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ON BENNETT'S CONJECTURE AND COMPLETE MONOTONICITY

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Dedicated to the memory of
Grahame Bennett (1945–2016)

(*Communicated by Z. Ditzian*)

Abstract. Bennett [1] gave a generalization of Schur's theorem in order to study various moment-preserving transformations. Recently, Su [5] confirmed a monotonicity conjecture of Bennett which is related to the generalized Schur's theorem and Haber's inequality. In this paper we present a short proof of this result which is based on a combinatorial identity. Moreover, we show that the function in Bennett's conjecture is not only monotonically decreasing but completely monotonic. Furthermore, we give its explicit representation as a Laplace integral which implies the complete monotonicity. Finally, we prove a multivariate version of the above-mentioned combinatorial identity.

1. Introduction and main result

Bennett [1] gave a generalization of Schur's theorem and utilized its special cases to study various moment-preserving transformations. See [2, p. 164] for the original form of Schur's theorem and [1] for the application of the generalized Schur's theorem to the study of moment sequences. Note that various moment sequences arise naturally in many branches of mathematics and have been extensively studied. The reader is referred to [4, 7] for the broad background of moment sequences and [3, 6] for the latest work on some moment sequences.

Let n be a fixed nonnegative integer and x,y be fixed nonnegative real numbers. In this paper we study the univariate function

$$F_n(a) \equiv F_n(x,y;a) = \binom{n+2a-1}{n}^{-1} \sum_{k=0}^{n} \binom{k+a-1}{k} x^k \binom{n-k+a-1}{n-k} y^{n-k}, \quad (1)$$

which is well-defined if $-2a \notin \{0,1,\ldots,n-1\}$.

While considering one special case of the generalized Schur's theorem, Bennett proposed the following conjecture [1, p. 31].

CONJECTURE 1. *For any $n \in \mathbb{N}$ and $x,y > 0$, the univariate function $F_n(a)$ is monotonically decreasing on $(0,+\infty)$.*

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Very recently, X.-T. Su [5, Theorem 1] affirmed Bennett's conjecture in positive by showing the following result.

THEOREM 2. (Su (2018)) *For $a>0$, the function $F_n(a)$ strictly decreases if $n \geqslant 2$, $x \neq y$ and $x,y>0$. Otherwise, $F_n(a)$ is a constant function.*

The purpose of this article is to provide a short proof of Theorem 2 generalizing it to the larger interval $(-1/2,+\infty)$. Moreover, we prove that the function $F_n(a)$ is completely monotonic, for $a>-1/2$. Recall that a function f is called completely monotonic on an interval (a,b) if it satisfies $(-1)^k f^{(k)}(x) \geqslant 0$, for all $x \in (a,b)$ and $k=0,1,2,\ldots$. Obviously, a completely monotonic function is monotonically decreasing. Bennett's conjecture 1 is a consequence of our result. In this view, we deliver a new short proof of Theorem 2. Furthermore, we present a representation of $F_n(a)$ as a Laplace integral.

We derive the following main result.

THEOREM 3. *Let $x,y>0$. For any $n \in \mathbb{N}$, the function $F_n(a)$ given by Eq. (1) is completely monotonic on the interval $(-1/2,+\infty)$. Moreover, if $n \geqslant 2$ and $x \neq y$,*

$$(-1)^k F_n^{(k)}(a) > 0 \qquad (k=0,1,2,\ldots). \tag{2}$$

REMARK 1. In particular, Eq. (2) implies, that if $n \geqslant 2$, $x,y>0$ and $x \neq y$, the function $F_n(a)$ is strictly decreasing, for $a>-1/2$.

In what follows P_k $(k=0,1,2,\ldots)$ denote the Legendre polynomials satisfying the orthogonality condition $\int_{-1}^{+1} P_k(z) P_\ell(z) \, \mathrm{d}z = 0$ if $k \neq \ell$. Recall the Rodrigues formula

$$P_k(z) = \frac{1}{2^k k!} \left(\frac{d}{dz}\right)^k \left(z^2-1\right)^k. \tag{3}$$

The next result represents $F_n(a)$ as a Laplace integral.

