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ON ITERATED AND BILINEAR INTEGRAL HARDY–TYPE OPERATORS

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Dedicated to the 70th anniversary of Professor Josip Pečarić

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Abstract. We characterize the weighted inequalities on Lebesgue cone of all nonnegative functions on the semi-axis for iterated integral operators with Oinarov kernels.

1. Introduction

The study of the weighted inequalities for iterated integral operators, including on the cone of monotone functions, has recently attracted a lot of interest. In particular, we can point out the articles [2]-[9],[13][1], [15]-[16], [18]-[20], [23], [25]-[29] devoted to this topic.

Let us introduce some notations. Denote $\mathfrak{M}$ the set of all Lebesgue measurable functions on $\mathbb{R}_+ := [0;\infty)$, and let $\mathfrak{M}^+ \subset \mathfrak{M}$ be the subset of all nonnegative functions.

For $0 < p \leqslant \infty$ and $v \in \mathfrak{M}^+$ we define weighted Lebesgue space

$$L_v^p := \left\{ f \in \mathfrak{M} : \|f\|_{p,v} := \left(\int_0^\infty |f(x)|^p v(x)dx \right)^{\frac{1}{p}} < \infty \right\},$$

$$L_v^\infty := \left\{ f \in \mathfrak{M} : \|f\|_{\infty,v} := \operatorname*{ess\,sup}_{x \geqslant 0} v(x)|f(x)| < \infty \right\}.$$

Suppose that $u, v, w \in \mathfrak{M}^+$, $0 < q, p, r < \infty$, and a kernel $k : [0,\infty)^2 \to [0,\infty)$ is a Borel function satisfying the following Oinarov condition [14]: $k(x,y) = 0$ for $0 \leqslant x < y$ and $k(x,y) \geqslant 0$ for $0 \leqslant y \leqslant x$ and

$$D^{-1}((k(x,z)+k(z,y)) \leqslant k(x,y) \leqslant D((k(x,z)+k(z,y)),\ x \geqslant z \geqslant y \geqslant 0 \qquad (1)$$

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with a constant $D \geqslant 1$ independent on x,z,y. Without a loss of generality, we may and shall assume that $k(x,y)$ is non-decreasing with respect to x and non-increasing with respect to y while the second variable is fixed. It follows from (1) that

$$k_0(x,y) := \sup_{y\leqslant s\leqslant x}\left[\sup_{s\leqslant z\leqslant x} k(z,s)\right] \approx k(x,y).$$

and we can replace $k(x,y)$ by $k_0(x,y)$ which have these properties (see [21], p S45).

We consider the weighted inequality

$$\|Rf\|_{r,u} \leqslant C\|f\|_{p,v},\ f\in\mathfrak{M}^+, \tag{2}$$

where the integral operator R has one of the following form:

$$Tf(x) := \left(\int_x^\infty k_1(y,x)w(y)\left(\int_0^y k_2(y,z)f(z)dz\right)^q dy\right)^{\frac{1}{q}}, \tag{3}$$

$$\mathscr{T}f(x) := \left(\int_0^x k_1(x,y)w(y)\left(\int_y^\infty k_2(z,y)f(z)dz\right)^q dy\right)^{\frac{1}{q}}, \tag{4}$$

$$Sf(x) := \left(\int_x^\infty k_1(y,x)w(y)\left(\int_y^\infty k_2(z,y)f(z)dz\right)^q dy\right)^{\frac{1}{q}}, \tag{5}$$

$$\mathscr{S}f(x) := \left(\int_0^x k_1(x,y)w(y)\left(\int_0^y k_2(y,z)f(z)dz\right)^q dy\right)^{\frac{1}{q}}. \tag{6}$$

The first result was given in [2] for the operator $\mathscr{T}$, when $k_i(x,y)\equiv 1, i=1,2;$ $p>1$ and the case $p=1$ was treated in [3] provided additional relations between weights w and u. Alternative reduction method was suggested in [19] and developed in [20]. This method comprises the case of one or two Oinarov kernels, all possible values of summation parameters and weights, inaccessible by discretization–anti-discretization method [2], [3]. However, new method uses an auxiliary function depended on a weight u, which motivates further investigations (see [4], [7]-[9]).

In this paper we obtain explicit criteria, that is with no reduction procedure or using an auxiliary function, for the inequality (2) provided R is any of the above iterated integrals and $r>q$. The case $r=q$ is reduced to Hardy-type inequalities which completely studied in [14], [24], [17], [10].

Well known application of iterated operators is the weighted inequalities with bilinear integral operators (see [1], [6], [8], [9], [18]). We demonstrate this idea in § 7, where we characterize bilinear Hardy-type inequality with Oinarov kernels extending the results of M. Křepela [9] and D. V. Prokhorov [18].

Throughout the article, products of the form $0\cdot\infty$ are assumed to be equal to 0. The sign $A\lesssim B$ means $A\leqslant cB$ with a constant c depending only on p, q and r; $A\approx B$ means that $A\lesssim B\lesssim A$. Also $\mathbb{Z}$ stands for the set of all integers, and χ_E denotes the characteristic function (indicator) of a set $E\subset(0,\infty)$. We use the symbols $:=$ and $=:$ for definition of new quantities. If $1\leqslant p\leqslant\infty$, then $p' := \frac{p}{p-1}$ for $1<p<\infty$, $p' := \infty$ for $p=1$ and $p' := 1$ for $p=\infty$.

2. Preliminaries

In this paper we use the classical results for Hardy-type inequalities (see for instance, [11], [12]).

THEOREM A. *Let* $0 < q < \infty$, $1 \leqslant p < \infty$. *Then the inequality of the form*

$$\left(\int_0^\infty \left(\int_0^x f\right)^q u(x)dx\right)^{\frac{1}{q}} \leqslant C\left(\int_0^\infty f^p v\right)^{\frac{1}{p}},$$

with the best constant C, *is valid for every* $f \in \mathfrak{M}^+$, *if and only if the following holds:*

(i) If $1 < p \leqslant q < \infty$, *then* $C \approx A_1$, *where*

$$A_1 := \sup_{x>0}\left(\int_x^\infty u(t)dt\right)^{\frac{1}{q}}\left(\int_0^x v^{1-p'}(t)dt\right)^{\frac{1}{p'}} < \infty.$$

(ii) If $1 \leqslant q < p < \infty$, *then* $C \approx A_2$, *where*

$$A_2 := \left(\int_0^\infty \left(\int_x^\infty u(t)dt\right)^{\frac{r}{q}}\left(\int_0^x v^{1-p'}(t)dt\right)^{\frac{r}{q'}} v^{1-p'}(x)dx\right)^{\frac{1}{r}} < \infty.$$

(iii) If $0 < q < 1 < p < \infty$, *then* $C \approx A_3$, *where*

$$A_3 := \left(\int_0^\infty \left(\int_x^\infty u(t)dt\right)^{\frac{r}{p}}\left(\int_0^x v^{1-p'}(t)dt\right)^{\frac{r}{p'}} u(x)x\right)^{\frac{1}{r}} < \infty.$$

(iv) If $0 < q < 1 = p$, *then* $C \approx A_4$, *where*

$$A_4 := \left(\int_0^\infty \left[\operatorname*{ess\,sup}_{0<t<x}\frac{1}{v(t)}\int_x^\infty u(t)dt\right]^{\frac{q}{1-q}} u(x)dx\right)^{\frac{1-q}{q}},$$

where $\frac{1}{r} := \frac{1}{q} - \frac{1}{p}$.

We need the following theorem (see [14], [24], [12]).

THEOREM B. *Let* $1 < q, p < \infty$. *Then the inequality with the best constant* C

$$\left(\int_0^\infty \left(\int_0^x k(x,y)f(y)dy\right)^q u(x)dx\right)^{\frac{1}{q}} \leqslant C\left(\int_0^\infty f^p v\right)^{\frac{1}{p}},$$

holds for every $f \in \mathfrak{M}^+$, *if and only if the following holds.*

(i) If $1 < p \leqslant q < \infty$, *then* $C \approx A_0 + A_1$, *where*

$$A_0 := \sup_{x>0} \left(\int_x^\infty k^q(t,x)u(t)dt \right)^{\frac{1}{q}} \left(\int_0^x v^{1-p'}(t)dt \right)^{\frac{1}{p'}} < \infty.$$

$$A_1 := \sup_{x>0} \left(\int_x^\infty u(t)dt \right)^{\frac{1}{q}} \left(\int_0^x k^{p'}(x,t)v^{1-p'}(t)dt \right)^{\frac{1}{p'}} < \infty.$$

(ii) If $1 < q < p < \infty$, *then* $C \approx B_0 + B_1$, *where*

$$B_0 := \left(\int_0^\infty \left(\int_x^\infty k^q(t,x)u(t)dt \right)^{\frac{r}{q}} \left(\int_0^x v^{1-p'}(t)dt \right)^{\frac{r}{q'}} v^{1-p'}(x)dx \right)^{\frac{1}{r}} < \infty.$$

$$B_1 := \left(\int_0^\infty \left(\int_x^\infty u(t)dt \right)^{\frac{r}{p}} u(x) \left(\int_0^x k^{p'}(x,t)v^{1-p'}(t)dt \right)^{\frac{r}{p'}} dx \right)^{\frac{1}{r}} < \infty.$$

REMARK 1. The similar results are valid for the inequalities

$$\left(\int_0^\infty \left(\int_x^\infty f \right)^q u(x)dx \right)^{\frac{1}{q}} \leqslant C \left(\int_0^\infty f^p v \right)^{\frac{1}{p}},$$

or

$$\left(\int_0^\infty \left(\int_x^\infty k(y,x)f(y)dy \right)^q u(x)dx \right)^{\frac{1}{q}} \leqslant C \left(\int_0^\infty f^p v \right)^{\frac{1}{p}}.$$

We omit details.

Also for solving the problem we often used duality property of L_v^p- spaces. Let us recall this property.

If $1 < p < \infty$, then for any $f \in \mathfrak{M}^+$ it holds that

$$\sup_{h \in \mathfrak{M}^+} \frac{\int_0^\infty f(x)h(x)dx}{\left(\int_0^\infty h^p(x)v(x)dx\right)^{\frac{1}{p}}} = \left(\int_0^\infty f^{p'}(x)v^{1-p'}(x)dx \right)^{\frac{1}{p'}}. \tag{7}$$

It implies the following corollary which we use later:

$$\sup_{h \in \mathfrak{M}^+} \frac{\int_0^t h(x)u(x)dx}{\left(\int_0^\infty h^p(x)u(x)dx\right)^{\frac{1}{p}}} = \left(\int_0^t u(x)dx \right)^{\frac{1}{p'}}. \tag{8}$$

3. Operator T

Let

$$V(t) := \int_0^t v^{1-p'},\, W(t) := \int_t^\infty w,\, U(t) := \int_0^t u,$$

$$V_{2,k}(t) := \int_0^t k_2^{p'}(t,y)v^{1-p'}(y)dy,\, W_1(t) := \int_t^\infty k_1(y,t)w(y)dy,$$

$$W_2(t) := \int_t^\infty k_2^q(y,t)w(y)dy,\, W_{12}(t) := \int_t^\infty k_1(y,t)k_2^q(y,t)w(y)dy.$$

REMARK 2. Let $\varphi \in \mathfrak{M}^{\uparrow}$. Without loss of generality we may and shall assume that φ is right-continuous. Then it is known (see [22], Chapter 12), that there exists a Borel measure, say η_φ, such that

$$\varphi(t) = \int_{[0,t]} d\eta_\varphi(s).$$

Thus, we may suppose that for the function $V_{2,k} \in \mathfrak{M}^{\uparrow}$ there is a Borel measure $dV_{2,k}$ such that $V_{2,k}(t) = \int_{[0,t]} dV_{2,k}(s)$.

THEOREM 1. *Let $1 < q,\ p,\ r < \infty,\ r > q$. Then for the best constant C_T of the inequality*

$$\left(\int_0^\infty \left(\int_x^\infty k_1(y,x)w(y)\left(\int_0^y k_2(y,t)f(t)dt\right)^q dy\right)^{\frac{r}{q}} u(x)dx\right)^{\frac{1}{r}} \leqslant C_T\left(\int_0^\infty f^p v\right)^{\frac{1}{p}},\quad f \in \mathfrak{M}^+, \tag{9}$$

the following holds.

