Mathematical
Inequalities
& Applications
Volume 22, Number 4 (2019), 1535–1548 doi:10.7153/mia-2019-22-106

EQUIVALENT INTEGRAL CONDITIONS RELATED TO BILINEAR HARDY–TYPE INEQUALITIES

SAIKAT KANJILAL, LARS-ERIK PERSSON* AND GULDARYA E. SHAMBILOVA

Dedicated to the 70th anniversary of Professor Josip Pečarić

(*Communicated by I. Perić*)

Abstract. Infinitely many, even scales of, equivalent conditions are derived to characterize the bilinear Hardy-type inequality under various ranges of parameters.

1. Introduction

Let $\mathfrak{M}$ denote the set of all Lebesgue measurable functions on (a,b), $-\infty \leqslant a < b \leqslant \infty$, $\mathfrak{M}^+ \subset \mathfrak{M}$ is the subset of all non-negative functions.

Let $u, v, v_i, \in \mathfrak{M}^+$, $0 < p, p_i, q \leqslant \infty, 1 \leqslant p_i, i = 1,2$. Denote $p_i' := \frac{p_i}{p_i - 1}$ for $1 < p_i < \infty$, $p_i' := 1$ for $p_i = \infty$ and $p_i' := \infty$ for $p_i = 1$, $i = 1,2$. We use the record $A \approx B$ that means $c_1 A \leqslant B \leqslant c_2 A$ with the constants c_1, c_2, depending only on irrelevant parameters.

We put

$$U(x) = \int_x^b u(t)\,dt, \quad V_i(x) = \int_a^x v_i^{1-p_i'}(t)\,dt, \quad i = 1,2.$$

It is well known that the best constant C for the Hardy inequality

$$\left(\int_a^b \left(\int_a^x f\right)^q u(x)dx\right)^{\frac{1}{q}} \leqslant C\left(\int_a^b f^p v\right)^{\frac{1}{p}}, f \in \mathfrak{M}^+, \tag{1}$$

is equivalent to the Muckenhoupt constant in the case $1 < p \leqslant q < \infty$

$$A_M := \sup_{a<x<b} U^{\frac{1}{q}}(x) V^{\frac{1}{p'}}(x) < \infty$$

Mathematics subject classification (2010): 26D10, 46E35.

Keywords and phrases: Inequalities, the Hardy inequality, bilinear Hardy-type inequalities, scales of equivalent conditions.

* Corresponding author.

or to the Mazya-Rozin constant for $0<q<p<\infty, 1<p<\infty, \frac{1}{r}:=\frac{1}{q}-\frac{1}{p}$

$$B_{MR}:=\left(\int_a^b U^{\frac{r}{p}}(x)V^{\frac{r}{p'}}(x)u(x)dx\right)^{\frac{1}{r}}<\infty$$

and it can be replaced by different (equivalent) constants. For instance, the constants A_M and B_{MR} can be replaced by the following [4]:

$$A_T:=\sup_{a<x<b}\left(\int_a^x uV^q\right)^{\frac{1}{q}}V^{-\frac{1}{p}}(x)<\infty,$$

$$B_{PS}:=\left(\int_a^b\left(\int_a^x uV^q\right)^{\frac{r}{p}}V^{q-\frac{r}{p}}(x)u(x)dx\right)^{\frac{1}{r}}<\infty.$$

More generally, it was discovered (see, e.g., [1], [5]) that these conditions are not unique and can be replaced by the scales of constants depending on a continuous parameter $s>0$ of a form:

$$A_M(s):=\sup_{a<x<b}\left(\int_x^b uV^{q(\frac{1}{p'}-s)}\right)^{\frac{1}{q}}V^s(x)<\infty, \tag{2}$$

$$A_T(s):=\sup_{a<x<b}\left(\int_a^x uV^{q(\frac{1}{p'}+s)}\right)^{\frac{1}{q}}V^{-s}(x)<\infty, \tag{3}$$

$$B_{MR}(s):=\left(\int_a^b\left(\int_x^b uV^{q(\frac{1}{p'}-s)}\right)^{\frac{r}{q}}V^{rs-1}(x)dV(x)\right)^{\frac{1}{r}}<\infty, \tag{4}$$

$$B_{PS}(s):=\left(\int_a^b\left(\int_a^x uV^{q(\frac{1}{p'}+s)}\right)^{\frac{r}{q}}V^{-rs-1}(x)dV(x)\right)^{\frac{1}{r}}<\infty. \tag{5}$$

Moreover, it yields that (see, e.g., [1], [3], [5]) $C\approx A_M(s)\approx A_T(s)$ for $1<p\leqslant q<\infty$ and $C\approx B_{MR}(s)\approx B_{PS}(s)$ for $0<q<p<\infty$, $p>1$ and $s>0$. Concerning history and references about such alternative conditions, even infinite scales of equivalent conditions, we refer to the book [3], Section 7, and the references therein e.g. [1], [5] and the PhD theses [6] and [7].

Recently, M. Krepela [2] considered the following bilinear Hardy-type inequality

$$\left(\int_a^b\left(\int_a^x f\right)^q\left(\int_a^x g\right)^q u(x)dx\right)^{\frac{1}{q}}\leqslant C\left(\int_a^b f^{p_1}v_1\right)^{\frac{1}{p_1}}\left(\int_a^b g^{p_2}v_2\right)^{\frac{1}{p_2}}, f,g\in\mathfrak{M}^+ \tag{6}$$

and characterized it for various range of parameters p_1, p_2, q. The results can be summarized in the following modified theorem:

THEOREM A. Let $0<q<\infty$, $1<p_1,p_2<\infty$. Then the inequality (6) with the best constant C holds for every $f,g\in\mathfrak{M}^+$, iff the following holds:

(i) If $1 < \max(p_1, p_2) \leqslant q < \infty$*, then* $C \approx A_1$*, where*

$$A_1 := \sup_{a<x<b} U^{\frac{1}{q}}(x) V_1^{\frac{1}{p_1'}}(x) V_2^{\frac{1}{p_2'}}(x) < \infty.$$

(ii) If $1 < p_1 \leqslant q < p_2 < \infty, \frac{1}{r_2} := \frac{1}{q} - \frac{1}{p_2}$*, then* $C \approx A_2$*, where*

$$A_2 := \sup_{a<x<b} V_1^{\frac{1}{p_1'}}(x) \left(\int_x^b U^{\frac{r_2}{q}} V_2^{\frac{r_2}{q'}} dV_2 \right)^{\frac{1}{r_2}}.$$

