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FULL RANGE BOUNDEDNESS OF BILINEAR HILBERT TRANSFORM ALONG CERTAIN POLYNOMIALS

DONG DONG

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Abstract. Let P and Q be two polynomials without constant term. Assume that the operator $B_{P,Q}(f,g)(x) = \int f(x-P(t))g(x-Q(t))\frac{dt}{t}$ is bounded from $L^{p_1} \times L^{p_2}$ into L^r, $p_1, p_2 \in (1,\infty)$, $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r}$. It is proved that if $P'(t) > 0$ for all $t \neq 0$, then $r \geqslant \frac{d}{d+1}$. Here d is the correlation degree of P and Q which is defined as the largest multiplicity of non-zero real roots of $P' - Q'$.

1. Background and main result

Let $P, Q : \mathbb{R} \to \mathbb{R}$ be two polynomials. The bilinear Hilbert transform along P and Q is defined by

$$B_{P,Q}(f,g)(x) = \int f(x-P(t))g(x-Q(t))\frac{dt}{t}, \; f,g \in \mathscr{S}(\mathbb{R}), \tag{1}$$

where $\mathscr{S}(\mathbb{R}^n), n \in \mathbb{N}$, denotes the Schwartz space on $\mathbb{R}^n$. Such an operator is a natural bilinear analogue of the linear singular Radon transform (a.k.a. Hilbert transform along curves) of the type

$$H_{P,Q}(f)(x_1,x_2) = \int f(x_1 - P(t), x_2 - Q(t))\frac{dt}{t}, \; f \in \mathscr{S}(\mathbb{R}^2),$$

which has been studied intensively since 1960s (see, for example, the work of Stein-Wainger [16], Christ et.al. [3] and the references therein). The study of $B_{P,Q}$ is still relatively new. The fundamental question is for what values p_1, p_2, r with $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r}$ is $B_{P,Q}$ bounded from $L^{p_1} \times L^{p_2}$ into L^r. To this end, we can without loss of generality assume that P and Q contain no constant term by a translation argument. In the rest of this article, we will always assume that p_1, p_2 lie in $(1,\infty)$ and r is determined by $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2}$.

Although for many pairs of polynomials P and Q the boundedness of $B_{P,Q}$ has been obtained for a large range of exponents p_1, p_2, r, the full range of the exponents remain unknown in some cases. For example, when $P(t) = t$ and $Q(t) = \alpha t$, $\alpha \neq 0, 1$,

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$B_{P,Q}$ is the famous bilinear Hilbert transform, which is proved by Lacey and Thiele [8, 9] to be bounded into L^r for $r > \frac{2}{3}$ (also see [6, 10, 17] for some uniform-in-α estimates). The lower bound $\frac{2}{3}$ of r is necessary in Lacey-Thiele's proof, but it may not be necessary for the boundedness of the bilinear Hilbert transform. Whether $r > \frac{2}{3}$ can be dropped in Lacey-Thiele's theorem is still an open problem; see [1, 2, 4] for some progress on this issue.

X. Li [11, 12] is the first to consider nonlinear polynomials in $B_{P,Q}$. He proved the $L^2 \times L^2 \to L^1$-boundedness of $B_{P,Q}$ when $P(t) = t$ and $Q(t) = t^d$ ($d \geqslant 2$ is an integer). Later on, Li and Xiao [13] established the full-range (up to the endpoint) boundedness of $B_{P,Q}$ when $P(t) = t$ and Q a polynomial without linear term (see also [7, 14, 15] for the case $P(t) = t$ and Q a smooth non-flat curve). Recently, the author [5] extended Li-Xiao's result to allow both P and Q to be nonlinear in $B_{P,Q}$. He proved the following theorem:

THEOREM 1. ([5]) *Given two polynomials P and Q without constant term, we can always write them as*

$$P(t) = a_{d_1}t^{d_1} + a_{d_1-1}t^{d_1-1} + \ldots + a_{e_1}t^{e_1}, 1 \leqslant e_1 \leqslant d_1, a_{d_1}a_{e_1} \neq 0 \tag{2}$$

$$Q(t) = b_{d_2}t^{d_2} + b_{d_2-1}t^{d_2-1} + \ldots + b_{e_2}t^{e_2}, 1 \leqslant e_2 \leqslant d_2, b_{d_2}b_{e_2} \neq 0. \tag{3}$$

Assume $d_1 \neq d_2$ and $e_1 \neq e_2$. Then there is a constant $C_{P,Q}$ depending on P and Q (and of course p,q,r) such that $B_{P,Q}(f,g)(x) = \int f(x-P(t))g(x-Q(t))\frac{dt}{t}$ is bounded from $L^{p_1} \times L^{p_2}$ into L^r, whenever $p_1, p_2 \in (1,\infty)$, $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2}$, and $r > \frac{d}{d+1}$, where d is the correlation degree of P and Q.