(i) If $1 < p \leqslant q < \infty$, then $C_T \approx \mathbb{F}_{11} + \mathbb{F}_{12} + \mathbb{F}_{13} + \mathbb{F}_{14} + \mathbb{F}_{21} + \mathbb{F}_{22} + \mathbb{F}_{23}$, where

$$\mathbb{F}_{11} = \sup_{t>0} V^{\frac{1}{p'}}(t) W_{12}^{\frac{1}{q}}(t) U^{\frac{1}{r}}(t),$$

$$\mathbb{F}_{12} = \sup_{t>0} V^{\frac{1}{p'}}(t)\left(\int_t^\infty W_{12}^{\frac{r}{q}}(s)u(s)\,ds\right)^{\frac{1}{r}},$$

$$\mathbb{F}_{13} = \sup_{t>0}\left(\int_0^t k_1^{\frac{r}{q}}(t,s)u(s)\,ds\right)^{\frac{1}{r}} W_2^{\frac{1}{q}}(t) V^{\frac{1}{p'}}(t),$$

$$\mathbb{F}_{14} = \sup_{t>0}\left(\int_t^\infty k_2^r(s,t) W_1^{\frac{r}{q}}(s)u(s)\,ds\right)^{\frac{1}{r}} V^{\frac{1}{p'}}(t),$$

$$\mathbb{F}_{21} = \sup_{t>0} V_{2,k}^{\frac{1}{p'}}(t) W_1^{\frac{1}{q}}(t) U^{\frac{1}{r}}(t),$$

$$\mathbb{F}_{22} = \sup_{t>0}\left(\int_0^t k_1^{\frac{r}{q}}(t,s)u(s)\,ds\right)^{\frac{1}{r}} W^{\frac{1}{q}}(t) V_{2,k}^{\frac{1}{p'}}(t),$$

$$\mathbb{F}_{23}=\sup_{t>0}V_{2,k}^{\frac{1}{p'}}(t)\left(\int_t^\infty W_1^{\frac{r}{q}}(s)u(s)\,ds\right)^{\frac{1}{r}}.$$

(ii) If $1<q<p<\infty$, $\frac{1}{s}:=\frac{1}{q}-\frac{1}{p}$, *then* $C_T\approx\mathbb{G}_{11}+\mathbb{G}_{12}+\mathbb{G}_{13}+\mathbb{G}_{14}+\mathbb{G}_{21}+\mathbb{G}_{22}+\mathbb{G}_{23}$, *where for* $p\leqslant r$

$$\mathbb{G}_{11}=\sup_{t>0}\left(\int_t^\infty W_{12}^{\frac{s}{q}}dV^{\frac{s}{p'}}\right)^{\frac{1}{s}}U^{\frac{1}{r}}(t),$$

$$\mathbb{G}_{12}=\sup_{t>0}\left(\int_t^\infty k_1^{\frac{s}{q}}(x,t)W_2^{\frac{s}{q}}(x)dV^{\frac{s}{p'}}(x)\right)^{\frac{1}{s}}U^{\frac{1}{r}}(t)$$

$$+\sup_{t>0}\left(\int_t^\infty W_2^{\frac{s}{q}}dV^{\frac{s}{p'}}\right)^{\frac{1}{s}}\left(\int_0^t k_1^{\frac{r}{q}}(t,s)u(s)ds\right)^{\frac{1}{r}},$$

$$\mathbb{G}_{13}=\sup_{t>0}\left(\int_t^\infty W_{12}^{\frac{r}{q}}u\right)^{\frac{1}{r}}V^{\frac{1}{p'}}(t),$$

$$\mathbb{G}_{14}=\sup_{t>0}\left(\int_0^t k_2^s(t,x)\,dV^{\frac{s}{p'}}(x)\right)^{\frac{1}{s}}\left(\int_t^\infty uW_1^{\frac{r}{q}}\right)^{\frac{1}{r}}$$

$$+\sup_{t>0}V^{\frac{1}{p'}}(t)\left(\int_t^\infty k_2^r(x,t)u(x)W_1^{\frac{r}{q}}(x)dx\right)^{\frac{1}{r}},$$

$$\mathbb{G}_{21}=\sup_{t>0}\left(\int_t^\infty W_1^{\frac{s}{q}}dV_{2,k}^{\frac{s}{p'}}\right)^{\frac{1}{s}}U^{\frac{1}{r}}(t),$$

$$\mathbb{G}_{22}=\sup_{t>0}\left(\int_t^\infty k_1^{\frac{s}{q}}(x,t)W^{\frac{s}{q}}(x)dV_{2,k}^{\frac{s}{p'}}(x)\right)^{\frac{1}{s}}U^{\frac{1}{r}}(t)$$

$$+\sup_{t>0}\left(\int_t^\infty W^{\frac{s}{q}}dV_{2,k}^{\frac{s}{p'}}\right)^{\frac{1}{s}}\left(\int_0^t k_1^{\frac{r}{q}}(t,s)u(s)ds\right)^{\frac{1}{r}},$$

$$\mathbb{G}_{23}=\sup_{t>0}V_{2,k}^{\frac{1}{p'}}(t)\left(\int_t^\infty W_1^{\frac{r}{q}}u\right)^{\frac{1}{r}}.$$

For $r<p$, $\frac{1}{s_1}:=\frac{1}{r}-\frac{1}{p}$,

$$\mathbb{G}_{11}=\left(\int_0^\infty\left(\int_t^\infty W_{12}^{\frac{s}{q}}dV^{\frac{s}{p'}}\right)^{\frac{s_1}{s}}dU^{\frac{s_1}{r}}(t)\right)^{\frac{1}{s_1}},$$

$$\mathbb{G}_{12}=\left(\int_0^\infty\left(\int_t^\infty k_1^{\frac{s}{q}}(x,t)W_2^{\frac{s}{q}}(x)dV^{\frac{s}{p'}}(x)\right)^{\frac{s_1}{s}}dU^{\frac{s_1}{r}}(t)\right)^{\frac{1}{s_1}}$$

$$+\left(\int_0^\infty\left(\int_0^t k_1^{\frac{r}{q}}(t,s)u(s)ds\right)^{\frac{s_1}{r}}d\left(-\left(\int_t^\infty W_2^{\frac{s}{q}}dV^{\frac{s}{p'}}\right)^{\frac{s_1}{s}}\right)\right)^{\frac{1}{s_1}},$$

$$\mathbb{G}_{13} = \left(\int_0^\infty V^{\frac{s_1}{p'}}(t) d\left(- \left(\int_t^\infty W_{12}^{\frac{r}{q}} u \right)^{\frac{s_1}{r}} \right) \right)^{\frac{1}{s_1}},$$

$$\mathbb{G}_{14} = \left(\int_0^\infty \left(\int_0^t k_2^s(t,x) dV^{\frac{s}{p'}}(x) \right)^{\frac{s_1}{s}} d\left(- \left(\int_t^\infty u W_1^{\frac{r}{q}} \right)^{\frac{s_1}{r}} \right) \right)^{\frac{1}{s_1}}$$

$$+ \left(\int_0^\infty \left(\int_t^\infty k_2^r(x,t) u(x) W_1^{\frac{r}{q}}(x) dx \right)^{\frac{s_1}{r}} dV^{\frac{s_1}{p'}}(t) \right)^{\frac{1}{s_1}},$$

$$\mathbb{G}_{21} = \left(\int_0^\infty \left(\int_t^\infty W_1^{\frac{s}{q}} dV_{2,k}^{\frac{s}{p'}} \right)^{\frac{s_1}{s}} dU^{\frac{s_1}{r}}(t) \right)^{\frac{1}{s_1}},$$

$$\mathbb{G}_{22} = \left(\int_0^\infty \left(\int_t^\infty k_1^{\frac{s}{q}}(x,t) W^{\frac{s}{q}}(x) dV_{2,k}^{\frac{s}{p'}}(x) \right)^{\frac{s_1}{s}} dU^{\frac{s_1}{r}}(t) \right)^{\frac{1}{s_1}}$$

$$+ \left(\int_0^\infty \left(\int_0^t k_1^{\frac{r}{q}}(t,s) u(s) ds \right)^{\frac{s_1}{r}} d\left(- \left(\int_t^\infty W^{\frac{s}{q}} dV_{2,k}^{\frac{s}{p'}} \right)^{\frac{s_1}{s}} \right) \right)^{\frac{1}{s_1}},$$

$$\mathbb{G}_{23} = \left(\int_0^\infty V_{2,k}^{\frac{s_1}{p'}}(t) d\left(- \left(\int_t^\infty W_1^{\frac{r}{q}} u \right)^{\frac{s_1}{r}} \right) \right)^{\frac{1}{s_1}}.$$

Proof. We have

$$C_T = \sup_f \|f\|_{p,v}^{-1} \left[\left(\int_0^\infty \left(\int_x^\infty k_1(y,x) w(y) \left(\int_0^y k_2(y,t) f(t) dt \right)^q dy \right)^{\frac{r}{q}} u(x) dx \right)^{\frac{q}{r}} \right]^{\frac{1}{q}}$$

$$\overset{(7)}{=} \sup_f \|f\|_{p,v}^{-1} \left[\sup_h \frac{\int_0^\infty \left(\int_x^\infty k_1(y,x) w(y) \left(\int_0^y k_2(y,t) f(t) dt \right)^q dy \right) h(x) u(x) dx}{\left(\int_0^\infty h^{\frac{r}{r-q}} u \right)^{1-\frac{q}{r}}} \right]^{\frac{1}{q}}$$

$$= \sup_h \left(\int_0^\infty h^{\frac{r}{r-q}} u \right)^{\frac{1}{r}-\frac{1}{q}} \sup_f \frac{\left(\int_0^\infty \left(\int_0^y k_2(y,t) f(t) dt \right)^q \left(\int_0^y k_1(y,x) h(x) u(x) dx \right) w(y) dy \right)^{\frac{1}{q}}}{\|f\|_{p,v}}.$$

The first relation follows by the duality property (7), the second relation holds by the Fubini theorem.

Case I: $1 < p \leqslant q < r < \infty$. Applying theorem B (i) we find

$$C_T \approx \sup_h \left(\int_0^\infty h^{\frac{r}{r-q}} u \right)^{\frac{1}{r}-\frac{1}{q}} (\mathscr{F}_1(h) + \mathscr{F}_2(h)),$$

where

$$\mathscr{F}_1(h) := \sup_{t>0} \left(\int_t^\infty k_2^q(y,t) w(y) \left(\int_0^y k_1(y,x) h(x) u(x) dx \right) dy \right)^{\frac{1}{q}} V^{\frac{1}{p'}}(t),$$

$$\mathscr{F}_2(h) := \sup_{t>0} \left(\int_t^\infty w(y) \left(\int_0^y k_1(y,x) h(x) u(x) dx \right) dy \right)^{\frac{1}{q}} V_{2,k}^{\frac{1}{p'}}(t).$$

Using Fubini's theorem and the property of Oinarov kernels, we have

$$\mathscr{F}_1(h) \approx \mathscr{F}_{11}(h) + \mathscr{F}_{12}(h) + \mathscr{F}_{13}(h) + \mathscr{F}_{14}(h),$$

where

$$\mathscr{F}_{11}(h) = \sup_{t>0} \left(\int_0^t hu \right)^{\frac{1}{q}} W_{12}^{\frac{1}{q}}(t) V^{\frac{1}{p'}}(t),$$

$$\mathscr{F}_{12}(h) = \sup_{t>0} \left(\int_t^\infty h(s) u(s) W_{12}(s) ds \right)^{\frac{1}{q}} V^{\frac{1}{p'}}(t),$$

$$\mathscr{F}_{13}(h) = \sup_{t>0} \left(\int_0^t k_1(t,s) h(s) u(s) ds \right)^{\frac{1}{q}} W_2^{\frac{1}{q}}(t) V^{\frac{1}{p'}}(t),$$

$$\mathscr{F}_{14}(h) = \sup_{t>0} \left(\int_t^\infty k_2(s,t)^q h(s) u(s) W_1(s) ds \right)^{\frac{1}{q}} V^{\frac{1}{p'}}(t).$$

Then $C_T \approx C_1 + C_2$, where

$$C_1 \approx \sum_{j=1}^4 \mathbb{F}_{1j}, C_2 \approx \sum_{j=1}^3 \mathbb{F}_{2j}, \mathbb{F}_{ij} := \sup_h \left(\int_0^\infty h^{\frac{r}{r-q}} u \right)^{\frac{1}{r} - \frac{1}{q}} \mathscr{F}_{ij}(h).$$