(iii) If $1 < p_2 \leqslant q < p_1 < \infty, \frac{1}{r_1} := \frac{1}{q} - \frac{1}{p_1}$*, then* $C \approx A_3$*, where*

$$A_3 := \sup_{a<x<b} V_2^{\frac{1}{p_2'}}(x) \left(\int_x^b U^{\frac{r_1}{q}} V_1^{\frac{r_1}{q'}} dV_1 \right)^{\frac{1}{r_1}}.$$

(iv) Let $0 < q < \min(p_1, p_2) < \infty, \min(p_1, p_2) > 1, \frac{1}{r_i} := \frac{1}{q} - \frac{1}{p_i},\ i = 1,2$ *and* $\frac{1}{q} \leqslant \frac{1}{p_1} + \frac{1}{p_2}$*. Then* $C \approx A_4 + A_5$*, where*

$$A_4 := \sup_{a<x<b} V_1^{\frac{1}{p_1'}}(x) \left(\int_x^b U^{\frac{r_2}{q}} V_2^{\frac{r_2}{q'}} dV_2 \right)^{\frac{1}{r_2}},$$

and

$$A_5 := \sup_{a<x<b} V_2^{\frac{1}{p_2'}}(x) \left(\int_x^b U^{\frac{r_1}{q}} V_1^{\frac{r_1}{q'}} dV_1 \right)^{\frac{1}{r_1}}.$$

(v) Let $0 < q < \min(p_1, p_2) < \infty, \min(p_1, p_2) > 1, \frac{1}{r_i} := \frac{1}{q} - \frac{1}{p_i}, i = 1,2,\ \frac{1}{q} > \frac{1}{p_1} + \frac{1}{p_2}$ *and* $\frac{1}{\kappa} := \frac{1}{q} - \frac{1}{p_1} - \frac{1}{p_2}$*. Then* $C \approx A_6 + A_7$*, where*

$$A_6 := \left(\int_a^b \left(\int_x^b U^{\frac{r_2}{q}}(y) V_2^{\frac{r_2}{q'}}(y) dV_2(y) \right)^{\frac{\kappa}{r_2}} V_1^{\frac{\kappa}{r_2'}}(x) dV_1(x) \right)^{\frac{1}{\kappa}},$$

and

$$A_7 := \left(\int_a^b \left(\int_x^b U^{\frac{r_1}{q}}(y) V_1^{\frac{r_1}{q'}}(y) dV_1(y) \right)^{\frac{\kappa}{r_1}} V_2^{\frac{\kappa}{r_1'}}(x) dV_2(x) \right)^{\frac{1}{\kappa}}.$$

Inspired by the works in [1] and [5] (see also [3]), in this paper we derive infinite many equivalent conditions to characterize the inequality (6) for each of the cases (i)-(v). More exactly, we prove that each of the Krepela constants $A_1 - A_7$ can be replaced by infinite many other equivalent constants, even scales of constants.

2. The case $1 < \max(p_1, p_2) \leqslant q < \infty$

We begin this section with the following:

THEOREM 1. *Let* $1 < \max(p_1, p_2) \leqslant q < \infty$. *Then* $C \approx A_1^{(1)}(s_1, s_2) \approx A_1^{(2)}(s_1, s_2)$, $\forall s_1, s_2 > 0$, *where*

$$A_1^{(1)}(s_1, s_2) := \sup_{a<x<b} V_1^{s_1}(x) V_2^{s_2}(x) \left(\int_x^b u V_1^{q(\frac{1}{p_1'} - s_1)} V_2^{q(\frac{1}{p_2'} - s_2)} \right)^{\frac{1}{q}}$$

and

$$A_1^{(2)}(s_1, s_2) = \sup_{a<x<b} V_1^{-s_1}(x) V_2^{-s_2}(x) \left(\int_a^x u V_1^{q(\frac{1}{p_1'} + s_1)} V_2^{q(\frac{1}{p_2'} + s_2)} \right)^{\frac{1}{q}}.$$

In this Theorem and also all other Theorems and proofs in this paper we let C denote the sharp constant in (6).

Proof. We have

$$\begin{aligned} C &= \sup_g \sup_f \frac{\left(\int_a^b \left(\int_a^t f \right)^q \left(\int_a^t g \right)^q u(t) dt \right)^{\frac{1}{q}}}{\|f\|_{p_1, v_1} \|g\|_{p_2, v_2}} \\ &\overset{(2)}{\approx} \sup_g \|g\|_{p_2, v_2}^{-1} \sup_{a<x<b} \left(\int_x^b \left(\int_a^y g \right)^q u(y) V_1^{q(\frac{1}{p_1'} - s_1)}(y) dy \right)^{\frac{1}{q}} V_1^{s_1}(x) \\ &\overset{(2)}{\approx} \sup_{a<x<b} V_1^{s_1}(x) \left(\sup_{x<y<b} \left(\int_y^b u V_1^{q(\frac{1}{p_1'} - s_1)} V_2^{q(\frac{1}{p_2'} - s_2)} \right)^{\frac{1}{q}} V_2^{s_2}(y) \right) = A_1^{(1)}(s_1, s_2). \end{aligned}$$

Similarly, by using (3) twice, we find that $C \approx A_1^{(2)}(s_1, s_2)$ so the proof is complete. □

REMARK 1. The Krepela constant A_1 appears just as a point $(\frac{1}{p_1'}, \frac{1}{p_2'})$ on our first scale of equivalent constants, i.e. $A_1^{(1)}(\frac{1}{p_1'}, \frac{1}{p_2'}) = A_1$.

Theorem 1 gives two scales of equivalent conditions, namely $A_1^{(1)}(s_1, s_2) < \infty$ and $A_1^{(2)}(s_1, s_2) < \infty$, characterizing the inequality (6). Below, we prove that also these scales of equivalent conditions are not unique and can in fact be replaced by several more equivalent conditions for the inequality (6) to hold. Unlike in Theorem 1, here we do not link any of the conditions with the inequality (6). Rather it is provided that each of these conditions is equivalent to the condition $A_1 < \infty$.