Here the *correlation degree* of P and Q is defined as the largest multiplicity of non-zero real roots of $P' - Q'$.

The author also observed in [5] that the lower bound $\frac{d}{d+1}$ for r may not be the best possible for some pairs of polynomials. For instance, when $P(t) = t^6$ and $Q(t) = 3t^4 - 3t^2$, $B_{P,Q}$ is the zero operator (as both P and Q are even), which is trivially bounded into L^r for any $r > \frac{1}{2}$. However, the correlation degree of P and Q is 2, which gives a lower bound $\frac{2}{3}$ by Theorem 1. It is then natural to ask

OPEN PROBLEM 2. For which pairs of polynomials P and Q is the condition $r > \frac{d}{d+1}$ necessary to the L^r-boundedness of $B_{P,Q}$?

One can further ask:

OPEN PROBLEM 3. Given polynomials P and Q, what is the minimal value of r that guarantees the L^r-boundedness of $B_{P,Q}$?

The purpose of this note is to give partial answers to these questions. We show that for many pairs of polynomials, $r > \frac{d}{d+1}$ is indeed the largest (up to the endpoint) range for the L^r-boundedness of $B_{P,Q}$. More precisely, THEOREM 4. *Let P and Q be two polynomials without constant term. Assume that the operator $B_{P,Q}(f,g)(x)=\int f(x-P(t))g(x-Q(t))\frac{dt}{t}$ is bounded from $L^{p_1}\times L^{p_2}$ into L^r, $p_1,p_2\in(1,\infty)$, $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{r}$. If $P'(t)>0$ for all $t\neq 0$, then $r\geqslant\frac{d}{d+1}$, where d is the correlation degree of P and Q.*

REMARKS.

(1). By similar arguments, the criterion in the above theorem also includes the case $P'(t)<0$ for all $t\neq 0$. By symmetry, the theorem also holds if the condition is imposed on the polynomial Q.

(2). Theorem 4 itself answers partially Open Problem 2. Together with Theorem 1, it also answers Open Problem 3 for a large range of pairs of polynomials.

(3). We believe that the criterion given in Theorem 4 could be weakened. The weakest condition is conjectured as follows:

CONJECTURE 5. Let P and Q be two polynomials without constant term. Assume that the operator $B_{P,Q}(f,g)(x)=\int f(x-P(t))g(x-Q(t))\frac{dt}{t}$ is bounded from $L^{p_1}\times L^{p_2}$ into L^r for $p_1,p_2\in(1,\infty)$ and $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{r}$. Then $r\geqslant\frac{d}{d+1}$, where d is the correlation degree of P and Q, as long as not both P and Q are even.

The criterion in Theorem 4 essentially says that P is strictly monotonic and thus graphically P is similar to an odd function. How to build the bridge between "odd" (in Theorem 4) and "not even" (in Conjecture 5) seems to be a very difficult problem. It is also possible that Conjecture 5 is in fact false, and Theorem 4 could be the best answer to Open Problem 2.

2. Proof of the Theorem

In our proof we will use C to denote a positive large constant whose value may change from line to line. Such constant may depend on the polynomials P and Q. $A\lesssim B$ is short for $A\leqslant CB$ and $A\ll B$ means $CA\leqslant B$ for some large C. χ_E will be used to denote the indicator function of the set E. We will follow a strategy developed in [13].

Given two polynomials P and Q without constant term, assume the correlation degree of P and Q is d. Let t_0 be a non-zero real root of $P'-Q'$ with multiplicity d. We may assume $t_0>0$, as the other case can be handled in a similar way. The assumption $P'(t)>0$ for all non-zero t implies that P is strictly increasing. Therefore, we have $P(t_0)>0$ as $P(0)=0$. Let $0<\delta\ll P(t_0)$ be small. We will use the following special choice of f and g:

$$\begin{cases} f=\chi_{[-\delta,\delta]}, \\ g=\chi_{[P(t_0)-Q(t_0)-\delta,P(t_0)-Q(t_0)+\delta]}. \end{cases} \tag{1}$$

By the boundedness of $B_{P,Q}$ and straightforward calculations of the norms of f and g, we have

$$\|B_{P,Q}(f,g)\|_r\lesssim\|f\|_{p_1}\|g\|_{p_2}\lesssim\delta^{\frac{1}{r}} \tag{2}$$

