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A HARDY–TYPE INEQUALITY WITH AHARONOV–BOHM MAGNETIC FIELD ON THE POINCARÉ DISK

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Abstract. A version of Aharonov-Bohm magnetic field on the Poincaré disk is introduced, then a Hardy-type inequality with Aharonov-Bohm magnetic field is proved.

1. Introduction

The classical Hardy inequality in $\mathbb{R}^N$ says that for all $f \in C_0^\infty(\mathbb{R}^N)$ and $N \geqslant 3$,

$$\int_{\mathbb{R}^N} |\nabla f|^2 dx \geqslant \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{f^2}{|x|^2} dx. \tag{1.1}$$

After the seminal work of Carron [4], inequality (1.1) has been generalized to Riemannian manifolds intensively by several authors [2],[3], [7], [11], [12], [13], [18]. Hardy's inequalities were also pursued for some subelliptic operators (see, e.g., [5], [6], [8], [9], [10], [16],) in particular, for the sub-Laplacian on the Heisenberg group and Grushin operators. But if $N = 2$, the Hardy inequality (1.1) becomes trivial. However Laptev and Weidl [15] have noticed that for the Aharonov-Bohm magnetic forms in two dimensional Euclidean space, the Hardy inequality still holds. In fact, let $\beta\mathbf{a}$ be the Aharonov-Bohm magnetic field

$$\beta\mathbf{a} = \beta\left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right), \qquad \beta \in \mathbb{R}$$

then for all $u \in C_0^\infty(\mathbb{R}^2\backslash\{0\})$,

$$\int_{\mathbb{R}^2} |(\nabla + i\beta\mathbf{a})u|^2 dx \geqslant \min_{k\in\mathbb{Z}} |k+\beta|^2 \int_{\mathbb{R}^2} \frac{|u|^2}{|x|^2} dx. \tag{1.2}$$

Recently Aermark and Laptev introduced a version of the Aharonov-Bohm magnetic field for a Grushin subelliptic operator and they proved an improved Hardy inequality in [1].

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Motivated by the above works, following the perturbed Aharonov-Bohm Hamiltonian on the hyperbolic plane $\mathbb{H}$ [14], we introduce a suitable notion of the Aharonov-Bohm magnetic field $\mathscr{A}$ for the Poincaré disk $\mathbb{B}$ and obtain a Hardy-type inequality associated with $\mathscr{A}$ in this note.

Let $\mathbb{H}$ be the upper plane $\{z = x+iy, y>0\}$ equipped with the metric

$$ds^2 = \frac{dx^2+dy^2}{y^2}$$

and the Poincaré disk $\mathbb{B}$ be the unit disk $B = \{x = (x_1,x_2) : x_1^2+x_2^2 < 1\}$ in $\mathbb{R}^2$ with metric

$$ds^2 = 4\frac{dx_1^2+dx_2^2}{(1-(x_1^2+x_2^2))^2}. \tag{1.3}$$

Here and in what follows we use the notation $r = \sqrt{x_1^2+x_2^2}$. The Riemannian measure $dV_{\mathbb{B}}$ on the Poincaré disk $\mathbb{B}$ is

$$dV_{\mathbb{B}} = \frac{4}{(1-r^2)^2}dx, \tag{1.4}$$

where dx is the usual Lebesgue measure on Euclidean plane. We also have

$$\int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^2 dV_{\mathbb{B}} = \int_B |\nabla u|^2 dx, \tag{1.5}$$

$$\nabla_{\mathbb{B}} = \left(\frac{1-r^2}{2}\right)\nabla, \tag{1.6}$$

where $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)$ is the usual gradient in Euclidean plane [17]. $|\nabla_{\mathbb{B}} u|^2 = \langle \nabla_{\mathbb{B}} u, \nabla_{\mathbb{B}} u\rangle$ where $\langle\cdot,\cdot\rangle$ denotes the inner product induced by the metric (1.3).