THEOREM 2. *Let* $1 < \max(p_1, p_2) \leqslant q < \infty$. *Then* $A_1 \approx A_1^{(3)}(s) \approx A_1^{(4)}(s) \approx A_1^{(5)}(s) \approx A_1^{(6)}(s_1, s_2)$, $\forall s, s_1, s_2 > 0$, *where*

$$A_1 := \sup_{a<x<b} U^{1/q}(x) V_1^{1/p_1'}(x) V_2^{1/p_2'}(x),$$

$$A_1^{(3)}(s) := \sup_{a<x<b} \left(\int_x^b u(t) V_1^{\frac{1/p_1'}{1/q+s}}(t) V_2^{\frac{1/p_2'}{1/q+s}}(t)\, dt \right)^{1/q+s} U^{-s}(x),$$

$$A_1^{(4)}(s) := \sup_{a<x<b} \left(\int_a^x u(t) V_1^{\frac{1/p_1'}{1/q-s}}(t) V_2^{\frac{1/p_2'}{1/q-s}}(t)\, dt \right)^{1/q-s} U^{s}(x), \quad 1/q > s,$$

$$A_1^{(5)}(s) := \sup_{a<x<b} \left(\int_x^b u(t) V_1^{\frac{1/p_1'}{1/q-s}}(t) V_2^{\frac{1/p_2'}{1/q-s}}(t)\, dt \right)^{1/q-s} U^{s}(x), \quad 1/q < s$$

and

$$A_1^{(6)}(s_1,s_2) := \sup_{a<x<b} \left(\int_a^x v_1^{1-p_1'}(t) U^{\frac{1/2q}{1/p_1'+s_1}}(t)\, dt \right)^{1/p_1'+s_1} \left(\int_a^x v_2^{1-p_2'}(t) U^{\frac{1/2q}{1/p_2'+s_2}}(t)\, dt \right)^{1/p_2'+s_2} \times V_1^{-s_1}(x) V_2^{-s_2}(x).$$

Proof. Note that the function U is decreasing while V_1 and V_2 are increasing. First we prove that $A_1 \approx A_1^{(3)}(s)$. We have

$$\begin{aligned} U^{1/q}(x) V_1^{1/p_1'}(x) V_2^{1/p_2'}(x) &= U^{1/q+s}(x) V_1^{1/p_1'}(x) V_2^{1/p_2'}(x) U^{-s}(x) \\ &= \left(\int_x^b u(t)\, dt \right)^{1/q+s} V_1^{1/p_1'}(x) V_2^{1/p_2'}(x) U^{-s}(x) \\ &= \left(\int_x^b u(t) V_1^{\frac{1/p_1'}{1/q+s}}(x) V_2^{\frac{1/p_2'}{1/q+s}}(x)\, dt \right)^{1/q+s} U^{-s}(x) \\ &\leqslant \left(\int_x^b u(t) V_1^{\frac{1/p_1'}{1/q+s}}(t) V_2^{\frac{1/p_2'}{1/q+s}}(t)\, dt \right)^{1/q+s} U^{-s}(x). \end{aligned}$$

Hence,

$$\sup_{a<x<b} U^{1/q}(x) V_1^{1/p_1'}(x) V_2^{1/p_2'}(x) \leqslant \sup_{a<x<b} \left(\int_x^b u(t) V_1^{\frac{1/p_1'}{1/q+s}}(t) V_2^{\frac{1/p_2'}{1/q+s}}(t)\, dt \right)^{1/q+s} U^{-s}(x). \tag{7}$$

Moreover,

$$\begin{aligned} &\left(\int_x^b u(t) V_1^{\frac{1/p_1'}{1/q+s}}(t) V_2^{\frac{1/p_2'}{1/q+s}}(t)\, dt \right)^{1/q+s} U^{-s}(x) \\ &\leqslant \sup_{x<t<b} U^{1/q}(t) V_1^{1/p_1'}(t) V_2^{1/p_2'}(t) \left(\int_x^b u(t) U^{-\frac{1/q}{1/q+s}}(t)\, dt \right)^{1/q+s} U^{-s}(x) \\ &= \sup_{x<t<b} U^{1/q}(t) V_1^{1/p_1'}(t) V_2^{1/p_2'}(t) \left(\left(\frac{1/q+s}{s} \right) U^{\frac{s}{1/q+s}}(x) \right)^{1/q+s} U^{-s}(x) \end{aligned}$$

$$\leqslant \left(\frac{1/q+s}{s}\right)^{1/q+s} \sup_{a<t<b} U^{1/q}(t)V_1^{1/p_1'}(t)V_2^{1/p_2'}(t). \tag{8}$$

The equivalence $A_1 \approx A_1^{(3)}(s)$ now follows by taking supremum in (8) and using (7).

Next, we prove that $A_1 \approx A_1^{(4)}(s)$. Fix $x \in (a,b)$ and define $y \in (x,b)$ such that

$$\int_x^y u(t)\,dt = \int_y^b u(t)\,dt. \tag{9}$$

Then

$$\begin{aligned}
U^{1/q}(x) &= \left(\int_x^b u(t)\,dt\right)^{1/q} = \left(\int_x^y u(t)\,dt + \int_y^b u(t)\,dt\right)^{1/q} = 2^{1/q}\left(\int_y^b u(t)\,dt\right)^{1/q} \\
&= 2^{1/q}\left(\int_y^b u(t)\,dt\right)^{1/q-s}\left(\int_y^b u(t)\,dt\right)^{s} = 2^{1/q}\left(\int_x^y u(t)\,dt\right)^{1/q-s} U^s(y).
\end{aligned}$$

We have

$$\begin{aligned}
U^{1/q}(x)V_1^{1/p_1'}(x)V_2^{1/p_2'}(x) &= 2^{1/q}\left(\int_x^y u(t)\,dt\right)^{1/q-s} U^s(y)V_1^{1/p_1'}(x)V_2^{1/p_2'}(x) \\
&\leqslant 2^{1/q}\left(\int_x^y u(t)V_1^{\frac{1/p_1'}{1/q-s}}(t)V_2^{\frac{1/p_2'}{1/q-s}}(t)\,dt\right)^{1/q-s} U^s(y) \\
&\leqslant 2^{1/q}\left(\int_a^y u(t)V_1^{\frac{1/p_1'}{1/q-s}}(t)V_2^{\frac{1/p_2'}{1/q-s}}(t)\,dt\right)^{1/q-s} U^s(y).
\end{aligned}$$

Hence,

$$U^{1/q}(x)V_1^{1/p_1'}(x)V_2^{1/p_2'}(x) \leqslant 2^{1/q} \sup_{a<y<b}\left(\int_a^y u(t)V_1^{\frac{1/p_1'}{1/q-s}}(t)V_2^{\frac{1/p_2'}{1/q-s}}(t)\,dt\right)^{1/q-s} U^s(y), \tag{10}$$

and the estimate $A_1 < cA_1^{(4)}(s)$ follows.