In what follows, we aim to get a lower bound of $\|B_{P,Q}(f,g)\|_r$ in terms of powers of δ. Recall the expression of $B_{P,Q}$

$$B_{P,Q}(f,g)(x) = \int f(x-P(t))g(x-Q(t))\frac{dt}{t}.$$

We will achieve our goal by properly restricting x and t in the above definition of $B_{P,Q}(f,g)(x)$. Consider the interval

$$I = \left[P(t_0) - \frac{\delta^{\frac{1}{d+1}}}{A}, P(t_0) + \frac{\delta^{\frac{1}{d+1}}}{A}\right],$$

where A is a large constant to be determined later. We will only consider those $x \in I$ when calculating $\|B_{P,Q}(f,g)\|_r$. Clearly $x > 0$ when A is large. Also note that we can assume $t > 0$, otherwise we would have $P(t) < 0$ and

$$|x - P(t)| > x \geqslant P(t_0) - \frac{\delta^{\frac{1}{d+1}}}{A} > \delta,$$

which implies $f(x-P(t)) = 0$ and $B_{P,Q}(f,g)(x) = 0$.

Since f,g are non-negative and t is positive, we can further restrict t in order to get a lower bound $B_{P,Q}(f,g)(x)$. For any $x \in I$, define

$$J_x = \left\{t > 0 : |P(t) - x| < \frac{\delta}{2}\right\}.$$

By the definition of f (1), $f(x-P(t)) = 1$ when $t \in J_x$. We claim that the same holds for g:

CLAIM 6. $g(x-Q(t)) = 1$ whenever $x \in I$ and $t \in J_x$.

Proof. Fix $x \in I$, and let $t \in J_x$. By the definitions of I and J_x,

$$|P(t) - P(t_0)| \leqslant |P(t) - x| + |P(t_0) - x| \leqslant \frac{\delta}{2} + \frac{\delta^{\frac{1}{d+1}}}{A} \lesssim \frac{\delta^{\frac{1}{d+1}}}{A} \quad (3)$$

for small δ. Invoke mean value theorem,

$$|t - t_0| = |P^{-1}(P(t)) - P^{-1}(P(t_0))| = |(P^{-1})'(\xi)||P(t) - P(t_0)| \quad (4)$$

for some $\xi \in \mathbb{R}$. By inverse function theorem, $|(P^{-1})'(\xi)|$ can never be ∞ as P' is never 0. Therefore, (3) and (4) give that

$$|t - t_0| \lesssim |P(t) - P(t_0)| \lesssim \frac{\delta^{\frac{1}{d+1}}}{A}, \quad (5)$$

Now using the assumption that t_0 is a root of $P' - Q'$ with multiplicity d, we see that

$$|x - Q(t) - (P(t_0) - Q(t_0))| \leqslant |x - P(t)| + |P(t) - Q(t) - (P(t_0) - Q(t_0))| \lesssim \frac{\delta}{2} + |t - t_0|^{d+1}$$

By (5), we can bound $|x - Q(t) - (P(t_0) - Q(t_0))|$ by

$$\frac{\delta}{2} + \frac{\delta}{A^{d+1}} < \delta \tag{6}$$

if A is chosen large enough. Hence $g(x - Q(t)) = 1$ by the definition of g (1). □

Applying mean value theorem and inverse function theorem again, we see that the measure of J_x is bounded below by

$$\left| P^{-1}\left(x + \frac{\delta}{2}\right) - P^{-1}\left(x - \frac{\delta}{2}\right) \right| \gtrsim \delta.$$

In sum, we have $f(x - P(t))g(x - Q(t)) = 1$ when $x \in I$ and t lies in an interval of length at least δ. Therefore,

$$|B_{P,Q}(f,g)(x)| \gtrsim \delta, \ x \in I.$$

Since the length of the interval I is $\gtrsim \delta^{\frac{1}{d+1}}$,

$$\|B_{P,Q}(f,g)\|_r \gtrsim \delta \cdot \delta^{\frac{1}{(d+1)r}}. \tag{7}$$

Combine (2) and (7), and we see that

$$\delta^{\frac{1}{r}} \gtrsim \delta^{1+\frac{1}{(d+1)r}}. \tag{8}$$

As (8) holds for arbitrarily small δ, we must have $r \geqslant \frac{d}{d+1}$. This finished the proof.

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Dong Dong
Center for Scientific Computation and Mathematical Modeling
University of Maryland
College Park, MD 20742, USA
e-mail: ddong12@cscamm.umd.edu