For $x = (x_1,x_2) \in \mathbb{B}\backslash\{0\}$, the Aharonov-Bohm magnetic field $\mathscr{A}$ on the Poincaré disk $\mathbb{B}$ is defined as:

$$\mathscr{A} = \left(-\frac{1-r^2}{2r}\sin\theta, \frac{1-r^2}{2r}\cos\theta\right) \tag{1.7}$$

where $x_1 = r\cos\theta, x_2 = r\sin\theta$, $\theta \in [0,2\pi), r \in (0,1)$. For any $x = (x_1,x_2) \in \mathbb{B}\backslash\{0\}$, the hyperbolic distance $\rho = \rho(x,0)$ between x and the origin is

$$\rho = \rho(x,0) = \log\left(\frac{1+r}{1-r}\right) \tag{1.8}$$

where $r = \sqrt{x_1^2+x_2^2}$.

Our main result in this paper is the following Hardy-type inequality with the Aharonov-Bohm magnetic field $\mathscr{A}$ on the Poincaré disk $\mathbb{B}$.

THEOREM 1.1. *For any $\alpha \in \mathbb{R}$ and any $u \in C_0^\infty(\mathbb{B}\backslash\{0\})$*

$$\int_{\mathbb{B}} \left|(\nabla_{\mathbb{B}} + i\alpha\mathscr{A})u\right|^2 dV_{\mathbb{B}} \geqslant \min_{k\in\mathbb{Z}}|k+\alpha|^2 \int_{\mathbb{B}} \frac{|u|^2}{\rho^2} dV_{\mathbb{B}}. \tag{1.9}$$

The proof of Theorem 1.1 will be given in the next section.

2. Proof of Theorem 1.1

With (1.4), (1.8), the Riemannian measure $dV_{\mathbb{B}}$ can be written as

$$dV_{\mathbb{B}} = \frac{4rdrd\theta}{(1-r^2)^2} = \sinh\rho d\rho d\theta. \tag{2.1}$$

Because of (1.7), (1.8) and (2.1), we have for any $u \in C_0^\infty(\mathbb{B}\backslash\{0\})$

$$\begin{aligned}\int_{\mathbb{B}} \left|(\nabla_{\mathbb{B}} + i\alpha\mathscr{A})u\right|^2 dV_{\mathbb{B}} =& \int_0^{2\pi}\int_0^{+\infty} \left(\left|\frac{\partial u}{\partial\rho}\right|^2 + \frac{1}{\sinh^2\rho}\left|\frac{\partial u}{\partial\theta} + i\alpha u\right|^2\right)\sinh\rho d\rho d\theta \\ =& \mathrm{I} + \mathrm{II},\end{aligned} \tag{2.2}$$

where

$$\mathrm{I} = \int_0^{2\pi}\int_0^{+\infty} \left|\frac{\partial u}{\partial\rho}\right|^2 \sinh\rho d\rho d\theta, \tag{2.3}$$

$$\mathrm{II} = \int_0^{2\pi}\int_0^{+\infty} \frac{1}{\sinh\rho}\left|\frac{\partial u}{\partial\theta} + i\alpha u\right|^2 d\rho d\theta. \tag{2.4}$$

The following Lemma 2.1 and Lemma 2.2 hold for I and II. Lemma 2.1 is called Leray inequality (see, e.g., [17]) in the literature. However for the sake of completeness we give the proof of it here.

LEMMA 2.1. *For any* $u \in C_0^\infty(\mathbb{B}\backslash\{0\})$,

$$\mathrm{I} \geqslant \frac{1}{4}\int_{\mathbb{B}} \frac{|u|^2}{\log^2\left(\tanh(\rho/2)\right)} \cdot \frac{1}{\sinh^2\rho} dV_{\mathbb{B}} \tag{2.5}$$

where $dV_{\mathbb{B}} = \sinh\rho d\rho d\theta$.

Proof. Using $\rho = \log\left(\frac{1+r}{1-r}\right)$, by abuse of notation we write $u(\rho,\theta) = u(r,\theta)$. Thus

$$\mathrm{I} = \int_0^{2\pi}\int_0^1 \left|\frac{\partial u}{\partial r}\right|^2 rdrd\theta.$$

Let $u = v(-\log r)^{-1/2}$,

$$\left|\frac{\partial u}{\partial r}\right|^2 = \left|\frac{\partial v}{\partial r}\right|^2(-\log r) + \frac{1}{4}\frac{|v|^2}{r^2\log r} - \frac{1}{2r}\left(\frac{\partial v}{\partial r}\bar{v} + \frac{\partial\bar{v}}{\partial r}v\right). \tag{2.6}$$

