean.

(b) For $f(x) = \frac{1}{x}$ with $x > 0$ we get

$$\frac{2}{\frac{1}{\lambda a+(1-\lambda)b} + \frac{1-\lambda}{a} + \frac{\lambda}{b}} \leqslant L(a,b)$$
$$\leqslant \frac{2}{\frac{1}{\lambda a+(1-\lambda)b} + \frac{1-\lambda}{a} + \frac{\lambda}{b} - \frac{(b-a)^2(a+b)}{4a^2b^2}\max\{(1-\lambda)^2,\lambda^2\}},$$

where $L(a,b) = \frac{a-b}{\ln a - \ln b}$ is the logarithmic mean.

(c) For $f(x) = -\ln x$ with $x > 0$ it holds

$$(\lambda a+(1-\lambda)b)a^{1-\lambda}b^{\lambda} \leqslant I^2(a,b) \leqslant (\lambda a+(1-\lambda)b)a^{1-\lambda}b^{\lambda}e^{\frac{(b-a)^2}{4ab}\max\{(1-\lambda)^2,\lambda^2\}},$$

where $I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$ is the identric mean.

2.4. Close companions of Bullen's functional

Let p and q be two positive numbers. Related to Theorem 1, the following Bullen type inequality was established in [2] (see also [24, p.146]):

$$\frac{1}{y}\int_{A-y}^{A+y} f(x)\,\mathrm{d}x \leqslant \frac{pf(a)+qf(b)}{p+q} + f\left(\frac{pa+qb}{p+q}\right), \tag{2.8}$$

where $A = \frac{pa+qb}{p+q}$, $y > 0$ and

$$y \leqslant \frac{b-a}{p+q}\min\{p,q\}.$$

Thus, inspired by (2.8) we introduce a new Bullen type functional,

$$\mathscr{V}(f;a,b,c,y)=(b-a)\left[\left(\frac{b-c}{b-a}f(a)+\frac{c-a}{b-a}f(b)+f(c)\right)-\frac{1}{y}\int_{c-y}^{c+y}f(x)\,\mathrm{d}x\right],$$

where $a\leqslant c\leqslant b,\ y>0$ and

$$y\leqslant\min\{b-c,c-a\}.$$

The functional $\mathscr{V}(f;a,b,c,y)$ is positive for continuous convex functions f.

REMARK 3. If we apply the first inequality of Theorem 2 replacing a with $\lambda a+(1-\lambda)b-y$ and b with $\lambda a+(1-\lambda)b+y$, one has

$$\begin{aligned}\frac{my^2}{6}\leqslant&\frac{f(\lambda a+(1-\lambda)b-y)+f(\lambda a+(1-\lambda)b+y)}{2}+f(\lambda a+(1-\lambda)b)\\&-\frac{1}{y}\int\limits_{\lambda a+(1-\lambda)b-y}^{\lambda a+(1-\lambda)b+y}f(x)\,\mathrm{d}x\\\leqslant&\lambda f(a)+(1-\lambda)f(b)+f(\lambda a+(1-\lambda)b)-\frac{1}{y}\int\limits_{\lambda a+(1-\lambda)b-y}^{\lambda a+(1-\lambda)b+y}f(x)\,\mathrm{d}x,\end{aligned}$$

for $m\leqslant f''(x)$ on $[\lambda a+(1-\lambda)b-y,\lambda a+(1-\lambda)b+y]$. Let $c:=\lambda a+(1-\lambda)b$. Since $\lambda=\frac{b-c}{b-a}$, it holds

$$\frac{my^2}{6}\leqslant\frac{b-c}{b-a}f(a)+\frac{c-a}{b-a}f(b)+f(c)-\frac{1}{y}\int\limits_{c-y}^{c+y}f(x)\,\mathrm{d}x,$$

hence

$$\mathscr{V}(f;a,b,c,y)\geqslant\frac{my^2}{6}(b-a)$$

for $0<y<\min\{(b-c),(c-a)\}$.