Using the duality property (8), we have

$$\mathbb{F}_{11} = \sup_{t>0} V^{\frac{1}{p'}}(t) W_{12}^{\frac{1}{q}}(t) U^{\frac{1}{r}}(t),$$

$$\mathbb{F}_{12} = \sup_{t>0} V^{\frac{1}{p'}}(t) \left(\int_t^\infty W_{12}^{\frac{r}{q}}(s) u(s)\, ds \right)^{\frac{1}{r}},$$

$$\mathbb{F}_{13} = \sup_{t>0} \left(\int_0^t k_1^{\frac{r}{q}}(t,s) u(s)\, ds \right)^{\frac{1}{r}} W_2^{\frac{1}{q}}(t) V^{\frac{1}{p'}}(t),$$

$$\mathbb{F}_{14} = \sup_{t>0} \left(\int_t^\infty k_2^r(s,t) W_1^{\frac{r}{q}}(s) u(s)\, ds \right)^{\frac{1}{r}} V^{\frac{1}{p'}}(t).$$

Similarly, for $\mathscr{F}_2(h)$ we have

$$\mathscr{F}_2(h) \cong \mathscr{F}_{21}(h) + \mathscr{F}_{22}(h) + \mathscr{F}_{23}(h),$$

where

$$\mathscr{F}_{21}(h) = \sup_{t>0}\left(\int_0^y hu\right)^{\frac{1}{q}} W_1^{\frac{1}{q}}(t)V_{2,k}^{\frac{1}{p'}}(t),$$

$$\mathscr{F}_{22}(h) = \sup_{t>0}\left(\int_0^t k_1(t,s)h(s)u(s)ds\right)^{\frac{1}{q}} W^{\frac{1}{q}}(t)V_{2,k}^{\frac{1}{p'}}(t),$$

$$\mathscr{F}_{23}(h) = \sup_{t>0}\left(\int_t^\infty h(s)u(s)W_1(s)ds\right)^{\frac{1}{q}} V_{2,k}^{\frac{1}{p'}}(t).$$

Using the consequence of the duality property (8) for

$$\mathbb{F}_{2j} := \sup_h \left(\int_0^\infty h^{\frac{r}{r-q}}u\right)^{\frac{1}{r}-\frac{1}{q}} \mathscr{F}_{2j}(h),$$

we have

$$\mathbb{F}_{21} = \sup_{t>0} V_{2,k}^{\frac{1}{p'}}(t)W_1^{\frac{1}{q}}(t)U^{\frac{1}{r}}(t),$$

$$\mathbb{F}_{22} = \sup_{t>0}\left(\int_0^t k_1^{\frac{r}{q}}(t,s)u(s)\,ds\right)^{\frac{1}{r}} W^{\frac{1}{q}}(t)V_{2,k}^{\frac{1}{p'}}(t),$$

$$\mathbb{F}_{23} = \sup_{t>0} V_{2,k}^{\frac{1}{p'}}(t)\left(\int_t^\infty W_1^{\frac{r}{q}}(s)u(s)\,ds\right)^{\frac{1}{r}}.$$

Case II: $1<q<p,r,\ \frac{1}{s} := \frac{1}{q}-\frac{1}{p}$. Continuing Applying theorem B we find

$$C_T \approx \sup_h \left(\int_0^\infty h^{\frac{r}{r-q}}u\right)^{\frac{1}{r}-\frac{1}{q}} (\mathscr{G}_1(h)+\mathscr{G}_2(h)),$$

where

$$\mathscr{G}_1(h) := \left(\int_0^\infty \left(\int_t^\infty k_2^q(y,t)\left(\int_0^y k_1(y,x)h(x)u(x)dx\right)w(y)dy\right)^{\frac{s}{q}} dV^{\frac{s}{p'}}(t)\right)^{\frac{1}{s}}$$

$$\mathscr{G}_2(h) := \left(\int_0^\infty V_{2,k}^{\frac{s}{p'}}(t)\left(\int_t^\infty \left(\int_0^y k_1(y,x)h(x)u(x)dx\right)w(y)dy\right)^{\frac{s}{p}}\right.$$

$$\left.\left(\int_0^t k_1(t,x)h(x)u(x)dx\right)w(t)dt\right)^{\frac{1}{s}}$$

$$= \left(\int_0^\infty \left(\int_t^\infty \left(\int_0^y k_1(y,x)h(x)u(x)dx\right)w(y)dy\right)^{\frac{s}{q}} dV_{2,k}^{\frac{s}{p'}}(t)\right)^{\frac{1}{s}}.$$

Using the Fubini theorem and property of Oinarov kernels, we have

$$\begin{aligned}\mathscr{G}_1(h) \approx &\left(\int_0^\infty \left(\int_0^t hu\right)^{\frac{s}{q}} W_{12}^{\frac{s}{q}}(t)dV^{\frac{s}{p'}}(t)\right)^{\frac{1}{s}}\\ &+\left(\int_0^\infty \left(\int_0^t k_1(t,s)h(s)u(s)ds\right)^{\frac{s}{q}} W_2^{\frac{s}{q}}(t)dV^{\frac{s}{p'}}(t)\right)^{\frac{1}{s}}\\ &+\left(\int_0^\infty \left(\int_t^\infty h(s)u(s)W_{12}(s)ds\right)^{\frac{s}{q}} dV^{\frac{s}{p'}}(t)\right)^{\frac{1}{s}}\\ &+\left(\int_0^\infty \left(\int_t^\infty k_2^q(s,t)h(s)u(s)W_1(s)ds\right)^{\frac{s}{q}} dV^{\frac{s}{p'}}(t)\right)^{\frac{1}{s}}\\ =:&\mathscr{G}_{11}(h)+\mathscr{G}_{12}(h)+\mathscr{G}_{13}(h)+\mathscr{G}_{14}(h),\end{aligned}$$

where $\mathscr{G}_{11}(h)$ and $\mathscr{G}_{13}(h)$ are defined by theorem A and its dual, as well as $\mathscr{G}_{12}(h)$ and $\mathscr{G}_{14}(h)$ are determined by theorem B and its dual. Then

$$C_1 \approx \sum_{j=1}^4 \mathbb{G}_{1j}.$$

(i) If $p \leqslant r$, then

$$\begin{aligned}\mathbb{G}_{11} =&\sup_{t>0}\left(\int_t^\infty W_{12}^{\frac{s}{q}}dV^{\frac{s}{p'}}\right)^{\frac{1}{s}} U^{\frac{1}{r}}(t),\\ \mathbb{G}_{12} =&\sup_{t>0}\left(\int_t^\infty k_1^{\frac{s}{q}}(x,t)W_2^{\frac{s}{q}}(x)dV^{\frac{s}{p'}}(x)\right)^{\frac{1}{s}} U^{\frac{1}{r}}(t)\\ &+\sup_{t>0}\left(\int_t^\infty W_2^{\frac{s}{q}}dV^{\frac{s}{p'}}\right)^{\frac{1}{s}}\left(\int_0^t k_1^{\frac{r}{q}}(t,s)u(s)ds\right)^{\frac{1}{r}}.\\ \mathbb{G}_{13} =&\sup_{t>0}\left(\int_t^\infty W_{12}^{\frac{r}{q}}u\right)^{\frac{1}{r}} V^{\frac{1}{p'}}(t),\\ \mathbb{G}_{14} =&\sup_{t>0}\left(\int_0^t k_2^s(t,x)\,dV^{\frac{s}{p'}}(x)\right)^{\frac{1}{s}}\left(\int_t^\infty uW_1^{\frac{r}{q}}\right)^{\frac{1}{r}}\\ &+\sup_{t>0}V^{\frac{1}{p'}}(t)\left(\int_t^\infty k_2^r(x,t)u(x)W_1^{\frac{r}{q}}(x)dx\right)^{\frac{1}{r}}.\end{aligned}$$

(ii) If $r<p$ and $\frac{1}{s_1} := \frac{1}{r}-\frac{1}{p}$, then

$$\mathbb{G}_{11} = \left(\int_0^\infty \left(\int_t^\infty W_{12}^{\frac{s}{q}}dV^{\frac{s}{p'}}\right)^{\frac{s_1}{s}} dU^{\frac{s_1}{r}}(t)\right)^{\frac{1}{s_1}},$$

$$\mathbb{G}_{12} = \left(\int_0^\infty \left(\int_t^\infty k_1^{\frac{s}{q}}(x,t) W_2^{\frac{s}{q}}(x) dV^{\frac{s}{p'}}(x) \right)^{\frac{s_1}{s}} dU^{\frac{s_1}{r}}(t) \right)^{\frac{1}{s_1}}$$

$$+ \left(\int_0^\infty \left(\int_0^t k_1^{\frac{r}{q}}(t,s) u(s) ds \right)^{\frac{s_1}{r}} d\left(-\left(\int_t^\infty W_2^{\frac{s}{q}} dV^{\frac{s}{p'}} \right)^{\frac{s_1}{s}} \right) \right)^{\frac{1}{s_1}},$$

$$\mathbb{G}_{13} = \left(\int_0^\infty V^{\frac{s_1}{p'}}(t) d\left(-\left(\int_t^\infty W_{12}^{\frac{r}{q}} u \right)^{\frac{s_1}{r}} \right) \right)^{\frac{1}{s_1}},$$

$$\mathbb{G}_{14} = \left(\int_0^\infty \left(\int_0^t k_2^s(t,x) dV^{\frac{s}{p'}}(x) \right)^{\frac{s_1}{s}} d\left(-\left(\int_t^\infty u W_1^{\frac{r}{q}} \right)^{\frac{s_1}{r}} \right) \right)^{\frac{1}{s_1}}$$

$$+ \left(\int_0^\infty \left(\int_t^\infty k_2^r(x,t) u(x) W_1^{\frac{r}{q}}(x) dx \right)^{\frac{s_1}{r}} dV^{\frac{s_1}{p'}}(t) \right)^{\frac{1}{s_1}}.$$

Similarly, for $\mathscr{G}_2(h)$ we have

$$\mathscr{G}_2(h) \approx \mathscr{G}_{21}(h) + \mathscr{G}_{22}(h) + \mathscr{G}_{23}(h),$$

where

$$\mathscr{G}_{21}(h) = \left(\int_0^\infty \left(\int_0^t hu \right)^{\frac{s}{q}} W_1^{\frac{s}{q}}(t) dV_{2,k}^{\frac{s}{p'}}(t) \right)^{\frac{1}{s}},$$

$$\mathscr{G}_{22}(h) = \left(\int_0^\infty \left(\int_0^t k_1(t,s) h(s) u(s) ds \right)^{\frac{s}{q}} W^{\frac{s}{q}}(t) dV_{2,k}^{\frac{s}{p'}}(t) \right)^{\frac{1}{s}},$$

$$\mathscr{G}_{23}(h) = \left(\int_0^\infty \left(\int_t^\infty h u W_1 \right)^{\frac{s}{q}} dV_{2,k}^{\frac{s}{p'}}(t) \right)^{\frac{1}{s}}.$$

Then

$$C_2 \approx \sum_{j=1}^{3} \mathbb{G}_{2j}.$$

(i) If $p \leqslant r$, then

$$\mathbb{G}_{21} = \sup_{t>0} \left(\int_t^\infty W_1^{\frac{s}{q}} dV_{2,k}^{\frac{s}{p'}} \right)^{\frac{1}{s}} U^{\frac{1}{r}}(t),$$

$$\mathbb{G}_{22} = \sup_{t>0} \left(\int_t^\infty k_1^{\frac{s}{q}}(x,t) W^{\frac{s}{q}}(x) dV_{2,k}^{\frac{s}{p'}}(x) \right)^{\frac{1}{s}} U^{\frac{1}{r}}(t)$$

$$+ \sup_{t>0} \left(\int_t^\infty W^{\frac{s}{q}} dV_{2,k}^{\frac{s}{p'}} \right)^{\frac{1}{s}} \left(\int_0^t k_1^{\frac{r}{q}}(t,s) u(s) ds \right)^{\frac{1}{r}},$$

$$\mathbb{G}_{23} = \sup_{t>0} V_{2,k}^{\frac{1}{p'}}(t) \left(\int_t^\infty W_1^{\frac{r}{q}} u \right)^{\frac{1}{r}}.$$