Multiplying r on both sides of (2.6) and integrating on $(0,1)$, we obtain

$$\begin{aligned}\int_0^1 \left|\frac{\partial u}{\partial r}\right|^2 rdr =& \int_0^1 \left|\frac{\partial v}{\partial r}\right|^2(-\log r)rdr + \frac{1}{4}\int_0^1 \frac{|v|^2}{r\log r}dr \\ &- \frac{1}{2}\int_0^1 \left(\frac{\partial v}{\partial r}\bar{v} + \frac{\partial\bar{v}}{\partial r}v\right)dr.\end{aligned} \tag{2.7}$$

Since $u \in C_0^\infty(\mathbb{B}\backslash\{0\})$, v still has compact support in $\mathbb{B}\backslash\{0\}$ and $v(0,\theta) = v(1,\theta) = 0$ for every $\theta \in [0,2\pi)$. Thus

$$\int_0^1 \left(\frac{\partial v}{\partial r}\bar{v} + \frac{\partial \bar{v}}{\partial r} v \right) dr = \int_0^1 d(v\bar{v}) = (v\bar{v})\big|_0^1 = 0.$$

Hence (2.7) becomes

$$\begin{aligned} \int_0^1 \left|\frac{\partial u}{\partial r}\right|^2 r dr &= \int_0^1 \left|\frac{\partial v}{\partial r}\right|^2 (-\log r) r dr + \frac{1}{4}\int_0^1 \frac{|u|^2}{r\log^2 r} dr \\ &\geqslant \frac{1}{4}\int_0^1 \frac{|u|^2}{r^2\log^2 r} r dr. \end{aligned} \tag{2.8}$$

Integrating (2.8) on $[0,2\pi)$ with θ and using $\rho = \log\left(\frac{1+r}{1-r}\right)$ again, we obtain (2.5) $\square$

LEMMA 2.2. *For any* $u \in C_0^\infty(\mathbb{B}\backslash\{0\})$ *and any* $\alpha \in \mathbb{R}$,

$$\mathrm{II} \geqslant \min_{k\in\mathbb{Z}} |k+\alpha|^2 \int_{\mathbb{B}} \frac{|u|^2}{\sinh^2\rho} dV_{\mathbb{B}}. \tag{2.9}$$

Proof. Let us expand u by Fourier series

$$u(\rho,\theta) = \sum_{k=-\infty}^{\infty} u_k(\rho)\frac{e^{ik\theta}}{\sqrt{2\pi}},$$

and hence

$$\partial_\theta u(\rho,\theta) = \sum_{k=-\infty}^{\infty} iku_k(\rho)\frac{e^{ik\theta}}{\sqrt{2\pi}}.$$

Thus

$$\begin{aligned} \mathrm{II} &= \int_0^{2\pi}\int_0^{+\infty} \frac{1}{\sinh\rho}\Big|\sum_{k=-\infty}^{\infty}(ik+i\alpha)u_k(\rho)\frac{e^{ik\theta}}{\sqrt{2\pi}}\Big|^2 d\rho d\theta \\ &\geqslant \min_{k\in\mathbb{Z}}|k+\alpha|^2 \int_0^{2\pi}\int_0^{+\infty}\frac{|u|^2}{\sinh^2\rho}\sinh\rho\, d\rho d\theta \\ &= \min_{k\in\mathbb{Z}}|k+\alpha|^2 \int_{\mathbb{B}} \frac{|u|^2}{\sinh^2\rho} dV_{\mathbb{B}}. \quad \square \end{aligned}$$

From Lemma 2.1 and Lemma 2.2, we have

$$\mathrm{I}+\mathrm{II} \geqslant \int_{\mathbb{B}} \left(\frac{1}{4}\frac{|u|^2}{\sinh^2\rho\cdot\log^2\left(\tanh(\rho/2)\right)} + \min_{k\in\mathbb{Z}}|k+\alpha|^2\frac{|u|^2}{\sinh^2\rho}\right) dV_{\mathbb{B}},$$

i.e.,

$$\int_{\mathbb{B}} \left|(\nabla_{\mathbb{B}}+i\alpha\mathscr{A})u\right|^2 dV_{\mathbb{B}} \geqslant \int_{\mathbb{B}} \left(\frac{1}{4}\frac{1}{\sinh^2\rho\cdot\log^2\big(\tanh(\rho/2)\big)}+\min_{k\in\mathbb{Z}}|k+\alpha|^2\frac{1}{\sinh^2\rho}\right)|u|^2 dV_{\mathbb{B}}. \tag{2.10}$$

Furthermore since $\min_{k\in\mathbb{Z}}|k+\alpha|^2 \leqslant \frac{1}{4}$ for all $\alpha\in\mathbb{R}$, Theorem 1.1 can be reduced to the following theorem.