Moreover, since $1/q-s>0$, we have

$$\begin{aligned}
&\left(\int_a^x u(t)V_1^{\frac{1/p_1'}{1/q-s}}(t)V_2^{\frac{1/p_2'}{1/q-s}}(t)\,dt\right)^{1/q-s} U^s(x) \\
&\leqslant \sup_{a<t<x} U^{1/q}(t)V_1^{1/p_1'}(t)V_2^{1/p_2'}(t)\left(\int_a^x u(t)U^{-\frac{1/q}{1/q-s}}(t)\,dt\right)^{1/q-s} U^s(x) \\
&\leqslant \sup_{a<t<b} U^{1/q}(t)V_1^{1/p_1'}(t)V_2^{1/p_2'}(t)\left(-\int_a^x U^{-\frac{1/q}{1/q-s}}(t)\,dU(t)\right)^{1/q-s} U^s(x)
\end{aligned}$$

$$= \sup_{a<t<b} U^{1/q}(t)V_1^{1/p_1'}(t)V_2^{1/p_2'}(t)\left(\frac{1/q-s}{s}\right)^{1/q-s}\left(U^{-\frac{s}{1/q-s}}(x)-U^{-\frac{s}{1/q-s}}(a)\right)^{1/q-s}U^s(x)$$

$$\leqslant \left(\frac{1/q-s}{s}\right)^{1/q-s}\sup_{a<t<b} U^{1/q}(t)V_1^{1/p_1'}(t)V_2^{1/p_2'}(t). \tag{11}$$

By now just taking supremum in (11) and using (10), we obtain the equivalence $A_1 \approx A_1^{(4)}(s)$.

Next, we prove that $A_1 \approx A_1^{(5)}(s)$. Since $1/q-s<0$, we get

$$\begin{aligned} U^{1/q}(x)V_1^{1/p_1'}(x)V_2^{1/p_2'}(x) &= U^{1/q-s}(x)U^s(x)V_1^{1/p_1'}(x)V_2^{1/p_2'}(x) \\ &= \left(\int_x^b u(t)\,dt\right)^{1/q-s}U^s(x)V_1^{1/p_1'}(x)V_2^{1/p_2'}(x) \\ &\leqslant \left(\int_x^b u(t)V_1^{\frac{1/p_1'}{1/q-s}}(t)V_2^{\frac{1/p_2'}{1/q-s}}(t)\,dt\right)^{1/q-s}U^s(x). \end{aligned}$$

Therefore

$$\sup_{a<x<b} U^{1/q}(x)V_1^{1/p_1'}(x)V_2^{1/p_2'}(x) \leqslant \tag{12}$$

$$\sup_{a<x<b}\left(\int_x^b u(t)V_1^{\frac{1/p_1'}{1/q-s}}(t)V_2^{\frac{1/p_2'}{1/q-s}}(t)\,dt\right)^{1/q-s}U^s(x). \tag{13}$$

Now fix $x\in(a,b)$ and define $y\in(x,b)$ such that

$$\int_x^y u(t)\,dt = \int_y^b u(t)\,dt.$$

Then $U(x)=2U(y)$ and using this fact, we obtain

$$\left(\int_x^b u(t)V_1^{\frac{1/p_1'}{1/q-s}}(t)V_2^{\frac{1/p_2'}{1/q-s}}(t)\,dt\right)^{1/q-s}U^s(x) \leqslant \left(\int_x^y u(t)V_1^{\frac{1/p_1'}{1/q-s}}(t)V_2^{\frac{1/p_2'}{1/q-s}}(t)\,dt\right)^{1/q-s}U^s(x)$$

$$\leqslant \left(\int_x^y u(t)\,dt\right)^{1/q-s}V_1^{1/p_1'}(y)V_2^{1/p_2'}(y)U^s(x) = \left(\int_y^b u(t)\,dt\right)^{1/q-s}V_1^{1/p_1'}(y)V_2^{1/p_2'}(y)U^s(x)$$

$$= U^{1/q-s}(y)V_1^{1/p_1'}(y)V_2^{1/p_2'}(y)2^sU^s(y) = 2^sU^{1/q}(y)V_1^{1/p_1'}(y)V_2^{1/p_2'}(y)$$

$$\leqslant 2^s\sup_{a<y<b} U^{1/q}(y)V_1^{1/p_1'}(y)V_2^{1/p_2'}(y). \tag{14}$$

We now take supremum in (14) and use (13) and the equivalence $A_1 \approx A_1^{(5)}(s)$ follows.

Finally, we prove that $A_1 \approx A_1^{(6)}(s_1,s_2)$. It yields that

$$U^{1/q}(x)V_1^{1/p_1'}(x)V_2^{1/p_2'}(x)$$

$$
\begin{aligned}
&= U^{1/q}(x)V_1^{1/p_1'+s_1}(x)V_2^{1/p_2'+s_2}(x)V_1^{-s_1}(x)V_2^{-s_2}(x)\\
&= U^{1/q}(x)\left(\int_a^x v_1^{1-p_1'}(t)\,dt\right)^{1/p_1'+s_1}\left(\int_a^x v_2^{1-p_2'}(t)\,dt\right)^{1/p_2'+s_2}V_1^{-s_1}(x)V_2^{-s_2}(x)\\
&\leqslant \left(\int_a^x v_1^{1-p_1'}(t)U^{\frac{1/2q}{1/p_1'+s_1}}(t)\,dt\right)^{1/p_1'+s_1}\left(\int_a^x v_2^{1-p_2'}(t)U^{\frac{1/2q}{1/p_2'+s_2}}(t)\,dt\right)^{1/p_2'+s_2}V_1^{-s_1}(x)V_2^{-s_2}(x).
\end{aligned} \tag{15}
$$