(ii) If $r < p$ and $\frac{1}{s_1} := \frac{1}{r} - \frac{1}{p}$, then

$$\mathbb{G}_{21} = \left(\int_0^\infty \left(\int_t^\infty W_1^{\frac{s}{q}} dV_{2,k}^{\frac{s}{p'}} \right)^{\frac{s_1}{s}} dU^{\frac{s_1}{r}}(t) \right)^{\frac{1}{s_1}},$$

$$\mathbb{G}_{22} = \left(\int_0^\infty \left(\int_t^\infty k_1^{\frac{s}{q}}(x,t) W^{\frac{s}{q}}(x) dV_{2,k}^{\frac{s}{p'}}(x) \right)^{\frac{s_1}{s}} dU^{\frac{s_1}{r}}(t) \right)^{\frac{1}{s_1}}$$

$$+ \left(\int_0^\infty \left(\int_0^t k_1^{\frac{r}{q}}(t,s) u(s) ds \right)^{\frac{s_1}{r}} d\left(- \left(\int_t^\infty W^{\frac{s}{q}} dV_{2,k}^{\frac{s}{p'}} \right)^{\frac{s_1}{s}} \right) \right)^{\frac{1}{s_1}},$$

$$\mathbb{G}_{23} = \left(\int_0^\infty V_{2,k}^{\frac{s_1}{p'}}(t) d\left(- \left(\int_t^\infty W_1^{\frac{r}{q}} u \right)^{\frac{s_1}{r}} \right) \right)^{\frac{1}{s_1}}.$$

The proof is complete. □

CLAIM 1. Let $U(x) := \operatorname{ess\,sup}_{0<t<x} u(t)$, $T(f) \in \mathfrak{M}^{\downarrow}$. Then

$$\operatorname{ess\,sup}_{x>0} u(x) Tf(x) = \sup_{x>0} U(x) Tf(x).$$

Proof. For " $\leqslant$ " the proof is obvious.

$$\operatorname{ess\,sup}_{x>0} u(x) Tf(x) = \sup_{x>0} \operatorname{ess\,sup}_{0<s\leqslant x} u(s) Tf(s) \geqslant \sup_{x>0} U(x) Tf(x). \quad \square$$

CLAIM 2. Let $U_*(x) := \operatorname{ess\,sup}_{x<t<\infty} u(t)$, $\mathscr{T}(f) \in \mathfrak{M}^{\uparrow}$. Then

$$\operatorname{ess\,sup}_{x>0} u(x) \mathscr{T}f(x) = \sup_{x>0} U_*(x) \mathscr{T}f(x).$$

Let $v \in \mathfrak{M}^+$. We define $v^{\uparrow}(x) := \operatorname{ess\,sup}_{0<t<x} \frac{1}{v(t)}$. We denote

$$\mathbb{W}_2(t) := \int_t^\infty k_2(y,t) w(y) dy, \; \mathbb{W}_{12}(t) := \int_t^\infty k_1(y,t) k_2(y,t) w(y) dy.$$

REMARK 3. (i) If $p = 1$, $1 < q < r < \infty$, then $C_T \approx \mathbb{F}_1 + \mathbb{F}_2 + \mathbb{F}_3 + \mathbb{F}_4$, where

$$\mathbb{F}_1 = \sup_{t>0} v^{\uparrow}(t) W_{12}^{\frac{1}{q}}(t) U^{\frac{1}{r}}(t),$$

$$\mathbb{F}_2 = \sup_{t>0} v^{\uparrow}(t)\left(\int_t^{\infty} W_{12}^{\frac{r}{q}}(s)u(s)\,ds\right)^{\frac{1}{r}},$$

$$\mathbb{F}_3 = \sup_{t>0} v^{\uparrow}(t)W_2^{\frac{1}{q}}(t)\left(\int_0^{t} k_1^{\frac{r}{q}}(t,s)u(s)\,ds\right)^{\frac{1}{r}},$$

$$\mathbb{F}_4 = \sup_{t>0} v^{\uparrow}(t)\left(\int_t^{\infty} k_2^{r}(s,t)W_1^{\frac{r}{q}}(s)u(s)\,ds\right)^{\frac{1}{r}}.$$

(ii) If $p=q=1$, $1\leqslant r\leqslant\infty$, then $C_T \approx \mathbb{G}_1+\mathbb{G}_2+\mathbb{G}_3+\mathbb{G}_4$, where

$$\mathbb{G}_1 = \sup_{t>0} v^{\uparrow}(t)\left(\int_t^{\infty} \mathbb{W}_{12}^{r}u\right)^{\frac{1}{r}},$$

$$\mathbb{G}_2 = \sup_{t>0} v^{\uparrow}(t)\left(\int_t^{\infty} k_2^{r}(s,t)W_1^{r}(s)u(s)\,ds\right)^{\frac{1}{r}},$$

$$\mathbb{G}_3 = \sup_{t>0} R_{12}(t)\left(\int_0^{t} u\right)^{\frac{1}{r}},\ R_{12}(t) := \operatorname*{ess\,sup}_{t<s<\infty}\frac{\mathbb{W}_{12}(s)}{v(s)},$$

$$\mathbb{G}_4 = \sup_{t>0} R_2(t)\left(\int_0^{t} k_1^{r}(t,s)u(s)ds\right)^{\frac{1}{r}},\ R_2(t) := \operatorname*{ess\,sup}_{t<s<\infty}\frac{\mathbb{W}_2(s)}{v(s)}.$$

Proof.

(i) As well as in theorem 1 using the duality property (7) and Fubini theorem, we obtain

$$C_T = \sup_f \|f\|_{1,v}^{-1}\left[\left(\int_0^{\infty}\left(\int_x^{\infty} k_1(y,x)w(y)\left(\int_0^{y} k_2(y,t)f(t)dt\right)^{q}dy\right)^{\frac{r}{q}}u(x)dx\right)^{\frac{q}{r}}\right]^{\frac{1}{q}}$$

$$= \sup_h\left(\int_0^{\infty} h^{\frac{r}{r-q}}u\right)^{\frac{1}{r}-\frac{1}{q}}\sup_f\frac{\left(\int_0^{\infty}\left(\int_0^{y} k_2(y,t)f(t)dt\right)^{q}\left(\int_0^{y} k_1(y,x)h(x)u(x)dx\right)w(y)dy\right)^{\frac{1}{q}}}{\|f\|_{1,v}}.$$

Applying theorem 1.1 from [5] and claim 1, we have

$$= \sup_h\left(\int_0^{\infty} h^{\frac{r}{r-q}}u\right)^{\frac{1}{r}-\frac{1}{q}}\operatorname*{ess\,sup}_{t>0}\frac{1}{v(t)}\left(\int_t^{\infty} k_2(y,t)\left(\int_0^{y} k_1(y,x)h(x)u(x)dx\right)w(y)dy\right)^{\frac{1}{q}}$$

$$= \sup_h\left(\int_0^{\infty} h^{\frac{r}{r-q}}u\right)^{\frac{1}{r}-\frac{1}{q}}\sup_{t>0} v^{\uparrow}(t)\left(\int_t^{\infty} k_2(y,t)\left(\int_0^{y} k_1(y,x)h(x)u(x)dx\right)w(y)dy\right)^{\frac{1}{q}}$$

$$= \sup_h\left(\int_0^{\infty} h^{\frac{r}{r-q}}u\right)^{\frac{1}{r}-\frac{1}{q}}\mathscr{F}_1(h).$$

Using Fubini theorem and the property of Oinarov's kernels, we have

$$\mathscr{F}_1(h) \cong \mathscr{F}_1(h)+\mathscr{F}_2(h)+\mathscr{F}_3(h)+\mathscr{F}_4(h),$$

where

$$\mathscr{F}_1(h) = \sup_{t>0} \left(\int_0^t hu \right)^{\frac{1}{q}} W_{12}^{\frac{1}{q}}(t) v^{\uparrow}(t),$$

$$\mathscr{F}_2(h) = \sup_{t>0} \left(\int_t^\infty h(s)u(s)W_{12}(s)ds \right)^{\frac{1}{q}} v^{\uparrow}(t),$$

$$\mathscr{F}_3(h) = \sup_{t>0} \left(\int_0^t k_1(t,s)h(s)u(s)ds \right)^{\frac{1}{q}} W_2^{\frac{1}{q}}(t) v^{\uparrow}(t),$$

$$\mathscr{F}_4(h) = \sup_{t>0} \left(\int_t^\infty k_2(s,t)^q h(s)u(s)W_1(s)ds \right)^{\frac{1}{q}} v^{\uparrow}(t).$$

Then

$$C_T \approx \sum_{j=1}^4 \mathbb{F}_j, \mathbb{F}_j := \sup_h \left(\int_0^\infty h^{\frac{r}{r-q}} u \right)^{\frac{1}{r} - \frac{1}{q}} \mathscr{F}_j(h).$$

Using the consequence of the duality property (8), we have

$$\mathbb{F}_1 = \sup_{t>0} v^{\uparrow}(t) W_{12}^{\frac{1}{q}}(t) U^{\frac{1}{r}}(t),$$

$$\mathbb{F}_2 = \sup_{t>0} v^{\uparrow}(t) \left(\int_t^\infty W_{12}^{\frac{r}{q}}(s)u(s)\,ds \right)^{\frac{1}{r}},$$

$$\mathbb{F}_3 = \sup_{t>0} \left(\int_0^t k_1^{\frac{r}{q}}(t,s)u(s)\,ds \right)^{\frac{1}{r}} W_2^{\frac{1}{q}}(t) v^{\uparrow}(t),$$

$$\mathbb{F}_4 = \sup_{t>0} \left(\int_t^\infty k_2^r(s,t) W_1^{\frac{r}{q}}(s)u(s)\,ds \right)^{\frac{1}{r}} v^{\uparrow}(t).$$

(ii) For the case $p = q = 1$, $1 \leqslant r \leqslant \infty$ we have

$$Tf(x) = \int_x^\infty k_1(y,x)w(y) \left(\int_0^y k_2(y,t)f(t)dt \right) dy.$$

Using Fubini theorem and properties of Oinarov's kernels, we obtain

$$Tf(x) \approx \left(\int_0^x f \right) \mathbb{W}_{12}(x) + \left(\int_0^x k_2(x,t)f(t)dt \right) W_1(x) + \left(\int_x^\infty f\mathbb{W}_{12} \right) + \left(\int_x^\infty k_1(t,x)f(t)\mathbb{W}_2(t)dt \right).$$

We apply theorem 1.1 from [5], claim 1 and claim 2, then $C_T \approx \sum_{j=1}^4 \mathbb{G}_j$, where

$$\mathbb{G}_1 = \sup_{t>0} v^{\uparrow}(t) \left(\int_t^\infty \mathbb{W}_{12}^r u \right)^{\frac{1}{r}},$$

$$\mathbb{G}_2 = \sup_{t>0} v^{\uparrow}(t) \left(\int_t^\infty k_2^r(s,t) W_1^r(s) u(s)\, ds \right)^{\frac{1}{r}},$$

$$\mathbb{G}_3 = \sup_{t>0} R_{12}(t) \left(\int_0^t u \right)^{\frac{1}{r}}, R_{12}(t) := \operatorname*{ess\,sup}_{t<s<\infty} \frac{\mathbb{W}_{12}(s)}{v(s)},$$

$$\mathbb{G}_4 = \sup_{t>0} R_2(t) \left(\int_0^t k_1^r(t,s) u(s) ds \right)^{\frac{1}{r}}, R_2(t) := \operatorname*{ess\,sup}_{t<s<\infty} \frac{\mathbb{W}_2(s)}{v(s)}.$$

The proof is complete. □

4. Operator $\mathscr{T}$

Let

$$V_*(t) := \int_t^\infty v^{1-p'}, W_*(t) := \int_0^t w, U_*(t) := \int_t^\infty u,$$

$$V_{2,k*}(t) := \int_t^\infty k_2^{p'}(y,t) v^{1-p'}(y) dy, W_{1*}(t) := \int_0^t k_1(t,y) w(y) dy,$$

$$W_{2*}(t) := \int_0^t k_2^q(t,y) w(y) dy, W_{12*}(t) := \int_0^t k_1(t,y) k_2^q(t,y) w(y) dy.$$

REMARK 4. Let $\varphi \in \mathfrak{M}^{\downarrow}$. Without loss of generality we may and shall assume that φ is left-continuous. Then it is known (see [22], chapter 12), that there exists a Borel measure, say η_φ, such that

$$\varphi(t) = \int_{[t,\infty]} d\eta_\varphi(s).$$

Thus, we may suppose that for the function $V_{2,k*} \in \mathfrak{M}^{\downarrow}$ there is a Borel measure $dV_{2,k*}$ such that $V_{2,k*}(t) = \int_{[t,\infty]} dV_{2,k*}(s)$.