REMARK 4. The appearance of the left-hand side term of (2.8) enables us remark that one can use such functionals to obtain a priori estimates (rough, but intuitive) whereas more accurate results are requested at a later stage. Due to a rapid growth of computing power and to the recent developments of Computational Fluid Dynamics (CFD), practical mathematical models of fire dynamics are sometimes based on the application of Large Eddy Simulation (LES) techniques to fire. The accuracy with which the fire dynamics can be simulated depends on the number of cells considered for the simulation, a number which is limited by the computing power available. The equations for LES are derived by applying a low-pass filter, parameterized by a width Δ, to the transport equations for mass, momentum and energy. For details see [18, p.6]. The filtered fields in the LES equations are in fact cell means; e.g. the filtered density for a cell of width Δ is

$$\bar{\rho}(x,t) = \frac{1}{\Delta}\int_{x-\Delta/2}^{x+\Delta/2} \rho(r,t)\,\mathrm{d}r,$$

a formula which lets us see the fact that the functional $\mathscr{V}(f;a,b,c,y)$ can provide a mathematical tool for further insights in this domain whenever one deals with densities $\rho(r,t)$ which are convex with respect to the first variable.

For convex functions it also holds that

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \geqslant \frac{c-a}{b-a} f\left(\frac{a+c}{2}\right) + \frac{b-c}{b-a} f\left(\frac{b+c}{2}\right),$$

where $c \in [a,b]$. This is an immediate consequence of [9, Theorem 1.1] ($c = \lambda b + (1-\lambda)a$). Therefore, there also seems to be of interest the "complement" of Bullen's functional given by

$$\mathscr{S}(f;a,b,c) = \int_a^b f(x)\,\mathrm{d}x - (b-a)\left(\frac{c-a}{b-a} f\left(\frac{a+c}{2}\right) + \frac{b-c}{b-a} f\left(\frac{b+c}{2}\right)\right).$$

We note that $\mathscr{B}(f;a,b,c)$ corresponds to the right-hand side inequality given by Farissi, while $\mathscr{S}(f;a,b,c)$ emerges from its left-hand side. For $c = \frac{a+b}{2}$ we get

$$\mathscr{S}(f;a,b) = \int_a^b f(x)\,\mathrm{d}x - (b-a)\left(\frac{1}{2} f\left(\frac{3a+b}{4}\right) + \frac{1}{2} f\left(\frac{a+3b}{4}\right)\right)$$

which is positive for convex functions (see [25]). In [21, p.69] we see that

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \geqslant \frac{1}{2} f\left(\frac{a+b}{2} - \vartheta\right) + \frac{1}{2} f\left(\frac{a+b}{2} + \vartheta\right)$$

for all $\vartheta \in \left[0, \frac{b-a}{4}\right]$ (the maximal case $\vartheta = \frac{b-a}{4}$ reduces again to $\mathscr{S}(f;a,b) \geqslant 0$) and yields another general formula of a functional,

$$\mathscr{N}(f;a,b,\vartheta) = \int_a^b f(x)\,\mathrm{d}x - (b-a)\left(\frac{1}{2} f\left(\frac{a+b}{2} - \vartheta\right) + \frac{1}{2} f\left(\frac{a+b}{2} + \vartheta\right)\right)$$

where $\vartheta \in \left[0, \frac{b-a}{4}\right]$.

REMARK 5. In order to establish some estimates of the right-hand side of the Hermite-Hadamard inequality and estimates of some numerical integrations too, one can also define Iyengar [15] type functionals using the recipe above,

$$\mathscr{I}(f;a,b) = (b-a)\frac{f(a)+f(b)}{2} - \int_a^b f(x)\,\mathrm{d}x,$$

or a more general one,

$$\mathscr{I}(f;a,b,c) = (b-c)f(a) + (c-a)f(b) - \int_a^b f(x)\,\mathrm{d}x,$$

but these remain beyond the purpose of the present paper.

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