On the other hand

$$
\begin{aligned}
&\left(\int_a^x v_1^{1-p_1'}(t)U^{\frac{1/2q}{1/p_1'+s_1}}(t)\,dt\right)^{1/p_1'+s_1}\left(\int_a^x v_2^{1-p_2'}(t)U^{\frac{1/2q}{1/p_2'+s_2}}(t)\,dt\right)^{1/p_2'+s_2}V_1^{-s_1}(x)V_2^{-s_2}(x)\\
&= \left(\int_a^x U^{\frac{1/2q}{1/p_1'+s_1}}(t)\,dV_1(t)\right)^{1/p_1'+s_1}\left(\int_a^x U^{\frac{1/2q}{1/p_2'+s_2}}(t)\,dV_2(t)\right)^{1/p_2'+s_2}V_1^{-s_1}(x)V_2^{-s_2}(x)\\
&\leqslant \left(U^{\frac{1/2q}{1/p_1'+s_1}}(x)V_1(x)\right)^{1/p_1'+s_1}\left(U^{\frac{1/2q}{1/p_2'+s_2}}(x)V_2(x)\right)^{1/p_2'+s_2}V_1^{-s_1}(x)V_2^{-s_2}(x)\\
&= U^{1/2q}(x)V_1^{1/p_1'+s_1}(x)U^{1/2q}(x)V_2^{1/p_2'+s_2}(x)V_1^{-s_1}(x)V_2^{-s_2}(x)\\
&= U^{1/q}(x)V_1^{1/p_1'}(x)V_2^{1/p_2'}(x).
\end{aligned} \tag{16}
$$

The equivalence $A_1 \approx A_1^{(6)}(s_1,s_2)$ now follows by just combining (15) and (16). The proof is complete. □

3. The cases $1 < p_1 \leqslant q < p_2 < \infty$ and $1 < p_2 \leqslant q < p_1 < \infty$

We begin this section with the following:

THEOREM 3. *Let* $1 < p_1 \leqslant q < p_2 < \infty$ *and* $\frac{1}{r_2} := \frac{1}{q} - \frac{1}{p_2}$. *Then* $C \approx A_2^{(1)}(s_1,s_2) \approx A_2^{(2)}(s_1,s_2) \approx A_2^{(3)}(s_1,s_2) \approx A_2^{(4)}(s_1,s_2)$, $s_1,s_2 > 0$, *where*

$$
A_2^{(1)}(s_1,s_2) = \sup_{a<x<b} V_1^{s_1}(x)\left(\int_x^b\left(\int_y^b uV_1^{q(\frac{1}{p_1'}-s_1)}V_2^{q(\frac{1}{p_2'}-s_2)}\right)^{\frac{r_2}{q}}V_2^{r_2s_2-1}(y)dV_2(y)\right)^{\frac{1}{r_2}},
$$

$$
A_2^{(2)}(s_1,s_2) = \sup_{a<x<b} V_1^{-s_1}(x)\left(\int_a^x\left(\int_a^y uV_1^{q(\frac{1}{p_1'}+s_1)}V_2^{q(\frac{1}{p_2'}+s_2)}\right)^{\frac{r_2}{q}}V_2^{-r_2s_2-1}(y)dV_2(y)\right)^{\frac{1}{r_2}},
$$

$$
A_2^{(3)}(s_1,s_2) = \sup_{a<x<b} V_1^{s_1}(x)\left(\int_x^b\left(\int_x^y uV_1^{q(\frac{1}{p_1'}-s_1)}V_2^{q(\frac{1}{p_2'}+s_2)}\right)^{\frac{r_2}{q}}V_2^{-r_2s_2-1}(y)dV_2(y)\right)^{\frac{1}{r_2}}
$$

and

$$A_2^{(4)}(s_1,s_2) = \sup_{a<x<b} V_1^{-s_1}(x)\left(\int_a^x \left(\int_y^x uV_1^{q(\frac{1}{p_1'}+s_1)} V_2^{q(\frac{1}{p_2'}-s_2)}\right)^{\frac{r_2}{q}} V_2^{r_2 s_2-1}(y)dV_2(y)\right)^{\frac{1}{r_2}}.$$

Proof. We have

$$C = \sup_g \sup_f \frac{\left(\int_a^b \left(\int_a^t f\right)^q \left(\int_a^t g\right)^q u(t)dt\right)^{\frac{1}{q}}}{\|f\|_{p_1,v_1}\|g\|_{p_2,v_2}}$$

$$\overset{(2)}{\approx} \sup_g \|g\|_{p_2,v_2}^{-1} \sup_{a<x<b}\left(\int_x^b \left(\int_a^y g\right)^q u(y)V_1^{q(\frac{1}{p_1'}-s_1)}(y)dy\right)^{\frac{1}{q}} V_1^{s_1}(x)$$

$$\overset{(4)}{\approx} \sup_{a<x<b} V_1^{s_1}(x)\left(\int_x^b \left(\int_y^b uV_1^{q(\frac{1}{p_1'}-s_1)} V_2^{q(\frac{1}{p_2'}-s_2)}\right)^{\frac{r_2}{q}} V_2^{r_2 s_2-1}(y)dV_2(y)\right)^{\frac{1}{r_2}}.$$

The first relation follows by the formula (2), the second relation is due to formula (4). By now using (3) and (5) we have $C \approx A_2^{(2)}(s_1,s_2)$. Moreover, using (2) and (5) we find that $C \approx A_2^{(3)}(s_1,s_2)$. Similarly, by instead using (3) and (4) we conclude that $C \approx A_2^{(4)}(s_1,s_2)$. The proof is complete. □

The case $1 < p_2 \leqslant q < p_1 < \infty$ can be handled similarly to obtain the following result:

THEOREM 4. *Let* $1 < p_2 \leqslant q < p_1 < \infty$ *and* $\frac{1}{r_1} := \frac{1}{q} - \frac{1}{p_1}$. *Then* $C \approx A_3^{(1)}(s_1,s_2) \approx A_3^{(2)}(s_1,s_2) \approx A_3^{(3)}(s_1,s_2) \approx A_3^{(4)}(s_1,s_2)$, $\forall s_1,s_2 > 0$, *where*

$$A_3^{(1)}(s_1,s_2) = \sup_{a<x<b} V_2^{s_2}(x)\left(\int_x^b \left(\int_y^b uV_1^{q(\frac{1}{p_1'}-s_1)} V_2^{q(\frac{1}{p_2'}-s_2)}\right)^{\frac{r_1}{q}} V_1^{r_1 s_1-1}(y)dV_1(y)\right)^{\frac{1}{r_1}},$$

$$A_3^{(2)}(s_1,s_2) = \sup_{a<x<b} V_2^{-s_2}(x)\left(\int_a^x \left(\int_a^y uV_1^{q(\frac{1}{p_1'}+s_1)} V_2^{q(\frac{1}{p_2'}+s_2)}\right)^{\frac{r_1}{q}} V_1^{-r_1 s_1-1}(y)dV_1(y)\right)^{\frac{1}{r_1}},$$

$$A_3^{(3)}(s_1,s_2) = \sup_{a<x<b} V_2^{s_2}(x)\left(\int_x^b \left(\int_x^y uV_1^{q(\frac{1}{p_1'}+s_1)} V_2^{q(\frac{1}{p_2'}-s_2)}\right)^{\frac{r_1}{q}} V_1^{-r_1 s_1-1}(y)dV_1(y)\right)^{\frac{1}{r_1}}$$

and

$$A_3^{(4)}(s_1,s_2) = \sup_{a<x<b} V_2^{-s_2}(x)\left(\int_a^x \left(\int_y^x uV_1^{q(\frac{1}{p_1'}-s_1)} V_2^{q(\frac{1}{p_2'}+s_2)}\right)^{\frac{r_1}{q}} V_1^{r_1 s_1-1}(y)dV_1(y)\right)^{\frac{1}{r_1}}.$$

REMARK 2. The Krepela constants A_2 and A_3 show up just as $A_2 = A_2^{(1)}(\frac{1}{p_1'}, \frac{1}{p_2'})$ and $A_3 = A_3^{(1)}(\frac{1}{p_1'}, \frac{1}{p_2'})$.

4. The case $0 < q < \min(p_1, p_2) < \infty$

We begin this section with the following:

THEOREM 5. *Let* $0 < q < \min(p_1, p_2) < \infty$, $\min(p_1, p_2) > 1$, $\frac{1}{q} \leqslant \frac{1}{p_1} + \frac{1}{p_2}$ *and* $\frac{1}{r_i} := \frac{1}{q} - \frac{1}{p_i}$, $i = 1, 2$. *Then* $C \approx A_4^{(1)}(s_1, s_2) + A_4^{(2)}(s_1, s_2) \approx A_4^{(3)}(s_1, s_2) + A_4^{(4)}(s_1, s_2) \approx A_4^{(5)}(s_1, s_2) + A_4^{(6)}(s_1, s_2) \approx A_4^{(7)}(s_1, s_2) + A_4^{(8)}(s_1, s_2)$, *where*

$$A_4^{(1)}(s_1, s_2) := \sup_{a<x<b} V_1^{s_1}(x) \left(\int_x^b \left(\int_y^b u V_1^{q(\frac{1}{p_1'} - s_1)} V_2^{q(\frac{1}{p_2'} - s_2)} \right)^{\frac{r_2}{q}} V_2^{r_2 s_2 - 1}(y) dV_2(y) \right)^{\frac{1}{r_2}},$$

$$A_4^{(2)}(s_1, s_2) := \sup_{a<x<b} V_2^{s_2}(x) \left(\int_x^b \left(\int_y^b u V_1^{q(\frac{1}{p_1'} - s_1)} V_2^{q(\frac{1}{p_2'} - s_2)} \right)^{\frac{r_1}{q}} V_1^{r_1 s_1 - 1}(y) dV_1(y) \right)^{\frac{1}{r_1}},$$

$$A_4^{(3)}(s_1, s_2) := \sup_{a<x<b} V_1^{-s_1}(x) \left(\int_a^x \left(\int_a^y u V_1^{q(\frac{1}{p_1'} + s_1)} V_2^{q(\frac{1}{p_2'} + s_2)} \right)^{\frac{r_2}{q}} V_2^{-r_2 s_2 - 1}(y) dV_2(y) \right)^{\frac{1}{r_2}},$$

$$A_4^{(4)}(s_1, s_2) := \sup_{a<x<b} V_2^{-s_2}(x) \left(\int_a^x \left(\int_a^y u V_1^{q(\frac{1}{p_1'} + s_1)} V_2^{q(\frac{1}{p_2'} + s_2)} \right)^{\frac{r_1}{q}} V_1^{-r_1 s_1 - 1}(y) dV_1(y) \right)^{\frac{1}{r_1}},$$

$$A_4^{(5)}(s_1, s_2) := \sup_{a<x<b} V_1^{s_1}(x) \left(\int_x^b \left(\int_x^y u V_1^{q(\frac{1}{p_1'} - s_1)} V_2^{q(\frac{1}{p_2'} + s_2)} \right)^{\frac{r_2}{q}} V_2^{-r_2 s_2 - 1}(y) dV_2(y) \right)^{\frac{1}{r_2}},$$

$$A_4^{(6)}(s_1, s_2) := \sup_{a<x<b} V_2^{-s_2}(x) \left(\int_a^x \left(\int_y^x u V_1^{q(\frac{1}{p_1'} - s_1)} V_2^{q(\frac{1}{p_2'} + s_2)} \right)^{\frac{r_1}{q}} V_1^{r_1 s_1 - 1}(y) dV_1(y) \right)^{\frac{1}{r_1}},$$

$$A_4^{(7)}(s_1, s_2) := \sup_{a<x<b} V_1^{-s_1}(x) \left(\int_a^x \left(\int_y^x u V_1^{q(\frac{1}{p_1'} + s_1)} V_2^{q(\frac{1}{p_2'} - s_2)} \right)^{\frac{r_2}{q}} V_2^{r_2 s_2 - 1}(y) dV_2(y) \right)^{\frac{1}{r_2}}$$

and

$$A_4^{(8)}(s_1, s_2) := \sup_{a<x<b} V_2^{s_2}(x) \left(\int_x^b \left(\int_x^y u V_1^{q(\frac{1}{p_1'} + s_1)} V_2^{q(\frac{1}{p_2'} - s_2)} \right)^{\frac{r_1}{q}} V_1^{-r_1 s_1 - 1}(y) dV_1(y) \right)^{\frac{1}{r_1}}.$$