THEOREM 2. *Let* $1 < q,\ p,\ r < \infty$, $r > q$. *Then for the best constant* $C_{\mathscr{T}}$ *of the inequality*

$$\left(\int_0^\infty \left(\int_0^x k_1(x,y) w(y) \left(\int_y^\infty k_2(t,y) f(t) dt \right)^q dy \right)^{\frac{r}{q}} u(x) dx \right)^{\frac{1}{r}} \leqslant C_{\mathscr{T}} \left(\int_0^\infty f^p v \right)^{\frac{1}{p}},\ f \in \mathfrak{M}^+, \tag{10}$$

the following holds.

(i) If $1 < p \leqslant q < \infty$, *then* $C_{\mathscr{T}} \approx \mathbb{F}_{11}^* + \mathbb{F}_{12}^* + \mathbb{F}_{13}^* + \mathbb{F}_{14}^* + \mathbb{F}_{21}^* + \mathbb{F}_{22}^* + \mathbb{F}_{23}^*$, *where*

$$\mathbb{F}_{11}^* = \sup_{t>0} V_*^{\frac{1}{p'}}(t) \left(\int_0^t W_{12*}^{\frac{r}{q}}(s) u(s)\, ds \right)^{\frac{1}{r}},$$

$$\mathbb{F}_{12}^* = \sup_{t>0} V_*^{\frac{1}{p'}}(t) \left(\int_0^t k_2^r(t,s) W_{1*}^{\frac{r}{q}}(s) u(s)\, ds \right)^{\frac{1}{r}},$$

$$\mathbb{F}^*_{13} = \sup_{t>0} V_*^{\frac{1}{p'}}(t) W_{12*}^{\frac{1}{q}}(t) U_*^{\frac{1}{r}}(t),$$

$$\mathbb{F}^*_{14} = \sup_{t>0} V_*^{\frac{1}{p'}}(t) W_{2*}^{\frac{1}{q}}(t) \left(\int_t^\infty k_1^{\frac{r}{q}}(s,t) u(s)\, ds \right)^{\frac{1}{r}},$$

$$\mathbb{F}^*_{21} = \sup_{t>0} V_{2,k*}^{\frac{1}{p'}}(t) \left(\int_0^t W_{1*}^{\frac{r}{q}}(s) u(s)\, ds \right)^{\frac{1}{r}},$$

$$\mathbb{F}^*_{22} = \sup_{t>0} V_{2,k*}^{\frac{1}{p'}}(t) W_{1*}^{\frac{1}{q}}(t) U_*^{\frac{1}{r}}(t),$$

$$\mathbb{F}^*_{23} = \sup_{t>0} V_{2,k*}^{\frac{1}{p'}}(t) W_*^{\frac{1}{q}}(t) \left(\int_t^\infty k_1^{\frac{r}{q}}(s,t) u(s)\, ds \right)^{\frac{1}{r}}.$$

(ii) If $1 < q < p < \infty, \frac{1}{s} := \frac{1}{q} - \frac{1}{p}$, *then* $C_{\mathscr{T}} \approx \mathbb{G}^*_{11} + \mathbb{G}^*_{12} + \mathbb{G}^*_{13} + \mathbb{G}^*_{14} + \mathbb{G}^*_{21} + \mathbb{G}^*_{22} + \mathbb{G}^*_{23}$, *where for* $p \leqslant r$

$$\mathbb{G}^*_{11} = \sup_{t>0} V_*^{\frac{1}{p'}}(t) \left(\int_0^t W_{12*}^{\frac{r}{q}} u \right)^{\frac{1}{r}},$$

$$\mathbb{G}^*_{12} = \sup_{t>0} \left(\int_t^\infty k_2^s(x,t)\, d(-V_*^{\frac{s}{p'}}(x)) \right)^{\frac{1}{s}} \left(\int_0^t W_{1*}^{\frac{r}{q}} u \right)^{\frac{1}{r}}$$

$$+ \sup_{t>0} V_*^{\frac{1}{p'}}(t) \left(\int_0^t k_2^r(t,x) u(x) W_{1*}^{\frac{r}{q}}(x)\, dx \right)^{\frac{1}{r}},$$

$$\mathbb{G}^*_{13} = \sup_{t>0} \left(\int_0^t W_{12*}^{\frac{s}{q}}\, d(-V_*^{\frac{s}{p'}}) \right)^{\frac{1}{s}} U_*^{\frac{1}{r}}(t),$$

$$\mathbb{G}^*_{14} = \sup_{t>0} \left(\int_0^t k_1^{\frac{s}{q}}(t,x) W_{2*}^{\frac{s}{q}}(x)\, d(-V_*^{\frac{s}{p'}}(x)) \right)^{\frac{1}{s}} U_*^{\frac{1}{r}}(t),$$

$$+ \sup_{t>0} \left(\int_0^t W_{2*}^{\frac{s}{q}}\, d(-V_*^{\frac{s}{p'}}) \right)^{\frac{1}{s}} \left(\int_t^\infty k_1^{\frac{r}{q}}(s,t) u(s) ds \right)^{\frac{1}{r}},$$

$$\mathbb{G}^*_{21} = \sup_{t>0} V_{2,k*}^{\frac{1}{p'}}(t) \left(\int_0^t W_{1*}^{\frac{r}{q}} u \right)^{\frac{1}{r}},$$

$$\mathbb{G}^*_{22} = \sup_{t>0} \left(\int_0^t k_1^{\frac{s}{q}}(t,x) W_*^{\frac{s}{q}}(x)\, d(-V_{2,k*}^{\frac{s}{p'}}(x)) \right)^{\frac{1}{s}} U_*^{\frac{1}{r}}(t)$$

$$+ \sup_{t>0} \left(\int_0^t W_*^{\frac{s}{q}}\, d(-V_{2,k*}^{\frac{s}{p'}}) \right)^{\frac{1}{s}} \left(\int_t^\infty k_1^{\frac{r}{q}}(s,t) u(s) ds \right)^{\frac{1}{r}},$$

$$\mathbb{G}^*_{23} = \sup_{t>0} \left(\int_0^t W_{1*}^{\frac{s}{q}}\, d(-V_{2,k*}^{\frac{s}{p'}}) \right)^{\frac{1}{s}} U_*^{\frac{1}{r}}(t).$$

For $r < p$ *and* $\frac{1}{s_1} := \frac{1}{r} - \frac{1}{p}$

$$\mathbb{G}_{11}^* = \left(\int_0^\infty V_*^{\frac{s_1}{p'}}(t) d \left(\left(\int_0^t W_{12*}^{\frac{r}{q}} u \right)^{\frac{s_1}{r}} \right) \right)^{\frac{1}{s_1}},$$

$$\mathbb{G}_{12}^* = \left(\int_0^\infty \left(\int_t^\infty k_2^s(x,t) d(-V_*^{\frac{s}{p'}}(x)) \right)^{\frac{s_1}{s}} d \left(\int_0^t u W_{1*}^{\frac{r}{q}} \right)^{\frac{s_1}{r}} \right)^{\frac{1}{s_1}}$$

$$+ \left(\int_0^\infty \left(\int_0^t k_2^r(t,x) u(x) W_{1*}^{\frac{r}{q}}(x) dx \right)^{\frac{s_1}{r}} d(-V_*^{\frac{s_1}{p'}}(t)) \right)^{\frac{1}{s_1}},$$

$$\mathbb{G}_{13}^* = \left(\int_0^\infty \left(\int_0^t W_{12*}^{\frac{s}{q}} d(-V_*^{\frac{s}{p'}}) \right)^{\frac{s_1}{s}} U_*^{\frac{s_1}{p}}(t) u(t) dt \right)^{\frac{1}{s_1}},$$

$$\mathbb{G}_{14}^* = \left(\int_0^\infty \left(\int_0^t k_1^{\frac{s}{q}}(t,x) W_{2*}^{\frac{s}{q}}(x) d(-V_*^{\frac{s}{p'}}(x)) \right)^{\frac{s_1}{s}} U_*^{\frac{s_1}{p}}(t) u(t) dt \right)^{\frac{1}{s_1}}$$

$$+ \left(\int_0^\infty \left(\int_t^\infty k_1^{\frac{r}{q}}(s,t) u(s) ds \right)^{\frac{s_1}{r}} d \left(\int_0^t W_{2*}^{\frac{s}{q}} d(-V_*^{\frac{s}{p'}}) \right)^{\frac{s_1}{s}} \right)^{\frac{1}{s_1}},$$

$$\mathbb{G}_{21}^* = \left(\int_0^\infty V_{2*}^{\frac{s_1}{p'}}(t) d \left(\int_0^t W_{1*}^{\frac{r}{q}} u \right)^{\frac{s_1}{r}} \right)^{\frac{1}{s_1}},$$

$$\mathbb{G}_{22}^* = \left(\int_0^\infty \left(\int_0^t k_1^{\frac{s}{q}}(t,x) W_*^{\frac{s}{q}}(x) d(-V_{2,k*}^{\frac{s}{p'}}(x)) \right)^{\frac{s_1}{s}} U_*^{\frac{s_1}{p}}(t) u(t) dt \right)^{\frac{1}{s_1}}$$

$$+ \left(\int_0^\infty \left(\int_t^\infty k_1^{\frac{r}{q}}(s,t) u(s) ds \right)^{\frac{s_1}{r}} d \left(\int_0^t W_*^{\frac{s}{q}} d(-V_{2,k*}^{\frac{s}{p'}}) \right)^{\frac{s_1}{s}} \right)^{\frac{1}{s_1}},$$

$$\mathbb{G}_{23}^* = \left(\int_0^\infty \left(\int_0^t W_{1*}^{\frac{s}{q}} d(-V_{2,k*}^{\frac{s}{p'}}) \right)^{\frac{s_1}{s}} U_*^{\frac{s_1}{p}}(t) u(t) dt \right)^{\frac{1}{s_1}}.$$

Let $v \in \mathfrak{M}^+$. We define $v^{\downarrow}(x) := \operatorname*{ess\,sup}_{t<s<\infty} \frac{1}{v(t)}$. We denote

$$\mathbb{W}_{2*}(t) := \int_0^t k_2(t,y) w(y) dy, \ \mathbb{W}_{12*}(t) := \int_0^t k_1(t,y) k_2(t,y) w(y) dy.$$

REMARK 5. (i) If $p = 1$, $1 < q < r < \infty$, then $C_{\mathscr{T}} \approx \mathbb{F}_1^* + \mathbb{F}_2^* + \mathbb{F}_3^* + \mathbb{F}_4^*$, where

$$\mathbb{F}_1^* = \sup_{t>0} v^{\downarrow}(t) \left(\int_0^t W_{12*}^{\frac{r}{q}}(s) u(s) ds \right)^{\frac{1}{r}},$$

$$\mathbb{F}_2^* = \sup_{t>0} v^{\downarrow}(t)\left(\int_0^t k_2^r(t,s)W_{1*}^{\frac{r}{q}}(s)u(s)\,ds\right)^{\frac{1}{r}},$$

$$\mathbb{F}_3^* = \sup_{t>0} v^{\downarrow}(t)W_{12*}^{\frac{1}{q}}(t)U_*^{\frac{1}{r}}(t),$$

$$\mathbb{F}_4^* = \sup_{t>0} v^{\downarrow}(t)W_{2*}^{\frac{1}{q}}(t)\left(\int_t^\infty k_1^{\frac{r}{q}}(s,t)u(s)\,ds\right)^{\frac{1}{r}}.$$

(ii) If $p=q=1$, $1\leqslant r\leqslant\infty$, then $C_{\mathscr{T}} \approx \mathbb{G}_1^*+\mathbb{G}_2^*+\mathbb{G}_3^*+\mathbb{F}_4^*$, where

$$\mathbb{G}_1^* = \sup_{t>0} R_{2*}(t)\left(\int_t^\infty k_1^r(s,t)u(s)\,ds\right)^{\frac{1}{r}}, R_{2*}(t) := \operatorname*{ess\,sup}_{0<s<t}\frac{\mathbb{W}_{2*}(s)}{v(s)},$$

$$\mathbb{G}_2^* = \sup_{t>0} R_{12*}(t)\left(\int_t^\infty u\right)^{\frac{1}{r}}, R_{12*}(t) := \operatorname*{ess\,sup}_{0<s<t}\frac{\mathbb{W}_{12*}(s)}{v(s)},$$

$$\mathbb{G}_3^* = \sup_{t>0} v^{\downarrow}(t)\left(\int_0^t k_2^r(t,s)W_{1*}^{\frac{r}{q}}(s)u(s)\,ds\right)^{\frac{1}{r}},$$

$$\mathbb{G}_4^* = \sup_{t>0} v^{\downarrow}(t)\left(\int_0^t \mathbb{W}_{12*}^{\frac{r}{q}}(s)u(s)\,ds\right)^{\frac{1}{r}}.$$

5. Operator S

THEOREM 3. *Let $1<q,\ p,\ r<\infty$, $r>q$. Then for the best constant C_S of the inequality*

$$\left(\int_0^\infty\left(\int_x^\infty k_1(y,x)w(y)\left(\int_y^\infty k_2(t,y)f(t)dt\right)^q dy\right)^{\frac{r}{q}} u(x)dx\right)^{\frac{1}{r}} \leqslant C_S\left(\int_0^\infty f^p v\right)^{\frac{1}{p}},\ f\in\mathfrak{M}^+, \tag{11}$$

the following holds.