THEOREM 2.3. *For any all* $\rho\in(0,+\infty)$,

$$\frac{1}{\sinh^2\rho\cdot\log^2\big(\tanh(\rho/2)\big)}+\frac{1}{\sinh^2\rho}\geqslant\frac{1}{\rho^2}, \tag{2.11}$$

or

$$\rho^2\geqslant\sinh^2\rho\cdot\log^2\big(\tanh(\rho/2)\big)-\rho^2\cdot\log^2\big(\tanh(\rho/2)\big). \tag{2.12}$$

Proof. In order to prove (2.12), we consider the case $\rho\in(0,1]$ and $\rho\in[1,+\infty)$. In fact it suffices to prove Lemma 2.4 and Lemma 2.5 below. □

LEMMA 2.4. *For any* $\rho\in[1,+\infty)$, *we have*

$$\rho^2\geqslant\sinh^2\rho\cdot\log^2\big(\tanh(\rho/2)\big). \tag{2.13}$$

Proof. For any $\rho\in[1,+\infty)$, we have $\log\big(\tanh(\rho/2)\big)<0$. (2.13) is equivalent to

$$\rho>-\sinh\rho\cdot\log\big(\tanh(\rho/2)\big). \tag{2.14}$$

For all $\rho\in[1,+\infty)$, let

$$f(\rho)=\rho+\sinh\rho\cdot\log\big(\tanh(\rho/2)\big).$$

Because $e^{f(1)}=e(\tanh(1/2))^{\sinh 1}$, in order to show $f(1)>0$ we need to prove that $e(\tanh(1/2))^{\sinh 1}>0$, i.e., $e^2>\left(\frac{e+1}{e-1}\right)^{e-e^{-1}}$. Since $e\approx 2.718$, $e-e^{-1}<2.4$, it is enough to show $e^2>\left(\frac{e+1}{e-1}\right)^{2.4}$ or $e^5>\left(\frac{e+1}{e-1}\right)^6$. But $\left(\frac{e+1}{e-1}\right)^6<\left(\frac{3.8}{1.7}\right)^6<2.24^6<2.7^5<e^5$. Therefore $f(1)>0$.

It is sufficient to prove that $f'(\rho)\geqslant 0$ for all $\rho\in[1,+\infty)$. A simple calculation shows

$$f'(\rho)=2+\frac{1}{2}\frac{1+\tanh^2(\rho/2)}{1-\tanh^2(\rho/2)}\log\big(\tanh^2(\rho/2)\big), \tag{2.15}$$

Let $x=\tanh^2(\rho/2)$ in (2.15). For $x\in[\tanh^2(1/2),1)$, we set

$$g(x)=4+\frac{1+x}{1-x}\log x. \tag{2.16}$$

Then

$$g(x)\geqslant 4+\frac{2}{1-x}\log x. \tag{2.17}$$

Let

$$h(x) = 4 + \frac{2}{1-x}\log x. \quad x \in [\tanh^2(1/2), 1)$$

It is easy to see that

$$h'(x) = \frac{2}{x(1-x)^2}[(1-x) + x\log x].$$

and

$$[(1-x) + x\log x]' = \log x < 0, \quad \forall x \in [\tanh^2(1/2), 1).$$

Using L'hospital's rule, we also obtain

$$h'(1) = \lim_{x\to 1} \frac{2}{x(1-x)^2}[(1-x) + x\log x] = 1.$$

Thus $h'(x)$ is a decreasing function on $[\tanh^2(1/2), 1)$ and the minimal value of $h'(x)$ is 1. Hence

$$h(x) > 0, \quad \forall x \in [\tanh^2(1/2), 1).$$

From (2.17), we know that

$$g(x) \geqslant h(x) > 0, \quad \forall x \in [\tanh^2(1/2), 1),$$

i.e.,

$$f'(\rho) \geqslant 0, \quad \forall \rho \in [1, +\infty). \tag{2.18}$$

From (2.18) and $f(1) > 0$, we can conclude that (2.14) holds for all $\rho \in [1, +\infty)$. □