Proof. We have

$$
\begin{aligned}
C &= \sup_g \sup_f \frac{\left(\int_a^b \left(\int_a^t f\right)^q \left(\int_a^t g\right)^q u(t)dt\right)^{\frac{1}{q}}}{\|f\|_{p_1,v_1}\|g\|_{p_2,v_2}} \\
&\overset{(4)}{\approx} \sup_g \|g\|_{p_2,v_2}^{-1}\left[\left(\int_a^b \left(\int_x^b \left(\int_a^y g\right)^q u(y)V_1^{q(\frac{1}{p_1'}-s_1)}(y)dy\right)^{\frac{r_1}{q}} V_1^{r_1s_1-1}(x)dV_1(x)\right)^{\frac{q}{r_1}}\right]^{\frac{1}{q}} \\
&= \sup_g \|g\|_{p_2,v_2}^{-1}\left[\sup_h \frac{\int_a^b \left(\int_a^x g\right)^q \left(\int_a^x h\right) u(x)V_1^{q(\frac{1}{p_1'}-s_1)}(x)dx}{\left(\int_a^b h^{\frac{p_1}{q}} V_1^{(r_1s_1-1)(-\frac{p_1}{r_1})} v_1^{\frac{p_1'}{r_1}}\right)^{\frac{q}{p_1}}}\right]^{\frac{1}{q}} \\
&\overset{(4)}{\approx} \sup_h \frac{\left(\int_a^b \left(\int_x^b \left(\int_a^y h\right) u(y)V_1^{q(\frac{1}{p_1'}-s_1)}(y)V_2^{q(\frac{1}{p_2'}-s_2)}(y)dy\right)^{\frac{r_2}{q}} V_2^{r_2s_2-1}(x)dV_2(x)\right)^{\frac{1}{r_2}}}{\left(\int_a^b h^{\frac{p_1}{q}} V_1^{(r_1s_1-1)(-\frac{p_1}{r_1})} v_1^{\frac{p_1'}{r_1}}\right)^{\frac{1}{p_1}}} \\
&\approx \sup_h \frac{\left(\int_a^b \left(\int_a^x h\right)^{\frac{r_2}{q}} \left(\int_x^b uV_1^{q(\frac{1}{p_1'}-s_1)}V_2^{q(\frac{1}{p_2'}-s_2)}\right)^{\frac{r_2}{q}} V_2^{r_2s_2-1}(x)dV_2(x)\right)^{\frac{1}{r_2}}}{\left(\int_a^b h^{\frac{p_1}{q}} V_1^{(r_1s_1-1)(-\frac{p_1}{r_1})} v_1^{\frac{p_1'}{r_1}}\right)^{\frac{1}{p_1}}} \\
&\quad + \sup_h \frac{\left(\int_a^b \left(\int_x^b h(y)\left(\int_y^b uV_1^{q(\frac{1}{p_1'}-s_1)}V_2^{q(\frac{1}{p_2'}-s_2)}\right)dy\right)^{\frac{r_2}{q}} V_2^{r_2s_2-1}(x)dV_2(x)\right)^{\frac{1}{r_2}}}{\left(\int_a^b h^{\frac{p_1}{q}} V_1^{(r_1s_1-1)(-\frac{p_1}{r_1})} v_1^{\frac{p_1'}{r_1}}\right)^{\frac{1}{p_1}}} \\
&=: H_1 + H_2.
\end{aligned}
\tag{17}
$$

The first relation follows by the formula (4), the second relation is due to duality. For the third relation we use again formula (4), the fourth relation holds by the Fubini theorem. Finally, we have Hardy's constants with $\tilde{p} = \frac{p_1}{q}, \tilde{q} = \frac{r_2}{q}$, and for the case $\tilde{p} \leqslant \tilde{q}$, we obtain that $H_1 \approx A_4^{(1)}(s_1,s_2)$ and $H_2 \approx A_4^{(2)}(s_1,s_2)$. The proof follows by just using these facts. $\square$

REMARK 3. We note that $A_4^{(1)}(\frac{1}{p_1'},\frac{1}{p_2'}) = A_4$ and $A_4^{(2)}(\frac{1}{p_1'},\frac{1}{p_2'}) = A_5$, where A_4 and A_5 are the Krepela constants defined in Theorem A (iv).

Our final results reads:

THEOREM 6. *Let* $0 < q < \min(p_1,p_2) < \infty$, $\min(p_1,p_2) > 1$, $\frac{1}{q} > \frac{1}{p_1} + \frac{1}{p_2}$ *and* $\frac{1}{r_i} := \frac{1}{q} - \frac{1}{p_i}$, $i = 1,2$. *Let* $\frac{1}{\kappa} := \frac{1}{q} - \frac{1}{p_1} - \frac{1}{p_2}$. *Then* $C \approx A_5^{(1)}(s_1,s_2) + A_5^{(2)}(s_1,s_2) \approx A_5^{(3)}(s_1,s_2) + A_5^{(4)}(s_1,s_2) \approx A_5^{(5)}(s_1,s_2) + A_5^{(6)}(s_1,s_2) \approx A_5^{(7)}(s_1,s_2) + A_5^{(8)}(s_1,s_2)$, *where*

$$(A_5^{(1)}(s_1,s_2))^\kappa := \int_a^b \left(\int_t^b \left(\int_y^b u V_1^{q(\frac{1}{p_1'}-s_1)} V_2^{q(\frac{1}{p_2'}-s_2)} \right)^{\frac{r_2}{q}} V_2^{r_2 s_2 - 1}(y) dV_2(y) \right)^{\frac{\kappa}{r_2}} \times V_1^{\kappa s_1 - 1}(t) dV_1(t),$$

$$(A_5^{(2)}(s_1,s_2))^\kappa := \int_a^b \left(\int_t^b \left(\int_y^b u V_1^{q(\frac{1}{p_1'}-s_1)} V_2^{q(\frac{1}{p_2'}-s_2)} \right)^{\frac{r_1}{q}} V_1^{r_1 s_1 - 1}(y) dV_1(y) \right)^{\frac{\kappa}{r_1}} \times V_2(t)^{\kappa s_2 - 1} dV_2(t),$$

$$(A_5^{(3)}(s_1,s_2))^\kappa := \int_a^b \left(\int_a^t \left(\int_a^y u V_1^{q(\frac{1}{p_1'}+s_1)} V_2^{q(\frac{1}{p_2'}+s_2)} \right)^{\frac{r_2}{q}} V_2^{-r_2 s_2 - 1}(y) dV_2(y) \right)^{\frac{\kappa}{r_2}} \times d\left(-\left(\int_t^b V_1^{-r_1 s_1 - 1} dV_1 \right)^{\frac{\kappa}{r_1}} \right),$$