(i) If $1<p\leqslant q<\infty$, then $C_S\approx\mathfrak{F}_1+\mathfrak{F}_2$, where

$$\mathfrak{F}_1 = \sup_{t>0} V_*^{\frac{1}{p'}}(t)\left(\int_0^t\left(\int_x^t k_2^q(t,y)k_1(y,x)w(y)dy\right)^{\frac{r}{q}} u(x)\,dx\right)^{\frac{1}{r}},$$

$$\mathfrak{F}_2 = \sup_{t>0} V_{2,k*}^{\frac{1}{p'}}(t)\left(\int_0^t\left(\int_x^t k_1(y,x)w(y)dy\right)^{\frac{r}{q}} u(x)\,dx\right)^{\frac{1}{r}}.$$

(ii) If $1<q<p<\infty$, $\frac{1}{s}:=\frac{1}{q}-\frac{1}{p}$, then $C_S\approx\mathfrak{G}_1+\mathfrak{G}_2$, where for $p\leqslant r$

$$\mathfrak{G}_1 = \sup_{t>0} U^{\frac{1}{r}}(t)\left(\int_t^\infty\left(\int_t^x k_2^q(y,t)k_1(x,y)w(y)dy\right)^{\frac{s}{q}} d(-V_*^{\frac{s}{p'}}(x))\right)^{\frac{1}{s}}$$

$$+\sup_{t>0} V_*^{\frac{1}{p'}}(t)\left(\int_0^t\left(\int_x^t k_2^q(t,y)k_1(y,x)w(y)dy\right)^{\frac{r}{q}}u(x)dx\right)^{\frac{1}{r}},$$

$$\mathfrak{G}_2=\sup_{t>0} U^{\frac{1}{r}}(t)\left(\int_t^\infty\left(\int_t^x k_1(x,y)w(y)dy\right)^{\frac{s}{q}}d(-V_{2,k*}^{\frac{s}{p'}}(x))\right)^{\frac{1}{s}}$$

$$+\sup_{t>0} V_{2,k*}^{\frac{1}{p'}}(t)\left(\int_0^t\left(\int_x^t k_1(y,x)w(y)dy\right)^{\frac{r}{q}}u(x)dx\right)^{\frac{1}{r}}.$$

For $r<p$ and $\frac{1}{s_1}:=\frac{1}{r}-\frac{1}{p}$

$$\mathfrak{G}_1=\left(\int_0^\infty\left(\int_t^\infty\left(\int_t^x k_2^q(y,t)k_1(x,y)w(y)dy\right)^{\frac{s}{q}}d(-V_*^{\frac{s}{p'}}(x))\right)^{\frac{s_1}{s}}U^{\frac{s_1}{p}}(t)u(t)dt\right)^{\frac{1}{s_1}}$$

$$+\left(\int_0^\infty\left(\int_0^t\left(\int_x^t k_2^q(t,y)k_1(y,x)w(y)dy\right)^{\frac{r}{q}}u(x)dx\right)^{\frac{s_1}{r}}d(-V_*^{\frac{s_1}{p'}}(t))\right)^{\frac{1}{s_1}},$$

$$\mathfrak{G}_2=\left(\int_0^\infty\left(\int_t^\infty\left(\int_t^x k_1(x,y)w(y)dy\right)^{\frac{s}{q}}d(-V_{2,k*}^{\frac{s}{p'}}(x))\right)^{\frac{s_1}{s}}U^{\frac{s_1}{p}}(t)u(t)dt\right)^{\frac{1}{s_1}}$$

$$+\left(\int_0^\infty\left(\int_0^t\left(\int_x^t k_1(y,x)w(y)dy\right)^{\frac{r}{q}}u(x)dx\right)^{\frac{s_1}{r}}d(-V_{2,k*}^{\frac{s_1}{p'}}(t))\right)^{\frac{1}{s_1}}.$$

Proof. Using similar reasoning as in theorem 1, we have

$$C_S=\sup_f\|f\|_{p,v}^{-1}\left[\left(\int_0^\infty\left(\int_x^\infty k_1(y,x)w(y)\left(\int_y^\infty k_2(t,y)f(t)dt\right)^q dy\right)^{\frac{r}{q}}u(x)dx\right)^{\frac{q}{r}}\right]^{\frac{1}{q}}$$

$$\overset{(7)}{=}\sup_f\|f\|_{p,v}^{-1}\left[\sup_h\frac{\int_0^\infty\left(\int_x^\infty k_1(y,x)w(y)\left(\int_y^\infty k_2(t,y)f(t)dt\right)^q dy\right)h(x)u(x)dx}{\left(\int_0^\infty h^{\frac{r}{r-q}}u\right)^{1-\frac{q}{r}}}\right]^{\frac{1}{q}}$$

$$=\sup_h\left(\int_0^\infty h^{\frac{r}{r-q}}u\right)^{\frac{1}{r}-\frac{1}{q}}\sup_f\frac{\left(\int_0^\infty\left(\int_y^\infty k_2(t,y)f(t)dt\right)^q\left(\int_0^y k_1(y,x)h(x)u(x)dx\right)w(y)dy\right)^{\frac{1}{q}}}{\|f\|_{p,v}}.$$

Applying dual theorem to theorem B, we find

$$C_S\approx\sup_h\left(\int_0^\infty h^{\frac{r}{r-q}}u\right)^{\frac{1}{r}-\frac{1}{q}}(\mathscr{F}_1(h)+\mathscr{F}_2(h)),$$

where

$$\mathscr{F}_1(h) = \sup_{t>0} \left(\int_0^t k_2^q(t,y) w(y) \left(\int_0^y k_1(y,x) h(x) u(x) dx \right) dy \right)^{\frac{1}{q}} V_*^{\frac{1}{p'}}(t),$$
$$\mathscr{F}_2(h) = \sup_{t>0} \left(\int_0^t w(y) \left(\int_0^y k_1(y,x) h(x) u(x) dx \right) dy \right)^{\frac{1}{q}} V_{2,k*}^{\frac{1}{p'}}(t).$$

Using Fubini theorem, we have

$$\mathscr{F}_1(h) = \sup_{t>0} \left(\int_0^t \left(\int_x^t k_2^q(t,y) k_1(y,x) w(y) dy \right) h(x) u(x) dx \right)^{\frac{1}{q}} V_*^{\frac{1}{p'}}(t),$$
$$\mathscr{F}_2(h) = \sup_{t>0} \left(\int_0^t \left(\int_x^t k_1(y,x) w(y) dy \right) h(x) u(x) dx \right)^{\frac{1}{q}} V_{2,k*}^{\frac{1}{p'}}(t).$$

Then $C_S \approx C_1 + C_2$, where

$$C_i = \mathfrak{F}_i := \sup_h \left(\int_0^\infty h^{\frac{r}{r-q}} u \right)^{\frac{1}{r} - \frac{1}{q}} \mathscr{F}_i(h), \, i = 1, 2.$$

Using the consequence of the duality property (8), we have

$$\mathfrak{F}_1 = \sup_{t>0} V_*^{\frac{1}{p'}}(t) \left(\int_0^t \left(\int_x^t k_2^q(t,y) k_1(y,x) w(y) dy \right)^{\frac{r}{q}} u(x)\, dx \right)^{\frac{1}{r}},$$
$$\mathfrak{F}_2 = \sup_{t>0} V_{2,k*}^{\frac{1}{p'}}(t) \left(\int_0^t \left(\int_x^t k_1(y,x) w(y) dy \right)^{\frac{r}{q}} u(x)\, dx \right)^{\frac{1}{r}}.$$

Case II: $1 < q < p, r$, $\frac{1}{s} := \frac{1}{q} - \frac{1}{p}$.

$$\mathscr{G}_1(h) := \left(\int_0^\infty \left(\int_0^t k_2^q(t,y) w(y) \left(\int_0^y k_1(y,x) h(x) u(x) dx \right) dy \right)^{\frac{s}{q}} d(-V_*^{\frac{s}{p'}}(t)) \right)^{\frac{1}{s}},$$
$$\mathscr{G}_2(h) := \left(\int_0^\infty \left(\int_0^t w(y) \left(\int_0^y k_1(y,x) h(x) u(x) dx \right) dy \right)^{\frac{s}{q}} d(-V_{2,k*}^{\frac{s}{p'}}(t)) \right)^{\frac{1}{s}}.$$

Using Fubini theorem, we have

$$\mathscr{G}_1(h) = \left(\int_0^\infty \left(\int_0^t \left(\int_x^t k_2^q(t,y) k_1(y,x) w(y) dy \right) h(x) u(x) dx \right)^{\frac{s}{q}} d(-V_*^{\frac{s}{p'}}(t)) \right)^{\frac{1}{s}},$$
$$\mathscr{G}_2(h) = \left(\int_0^\infty \left(\int_0^t \left(\int_x^t k_1(y,x) w(y) dy \right) h(x) u(x) dx \right)^{\frac{s}{q}} d(-V_{2,k*}^{\frac{s}{p'}}(t)) \right)^{\frac{1}{s}}.$$

We use theorem B to obtain for $p \leqslant r$

$$\mathfrak{G}_1 = \sup_{t>0} U^{\frac{1}{r}}(t) \left(\int_t^\infty \left(\int_t^x k_2^q(y,t) k_1(x,y) w(y) dy \right)^{\frac{s}{q}} d(-V_*^{\frac{s}{p'}}(x)) \right)^{\frac{1}{s}}$$

$$+ \sup_{t>0} V_*^{\frac{1}{p'}}(t) \left(\int_0^t \left(\int_x^t k_2^q(t,y) k_1(y,x) w(y) dy \right)^{\frac{r}{q}} u(x) dx \right)^{\frac{1}{r}},$$

$$\mathfrak{G}_2 = \sup_{t>0} U^{\frac{1}{r}}(t) \left(\int_t^\infty \left(\int_t^x k_1(x,y) w(y) dy \right)^{\frac{s}{q}} d(-V_{2,k*}^{\frac{s}{p'}}(x)) \right)^{\frac{1}{s}}$$

$$+ \sup_{t>0} V_{2,k*}^{\frac{1}{p'}}(t) \left(\int_0^t \left(\int_x^t k_1(y,x) w(y) dy \right)^{\frac{r}{q}} u(x) dx \right)^{\frac{1}{r}}.$$