LEMMA 2.5. *For any $\rho \in (0, 1]$, we have*

$$\rho^2 \geqslant \sinh^2\rho \cdot \log^2\big(\tanh(\rho/2)\big) - \rho^2 \cdot \log^2\big(\tanh(\rho/2)\big). \tag{2.19}$$

Proof. For any $\rho \in (0, 1)$, $\rho - \log\big(\tanh(\rho/2)\big) \cdot \sqrt{\sinh^2\rho - \rho^2} \geqslant 0$. Thus (2.19) is equivalent to

$$\rho + \log\big(\tanh(\rho/2)\big) \cdot \sqrt{\sinh^2\rho - \rho^2} \geqslant 0.$$

or

$$1 + \log\big(\tanh(\rho/2)\big) \cdot \frac{\sqrt{\sinh^2\rho - \rho^2}}{\rho} \geqslant 0. \tag{2.20}$$

The minimum of $t\log t$ on $(0, 1)$ is $-\frac{1}{e}$. Thus for all $\rho \in (0, 1]$,

$$\begin{aligned} 1 + \log\big(\tanh(\rho/2)\big) \cdot \frac{\sqrt{\sinh^2\rho - \rho^2}}{\rho} &= 1 + \tanh(\rho/2) \cdot \log\big(\tanh(\rho/2)\big) \frac{\sqrt{\sinh^2\rho - \rho^2}}{\rho\tanh(\rho/2)} \\ &\geqslant 1 - \frac{1}{e} \frac{\sqrt{\sinh^2\rho - \rho^2}}{\rho\tanh(\rho/2)}. \end{aligned}$$

In order to prove (2.20), it is sufficient to show that for all $\rho \in (0,1]$,

$$e^2\rho^2 \cdot \tanh^2(\rho/2) - \sinh^2\rho + \rho^2 \geqslant 0. \tag{2.21}$$

The proof of (2.21) will be completed by Lemma 2.6 and Lemma 2.7 below. □

LEMMA 2.6. *For all* $\rho \in (0,1]$,

$$e^2\rho^2 \cdot \tanh^2(\rho/2) \geqslant \rho^2 \sinh^2\rho. \tag{2.22}$$

Proof. (2.22) is equivalent to

$$\cosh(\rho/2) \leqslant \sqrt{\frac{e}{2}}, \quad \forall \rho \in (0,1]. \tag{2.23}$$

Because $\big(\cosh(\rho/2)\big)' = \dfrac{1}{2}\sinh(\rho/2) \geqslant 0$, $\cosh(\rho/2)$ is increasing on $(0,1]$. It is easy to see that $1+\sqrt{2} < e$, i.e., $1+e^{-1} < \sqrt{2}$ or $\dfrac{1}{2}(e^{1/2}+e^{-1/2}) < \sqrt{\dfrac{e}{2}}$. Thus $\cosh(1/2) \leqslant \sqrt{\dfrac{e}{2}}$ and (2.23) is proved. □

LEMMA 2.7. *For all* $\rho \in (0,1]$,

$$\rho^2 \sinh^2\rho - \sinh^2\rho + \rho^2 \geqslant 0. \tag{2.24}$$

Proof. (2.24) is equivalent to

$$\rho \cdot \cosh\rho - \sinh\rho \leqslant 0, \quad \forall \rho \in (0,1]. \tag{2.25}$$

Let $h(\rho) = \rho \cdot \cosh\rho - \sinh\rho$, $\rho \in (0,1]$. $h'(\rho) = \rho \sinh\rho \geqslant 0$ and $h(\rho)$ is an increasing function on $(0,1]$. We also have $h(0) = 0$. Hence $h(\rho) \geqslant 0$ on $(0,1]$, i.e., (2.25) holds. □

Now Lemma 2.5 or (2.21) comes from (2.22) and (2.24). Combining Lemma 2.4 and Lemma 2.5, we complete the proof of Theorem 2.3.

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