$$(A_5^{(4)}(s_1,s_2))^\kappa := \int_a^b \left(\int_a^t \left(\int_a^y u V_1^{q(\frac{1}{p_1'}+s_1)} V_2^{q(\frac{1}{p_2'}+s_2)} \right)^{\frac{r_1}{q}} V_1^{-r_1 s_1 - 1}(y) dV_1(y) \right)^{\frac{\kappa}{r_1}} \times d\left(-\left(\int_t^b V_2^{-r_2 s_2 - 1} dV_2 \right)^{\frac{\kappa}{r_2}} \right),$$

$$(A_5^{(5)}(s_1,s_2))^\kappa := \int_a^b \left(\int_t^b \left(\int_t^y u V_1^{q(\frac{1}{p_1'}-s_1)} V_2^{q(\frac{1}{p_2'}+s_2)} \right)^{\frac{r_2}{q}} V_2^{-r_2 s_2 - 1}(y) dV_2(y) \right)^{\frac{\kappa}{r_2}} \times d\left(\left(\int_a^t V_1^{r_1 s_1 - 1} dV_1 \right)^{\frac{\kappa}{r_1}} \right),$$

$$(A_5^{(6)}(s_1,s_2))^\kappa := \int_a^b \left(\int_a^t \left(\int_y^t u V_1^{q(\frac{1}{p_1'}-s_1)} V_2^{q(\frac{1}{p_2'}+s_2)} \right)^{\frac{r_1}{q}} V_1^{r_1 s_1 - 1}(y) dV_1(y) \right)^{\frac{\kappa}{r_1}} \times d\left(-\left(\int_t^b V_2^{-r_2 s_2 - 1} dV_2 \right)^{\frac{\kappa}{r_2}} \right),$$

$$(A_5^{(7)}(s_1,s_2))^\kappa := \int_a^b \left(\int_a^t \left(\int_y^x u V_1^{q(\frac{1}{p_1'}+s_1)} V_2^{q(\frac{1}{p_2'}-s_2)} \right)^{\frac{r_2}{q}} V_2^{r_2 s_2 - 1}(y) dV_2(y) \right)^{\frac{\kappa}{r_2}} \times d\left(-\left(\int_t^b V_1^{-r_1 s_1 - 1} dV_1 \right)^{\frac{\kappa}{r_1}} \right)$$

and

$$(A_5^{(8)}(s_1,s_2))^\kappa := \int_a^b \left(\int_t^b \left(\int_t^y u V_1^{q(\frac{1}{p_1'}+s_1)} V_2^{q(\frac{1}{p_2'}-s_2)} \right)^{\frac{r_1}{q}} V_1^{-r_1 s_1 - 1}(y) dV_1(y) \right)^{\frac{\kappa}{r_1}} \times d\left(\left(\int_a^t V_2^{r_2 s_2 - 1} dV_2 \right)^{\frac{\kappa}{r_2}} \right).$$

Proof. The proof is absolutely analogous to that of Theorem 5 so we do not give the details. We only note as in this proof we have $C \approx H_1 + H_2$, where H_1 and H_2 are defined in (17). Next for the case $\tilde{q} < \tilde{p}$ the following holds $H_1 \approx A_5^{(1)}(s_1,s_2)$ and $H_2 \approx A_5^{(2)}(s_1,s_2)$. $\square$

REMARK 4. In this case the corresponding Krepela constants A_6 and A_7 are just recovered as $A_6 = A_5^{(1)}(\frac{1}{p_1'}, \frac{1}{p_2'})$ and $A_7 = A_5^{(2)}(\frac{1}{p_1'}, \frac{1}{p_2'})$.

REMARK 5. There are not many results concerning bilinear Hardy type inequalities then that by M. Krepela presented in Theorem A. However, some other results of this type can be found in the recent PhD thesis [8].

Acknowledgement. The first author acknowledges the support of the Science Foundation of India, project no. DST/INT/RUS/RSF/P-01. The work of the third author was partially supported by the Russian Fund for Basic Researches (project 19-01-00223).

REFERENCES

[1] A. GOGATISHVILI, A. KUFNER, L.-E. PERSSON AND A. WEDESTIG, *An equivalence theorem for integral conditions related to Hardy's inequality*, Real Analysis Exchange, **29**, 2 (2003), 1–14.

[2] M. KREPELA, *Iterating bilinear Hardy inequalities*, Proc. Edinb. Math. Soc. **60**, 4 (2017), 955–971.

[3] A. KUFNER, L.-E. PERSSON AND N. SAMKO, *Weighted Inequalities of Hardy Type*, Second Edition, World Scientific New Jersey, 2017.

[4] L.-E. PERSSON AND V. D. STEPANOV, *Weighted integral inequalities with the geometric mean operator*, J. Inequal. Appl. **7**, (2002), 727–746.

[5] L.-E. PERSSON, V. D. STEPANOV AND P. WALL, *Some scales of equivalent weight characterizations of Hardy's inequality: the case* $q < p$, Math. Inequal. Appl. **10**, 2 (2007), 267–279.

[6] A. WEDESTIG, *Weighted Inequalities of Hardy Type and their Limiting Inequalities*, Doctoral Thesis, Department of Mathematics, Luleå, University of Technology, Sweden, 2003.

[7] E. P. USHAKOVA, *Norm Inequalities of Hardy and Pólya-Knopp Types*, Doctoral Thesis, Department of Mathematics, Luleå, University of Technology, Sweden, 2006.

[8] G. E. SHAMBILOVA, *Some New Hardy-Type Inequalities on the Cone of Monotone Functions*, Doctoral Thesis, Department of Mathematics, Luleå, University of Technology, Sweden, 2018.

(Received September 11, 2018)

Saikat Kanjilal
Department of Mathematics
South Asian University
New Delhi, India
e-mail: saikatkanjilal@students.sau.ac.in

Lars-Erik Persson
UIT
The Arctic University of Norway
Narvik, Norway
Department of Mathematics and Computer Science
Karlstad University
65188, Karlstad, Sweden
e-mail: larserik6pers@gmail.com

Guldarya E. Shambilova
Moscow Institute of Physics and Technology
141701 Dolgoprudnyi, Moscow Region, Russia
e-mail: shambilova@mail.ru