For $r < p$ and $\frac{1}{s_1} := \frac{1}{r} - \frac{1}{p}$

$$\mathfrak{G}_1 = \left(\int_0^\infty \left(\int_t^\infty \left(\int_t^x k_2^q(y,t) k_1(x,y) w(y) dy \right)^{\frac{s}{q}} d(-V_*^{\frac{s}{p'}}(x)) \right)^{\frac{s_1}{s}} U^{\frac{s_1}{p}}(t) u(t) dt \right)^{\frac{1}{s_1}}$$

$$+ \left(\int_0^\infty \left(\int_0^t \left(\int_x^t k_2^q(t,y) k_1(y,x) w(y) dy \right)^{\frac{r}{q}} u(x) dx \right)^{\frac{s_1}{r}} d(-V_*^{\frac{s_1}{p'}}(t)) \right)^{\frac{1}{s_1}},$$

$$\mathfrak{G}_2 = \left(\int_0^\infty \left(\int_t^\infty \left(\int_t^x k_1(x,y) w(y) dy \right)^{\frac{s}{q}} d(-V_{2*}^{\frac{s}{p'}}(x)) \right)^{\frac{s_1}{s}} U^{\frac{s_1}{p}}(t) u(t) dt \right)^{\frac{1}{s_1}}$$

$$+ \left(\int_0^\infty \left(\int_0^t \left(\int_x^t k_1(y,x) w(y) dy \right)^{\frac{r}{q}} u(x) dx \right)^{\frac{s_1}{r}} d(-V_{2,k*}^{\frac{s_1}{p'}}(t)) \right)^{\frac{1}{s_1}}.$$

The proof is complete. □

REMARK 6. (i) If $p = 1$, $1 < q < r < \infty$, then $C_S \approx \mathfrak{F}_1$, where

$$\mathfrak{F}_1 = \sup_{t>0} v^{\downarrow}(t) \left(\int_0^t \left(\int_x^t k_2^q(t,y) k_1(y,x) w(y) dy \right)^{\frac{r}{q}} u(x)\, dx \right)^{\frac{1}{r}},$$

(ii) If $p = q = 1$, $1 \leqslant r \leqslant \infty$, then $C_S \approx \mathfrak{G}_1$, where

$$\mathfrak{G}_1 = \sup_{t>0} v^{\downarrow}(t) \left(\int_0^t \left(\int_x^t k_2(t,y) k_1(y,x) w(y) dy \right)^{r} u(x)\, dx \right)^{\frac{1}{r}}.$$

6. Operator $\mathscr{S}$

THEOREM 4. *Let $1 < q,\ p,\ r < \infty$, $r > q$. Then for the best constant $C_{\mathscr{S}}$ of the inequality*

$$\left(\int_0^\infty \left(\int_0^x k_1(x,y)w(y)\left(\int_0^y k_2(y,t)f(t)dt\right)^q dy\right)^{\frac{r}{q}} u(x)dx\right)^{\frac{1}{r}} \leqslant C_{\mathscr{S}}\left(\int_0^\infty f^p v\right)^{\frac{1}{p}},\ \ f \in \mathfrak{M}^+, \tag{12}$$

the following holds.

(i) If $1 < p \leqslant q < \infty$, then $C_{\mathscr{S}} \approx \mathfrak{F}_1^* + \mathfrak{F}_2^*$, where

$$\mathfrak{F}_1^* = \sup_{t>0} V^{\frac{1}{p'}}(t)\left(\int_t^\infty \left(\int_t^x k_2^q(y,t)k_1(x,y)w(y)dy\right)^{\frac{r}{q}} u(x)\,dx\right)^{\frac{1}{r}},$$

$$\mathfrak{F}_2^* = \sup_{t>0} V_2^{\frac{1}{p'}}(t)\left(\int_t^\infty \left(\int_t^x k_1(x,y)w(y)dy\right)^{\frac{r}{q}} u(x)\,dx\right)^{\frac{1}{r}}.$$

(ii) If $1 < q < p < \infty$, $\frac{1}{s} := \frac{1}{q} - \frac{1}{p}$, then $C_{\mathscr{S}} \approx \mathfrak{G}_1^* + \mathfrak{G}_2^*$, where for $p \leqslant r$

$$\mathfrak{G}_1^* = \sup_{t>0} U_*^{\frac{1}{r}}(t)\left(\int_0^t \left(\int_x^t k_2^q(t,y)k_1(y,x)w(y)dy\right)^{\frac{s}{q}} dV^{\frac{s}{p'}}(x)\right)^{\frac{1}{s}}$$
$$+ \sup_{t>0} V^{\frac{1}{p'}}(t)\left(\int_t^\infty \left(\int_t^x k_2^q(y,t)k_1(x,y)w(y)dy\right)^{\frac{r}{q}} u(x)dx\right)^{\frac{1}{r}}.$$

$$\mathfrak{G}_2^* = \sup_{t>0} U_*^{\frac{1}{r}}(t)\left(\int_0^t \left(\int_x^t k_1(y,x)w(y)dy\right)^{\frac{s}{q}} dV_{2,k}^{\frac{s}{p'}}(x)\right)^{\frac{1}{s}}$$
$$+ \sup_{t>0} V_{2,k}^{\frac{1}{p'}}(t)\left(\int_t^\infty \left(\int_t^x k_1(x,y)w(y)dy\right)^{\frac{r}{q}} u(x)dx\right)^{\frac{1}{r}}.$$

For $r < p$

$$\mathfrak{G}_1^* = \left(\int_0^\infty \left(\int_0^t \left(\int_x^t k_2^q(t,y)k_1(y,x)w(y)dy\right)^{\frac{s}{q}} dV^{\frac{s}{p'}}(x)\right)^{\frac{s_1}{s}} U_*^{\frac{s_1}{p}}(t)u(t)dt\right)^{\frac{1}{s_1}}$$
$$+ \left(\int_0^\infty \left(\int_t^\infty \left(\int_t^x k_2^q(y,t)k_1(x,y)w(y)dy\right)^{\frac{r}{q}} u(x)dx\right)^{\frac{s_1}{r}} dV^{\frac{s_1}{p'}}(t)\right)^{\frac{1}{s_1}},$$

$$\mathfrak{G}_2^* = \left(\int_0^\infty \left(\int_0^t \left(\int_x^t k_1(y,x)w(y)dy\right)^{\frac{s}{q}} dV_{2,k}^{\frac{s}{p'}}(x)\right)^{\frac{s_1}{s}} U_*^{\frac{s_1}{p}}(t)u(t)dt\right)^{\frac{1}{s_1}}$$

$$+\left(\int_0^\infty\left(\int_t^\infty\left(\int_t^x k_1(x,y)w(y)dy\right)^{\frac{r}{q}}u(x)dx\right)^{\frac{s_1}{r}}dV_{2,k}^{\frac{s_1}{p'}}(t)\right)^{\frac{1}{s_1}}.$$

REMARK 7. (i) If $p=1$, $1<q<r<\infty$, then $C_{\mathscr{S}}\approx\mathfrak{F}_1^*$, where

$$\mathfrak{F}_1^*=\sup_{t>0}v^{\uparrow}(t)\left(\int_t^\infty\left(\int_t^x k_2^q(y,t)k_1(x,y)w(y)dy\right)^{\frac{r}{q}}u(x)\,dx\right)^{\frac{1}{r}}.$$

(ii) If $p=q=1$, $1\leqslant r\leqslant\infty$, then $C_{\mathscr{S}}\approx\mathfrak{G}_1^*$, where

$$\mathfrak{G}_1^*=\sup_{t>0}v^{\uparrow}(t)\left(\int_t^\infty\left(\int_t^x k_2(y,t)k_1(x,y)w(y)dy\right)^{\frac{r}{q}}u(x)\,dx\right)^{\frac{1}{r}}.$$

7. Bilinear Hardy-type inequality

As application of the results for operator T we consider the characterization problem of the inequality

$$\left(\int_0^\infty[R_1fR_2g]^qw\right)^{\frac{1}{q}}\leqslant C\|f\|_{p_1,v_1}\|g\|_{p_2,v_2},\ f,g\in\mathfrak{M}^+, \tag{13}$$

where $R_i(x):=\int_0^x k_i(x,y)h(y)dy$, where k_i satisfies condition (1).

The case $1<\min(p_1,p_2)\leqslant q<\infty$ was explicitly solved in [18] and a reduction theorem was proved for the case $1<q<\min(p_1,p_2)$. We complement this case by explicit criteria.

Additionally to notations of § 3 we define

$$V_i(t):=\int_0^t v_i^{1-p_i'},\ i=1,2;\ \mathscr{V}_{2,k}(t):=\int_0^t k_2^{p_2'}(t,y)v_2^{1-p_2'}(y)dy,$$

and $\frac{1}{r_i}:=\frac{1}{q}-\frac{1}{p_i}$, $i=1,2$.

Following by a known scheme (see [1], [9]) we write for the least constant C in (13)

$$C=\sup_{0\neq g\in\mathfrak{M}^+}\|g\|_{p_2,v_2}^{-1}\sup_{0\neq f\in\mathfrak{M}^+}\frac{(\int_0^\infty[R_1fR_2g]^qw)^{\frac{1}{q}}}{\|f\|_{p_1,v_1}}=:\sup_g\|g\|_{p_2,v_2}^{-1}\mathscr{F}(g).$$

Let $1<q<\min(p_1,p_2)$ and suppose first that $1<q<p_1$. For a fixed $g\in\mathfrak{M}^+$, applying theorem B, we have

$$\mathscr{F}(g)\approx\mathscr{F}_1(g)+\mathscr{F}_2(g),$$

where

$$\mathscr{F}_1^{r_1}(g) := \int_0^\infty \left(\int_y^\infty k_1^q(x,y)(R_2 g(x))^q w(x)dx \right)^{\frac{r_1}{q}} V_1^{\frac{r_1}{q'}}(y) v_1^{1-p_1'}(y)dy,$$

$$\mathscr{F}_2^{r_1}(g) := \int_0^\infty \left(\int_y^\infty (R_2 g)^q w \right)^{\frac{r_1}{p_1}} K_1(y)(R_2 g)^q w(y)dy,$$

where

$$K_1(y) := \left(\int_0^y k_1^{p_1'}(y,s) v_1^{1-p_1'}(s)ds \right)^{\frac{r_1}{p_1'}}.$$

By remark 1 we assume that $K_1 \in \mathfrak{M}^\uparrow$, and $K_1(y) = \int_{[0,y]} dK_1$. Then

$$\mathscr{F}_2^{r_1}(g) \approx \int_0^\infty \left(\int_y^\infty (R_2 g)^q w \right)^{\frac{r_1}{q}} dK_1(y)$$

and, thus, $C \approx \mathbb{F} + \tilde{\mathbb{F}}$, where

$$\mathbb{F} := \sup_g \|g\|_{p_2,v_2}^{-1} \mathscr{F}_1(g);\ \tilde{\mathbb{F}} := \sup_g \|g\|_{p_2,v_2}^{-1} \mathscr{F}_2(g).$$

To characterize the functional $\mathbb{F}$ we use theorem 1.

a) If $1 < q < p_2 \leqslant r_1$, then

$$\mathbb{F} \approx \mathbb{F}_{11} + \mathbb{F}_{12} + \mathbb{F}_{13} + \mathbb{F}_{14} + \mathbb{F}_{21} + \mathbb{F}_{22} + \mathbb{F}_{23}, \tag{14}$$

where, with

$$W_{1,2}(t) := \int_t^\infty k_1^q(y,t) k_2^q(y,t) w(y)dy$$

and

$$W_1(t) := \int_t^\infty k_1^q(y,t) w(y)dy,$$

$$\mathbb{F}_{11} = \sup_{t>0} \left(\int_t^\infty W_{12}^{\frac{r_2}{q}} dV_2^{\frac{r_2}{p_2'}} \right)^{\frac{1}{r_2}} V_1^{\frac{1}{p_1'}}(t),$$

$$\mathbb{F}_{12} = \sup_{t>0} \left(\int_t^\infty k_1^{r_2}(x,t) W_2^{\frac{r_2}{q}}(x) dV_2^{\frac{r_2}{p_2'}}(x) \right)^{\frac{1}{r_2}} V_1^{\frac{1}{p_1'}}(t)$$

$$+ \sup_{t>0} \left(\int_t^\infty W_2^{\frac{r_2}{q}} dV_2^{\frac{r_2}{p_2'}} \right)^{\frac{1}{r_2}} \left(\int_0^t k_1^{r_1}(t,s) dV_1^{\frac{r_1}{p_1'}}(s) \right)^{\frac{1}{r_1}},$$

$$\mathbb{F}_{13} = \sup_{t>0} \left(\int_t^\infty W_{12}^{\frac{r_1}{q}} dV_1^{\frac{r_1}{p_1'}} \right)^{\frac{1}{r_1}} V_2^{\frac{1}{p_2'}}(t),$$

$$\mathbb{F}_{14} = \sup_{t>0} \left(\int_0^t k_2^{r_2}(t,x)\, dV_2^{\frac{r_2}{p_2'}}(x) \right)^{\frac{1}{r_2}} \left(\int_t^\infty W_1^{\frac{r_1}{q}} dV_1^{\frac{r_1}{p_1'}} \right)^{\frac{1}{r_1}}$$

$$+ \sup_{t>0} V_2^{\frac{1}{p_2'}}(t) \left(\int_t^\infty k_2^{r_1}(x,t) W_1^{\frac{r_1}{q}}(x) dV_1^{\frac{r_1}{p_1'}}(x) \right)^{\frac{1}{r_1}},$$

$$\mathbb{F}_{21} = \sup_{t>0} \left(\int_t^\infty W_1^{\frac{r_2}{q}} d\mathscr{V}_{2,k}^{\frac{r_2}{p_2'}} \right)^{\frac{1}{r_2}} V_1^{\frac{1}{p_1'}}(t),$$

$$\mathbb{F}_{22} = \sup_{t>0} \left(\int_t^\infty k_1^{r_2}(x,t) W^{\frac{r_2}{q}}(x) d\mathscr{V}_{2,k}^{\frac{r_2}{p_2'}}(x) \right)^{\frac{1}{r_2}} V_1^{\frac{1}{p_1'}}(t)$$

$$+ \sup_{t>0} \left(\int_t^\infty W^{\frac{r_2}{q}} d\mathscr{V}_{2,k}^{\frac{r_2}{p_2'}} \right)^{\frac{1}{r_2}} \left(\int_0^t k_1^{r_1}(t,s) dV_1^{\frac{r_1}{p_1'}}(s) \right)^{\frac{1}{r_1}},$$

$$\mathbb{F}_{23} = \sup_{t>0} \mathscr{V}_{2,k}^{\frac{1}{p_2'}}(t) \left(\int_t^\infty W_1^{\frac{r_1}{q}} dV_1^{\frac{r_1}{p_1'}} \right)^{\frac{1}{r_1}}.$$

b) If $1 < q < r_1 < p_2, \frac{1}{s} := \frac{1}{r_1} - \frac{1}{p_2}$, then

$$\mathbf{F} \approx \mathbf{F}_{11} + \mathbf{F}_{12} + \mathbf{F}_{13} + \mathbf{F}_{14} + \mathbf{F}_{21} + \mathbf{F}_{22} + \mathbf{F}_{23} =: \mathbf{F}, \tag{15}$$

where

$$\mathbf{F}_{11} = \left(\int_0^\infty \left(\int_t^\infty W_{12}^{\frac{r_2}{q}} dV_2^{\frac{r_2}{p_2'}} \right)^{\frac{s}{r_2}} dV_1^{\frac{s}{p_1'}}(t) \right)^{\frac{1}{s}},$$

$$\mathbf{F}_{12} = \left(\int_0^\infty \left(\int_t^\infty k_1^{r_2}(x,t) W_2^{\frac{r_2}{q}}(x) dV_2^{\frac{r_2}{p_2'}}(x) \right)^{\frac{s}{r_2}} dV_1^{\frac{s}{p_1'}}(t) \right)^{\frac{1}{s}}$$

$$+ \left(\int_0^\infty \left(\int_0^t k_1^{r_1}(t,s) dV_1^{\frac{r_1}{p_1'}}(s) \right)^{\frac{s}{r_1}} d\left(- \left(\int_t^\infty W_2^{\frac{r_2}{q}} dV_2^{\frac{r_2}{p_2'}} \right)^{\frac{s}{r_2}} \right) \right)^{\frac{1}{s}},$$

$$\mathbf{F}_{13} = \left(\int_0^\infty V_2^{\frac{s}{p_2'}}(t) \left(\int_t^\infty W_{12}^{\frac{r_1}{q}} dV_1^{\frac{r_1}{p_1'}} \right)^{\frac{s}{p_2}} W_{12}^{\frac{r_1}{q}}(t) dV_1^{\frac{r_1}{p_1'}}(t) \right)^{\frac{1}{s}},$$

$$\mathbf{F}_{14} = \left(\int_0^\infty \left(\int_0^t k_2^{r_2}(t,x) dV_2^{\frac{r_2}{p_2'}}(x) \right)^{\frac{s}{r_2}} \left(\int_t^\infty W_1^{\frac{r_1}{q}} dV_1^{\frac{r_1}{p_1'}} \right)^{\frac{s}{p_2}} W_1^{\frac{r_1}{q}}(t) dV_1^{\frac{r_1}{p_1'}}(t) \right)^{\frac{1}{s}}$$

$$+\left(\int_0^\infty\left(\int_t^\infty k_2^{r_1}(x,t)W_1^{\frac{r_1}{q}}(x)dV_1^{\frac{r_1}{p_1'}}(x)\right)^{\frac{s}{r_1}}dV_2^{\frac{s}{p_2'}}(t)\right)^{\frac{1}{s}},$$

$$\mathbf{F}_{21}=\left(\int_0^\infty\left(\int_t^\infty W_1 d\mathcal{V}_{2,k}^{\frac{r_2}{p_2'}}\right)^{\frac{s}{r_2}}dV_1^{\frac{s}{p_1'}}(t)\right)^{\frac{1}{s}},$$

$$\mathbf{F}_{22}=\left(\int_0^\infty\left(\int_t^\infty k_1^{r_2}(x,t)W^{\frac{r_2}{q}}(x)d\mathcal{V}_{2,k}^{\frac{r_2}{p_2'}}(x)\right)^{\frac{s}{r_2}}dV_1^{\frac{s}{p_1'}}(t)\right)^{\frac{1}{s}}$$

$$+\left(\int_0^\infty\left(\int_0^t k_1^{r_1}(t,s)dV_1^{\frac{r_1}{p_1'}}(s)\right)^{\frac{s}{r_1}}d\left(-\left(\int_t^\infty W^{\frac{r_2}{q}}d(\mathcal{V}_{2,k}^{\frac{r_2}{p_2'}})\right)^{\frac{s}{r_2}}\right)\right)^{\frac{1}{s}},$$

$$\mathbf{F}_{23}=\left(\int_0^\infty \mathcal{V}_{2,k}^{\frac{s}{p_2'}}(t)\left(\int_t^\infty W_1^{\frac{r_2}{q}}dV_1^{\frac{r_1}{p_1'}}\right)^{\frac{s}{p_2}}W_1^{\frac{r_2}{q}}(t)dV_1^{\frac{r_1}{p_1'}}(t)\right)^{\frac{1}{s}}.$$

Again we use theorem 1 for characterization of the functional $\tilde{\mathbb{F}}$.

a) If $1<q<p_2\leqslant r_1$ then

$$\tilde{\mathbb{F}}\approx\tilde{\mathbb{F}}_{11}+\tilde{\mathbb{F}}_{12}+\tilde{\mathbb{F}}_{13}++\tilde{\mathbb{F}}_{21}+\tilde{\mathbb{F}}_{22}, \tag{16}$$

where

$$\tilde{\mathbb{F}}_{11}=\sup_{t>0}\left(\int_t^\infty W_2^{\frac{r_2}{q}}dV_2^{\frac{r_2}{p_2'}}\right)^{\frac{1}{r_2}}K_1^{\frac{1}{r_1}}(t),$$

$$\tilde{\mathbb{F}}_{12}=\sup_{t>0}\left(\int_t^\infty W_2^{\frac{r_1}{q}}dK_1\right)^{\frac{1}{r_1}}V_2^{\frac{1}{p_2'}}(t),$$

$$\tilde{\mathbb{F}}_{13}=\sup_{t>0}\left(\int_0^t k_2^{r_2}(t,x)dV_2^{\frac{r_2}{p_2'}}(x)\right)^{\frac{1}{r_2}}\left(\int_t^\infty W^{\frac{r_1}{q}}dK_1\right)^{\frac{1}{r_1}}$$

$$+\sup_{t>0}V_2^{\frac{1}{p_2'}}(t)\left(\int_t^\infty k_2^{r_1}(x,t)W^{\frac{r_1}{q}}(x)dK_1(x)\right)^{\frac{1}{r_1}},$$

$$\tilde{\mathbb{F}}_{21}=\sup_{t>0}\left(\int_t^\infty W^{\frac{r_2}{q}}d\mathcal{V}_{2,k}^{\frac{r_2}{p_2'}}\right)^{\frac{1}{r_2}}K_1^{\frac{1}{r_1}}(t),$$

$$\tilde{\mathbb{F}}_{22}=\sup_{t>0}\mathcal{V}_{2,k}^{\frac{1}{p_2'}}(t)\left(\int_t^\infty W^{\frac{r_1}{q}}dK_1\right)^{\frac{1}{r_1}}.$$

b) If $1<q<r_1<p_2, \frac{1}{s}:=\frac{1}{r_1}-\frac{1}{p_2}$, then

$$\tilde{\mathbb{F}}\approx\tilde{\mathbf{F}}_{11}+\tilde{\mathbf{F}}_{12}+\tilde{\mathbf{F}}_{13}++\tilde{\mathbf{F}}_{21}+\tilde{\mathbf{F}}_{22}=:\tilde{\mathbf{F}}, \tag{17}$$

where

$$\tilde{\mathbf{F}}_{11} = \left(\int_0^\infty \left(\int_t^\infty W_2^{\frac{r_2}{q}} dV_2^{\frac{r_2}{p_2'}} \right)^{\frac{s}{r_2}} dK_1^{\frac{s}{r_1}}(t) \right)^{\frac{1}{s}},$$

$$\tilde{\mathbf{F}}_{12} = \left(\int_0^\infty V_2^{\frac{s}{p_2'}}(t) \left(\int_t^\infty W_2^{\frac{r_1}{q}} dK_1 \right)^{\frac{s}{p_2}} W_2^{\frac{r_1}{q}}(t) dK_1(t) \right)^{\frac{1}{s}},$$

$$\tilde{\mathbf{F}}_{13} = \left(\int_0^\infty \left(\int_0^t k_2^{r_2}(t,x) dV_2^{\frac{r_2}{p_2'}}(x) \right)^{\frac{s}{r_2}} \left(\int_t^\infty W^{\frac{r_1}{q}} dK_1 \right)^{\frac{s}{p_2}} W^{\frac{r_1}{q}}(t) dK_1(t) \right)^{\frac{1}{s}}$$

$$+ \left(\int_0^\infty \left(\int_t^\infty k_2^{r_1}(x,t) W^{\frac{r_1}{q}}(x) dK_1(x) \right)^{\frac{s}{r_1}} dV_2^{\frac{s}{p_2'}}(t) \right)^{\frac{1}{s}},$$

$$\tilde{\mathbf{F}}_{21} = \left(\int_0^\infty \left(\int_t^\infty W d\mathscr{V}_{2,k}^{\frac{r_2}{p_2'}} \right)^{\frac{s}{r_2}} dK_1^{\frac{s}{r_1}}(t) \right)^{\frac{1}{s}},$$

$$\tilde{\mathbf{F}}_{22} = \left(\int_0^\infty \mathscr{V}_{2,k}^{\frac{s}{p_2'}}(t) \left(\int_t^\infty W^{\frac{r_2}{q}} dK_1 \right)^{\frac{s}{p_2}} W^{\frac{r_2}{q}}(t) dK_1(t) \right)^{\frac{1}{s}}.$$

By summing up our investigations above we can now formulate our main result in this section.

THEOREM 5. *Let* $1 < q < \min(p_1, p_2) < \infty$ *and* $\frac{1}{r_i} := \frac{1}{q} - \frac{1}{p_i}$, $i = 1,2$. *Then the inequality* (13) *with the best constant* C *holds for every* $f, g \in \mathfrak{M}^+$, *if and if only if the following holds.*

(i) If $1 < q < p_2 \leqslant r_1 < \infty$, *then* $C \approx \mathbb{F} + \tilde{\mathbb{F}}$, *where* $\mathbb{F}$ *and* $\tilde{\mathbb{F}}$ *are defined by* (14) *and* (16), *respectively.*

(ii) If $1 < q < r_1 < p_2 < \infty$ *and* $\frac{1}{s} := \frac{1}{r_1} - \frac{1}{p_2}$, *then* $C \approx \mathbf{F} + \tilde{\mathbf{F}}$, *where* $\mathbf{F}$ *and* $\tilde{\mathbf{F}}$ *are defined by* (15) *and* (17), *correspondingly.*

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