THEOREM 4. *For $n \geqslant 2$ and $x,y \in \mathbb{R}$, the function $F_n(a)$ has, for $a>-1/2$, the representation*

$$F_n(a) = \left(\frac{x+y}{2}\right)^n + \int_0^\infty e^{-(2a+1)t} f_n(t) \, dt \tag{4}$$

as a Laplace integral, where

$$f_n(t) = \sum_{k=2}^n \binom{n}{k} \left(\frac{x-y}{2}\right)^k y^{n-k} e^{-(k-2)t} P'_{k-1}\left(e^t\right). \tag{5}$$

If $x,y \geqslant 0$ the function f_n satisfies $f_n(t) \geqslant 0$, for $t \geqslant 0$.

REMARK 2. Theorem 3 is a direct corollary of Theorem 4. This will be demonstrated below.

REMARK 3. According to Eqs. (4) and (5) it holds

$$\left(\frac{x+y}{2}\right)^{n}=\lim_{a\to+\infty}F_{n}(a)$$

(cf. Eq. (6)).

REMARK 4. We can rewrite Eq. (4) as a Laplace–Stieljes integral

$$F_{n}(a)=\left(\frac{x+y}{2}\right)^{n}+\int_{0}^{\infty}e^{-(2a+1)t}\,d\alpha_{n}(t)$$

with $\alpha_n(0)=0$ und $\alpha_n(t)=\left(\frac{x+y}{2}\right)^n+\int_0^t f_n(u)\,du$, for $t>0$.

REMARK 5. Since $P_k'(1)=\binom{k+1}{2}$, we have

$$f_{n}(0)=\binom{n}{2}\left(\frac{x-y}{2}\right)^{2}\left(\frac{x+y}{2}\right)^{n-2}.$$

Furthermore, we have the limit

$$\lim_{t\to+\infty}f_{n}(t)=\frac{n}{2}(x-y)y^{n-1}\left({}_2F_1\left(\frac{1}{2},1-n;1;1-\frac{x}{y}\right)-{}_2F_1\left(\frac{1}{2},1-n;2;1-\frac{x}{y}\right)\right)$$

in terms of hypergeometric functions ${}_2F_1$. This follows by

$$P_{k}(w)=2^{-k}\binom{2k}{k}w^{k}+O\left(w^{k-1}\right)\qquad(w\to+\infty).$$

We list some initial instances:

$$f_{2}(t)=\left(\frac{x-y}{2}\right)^{2},$$

$$f_{3}(t)=3\left(\frac{x-y}{2}\right)^{2}\frac{x+y}{2}\geqslant 0,\ \text{if}\ x+y\geqslant 0,$$

$$f_{4}(t)=6\left(\frac{x-y}{2}\right)^{2}\left(\frac{x+y}{2}\right)^{2}+\frac{3}{2}\left(1-e^{-2t}\right)\left(\frac{x-y}{2}\right)^{4}\geqslant 0,$$

$$f_{5}(t)=10\left(\frac{x-y}{2}\right)^{2}\left(\frac{x+y}{2}\right)^{3}+\frac{15}{2}\left(\frac{x-y}{2}\right)^{4}\frac{x+y}{2}\left(1-e^{-2t}\right)\geqslant 0,\ \text{if}\ x+y\geqslant 0.$$

2. Auxiliary results and proofs

First of all we gather some elementary properties of completely monotonic functions. Let $I \subseteq \mathbb{R}$ be an arbitrary interval.

(1) If $f_1, \ldots, f_m$ are completely monotonic functions on I, so any linear combination $c_1 f_1 + \cdots + c_m f_m$, for $c_1, \ldots, c_m \geqslant 0$.

(2) If $f_1, \ldots, f_m$ are completely monotonic functions on I, so is their product $f_1 \cdots f_m$. This follows by the Leibniz rule for differentiability of several functions.

(3) Finally, we recall the following criterion (see [7, Theorem 12A, Sect. 12, Chapter 4]):

LEMMA 1. *A necessary and sufficient condition that f should be completely monotonic on $[0,+\infty)$ is that*

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t),$$

where α is bounded and non-decreasing and the integral converges for $0 \leqslant x < +\infty$.

The first step in the proof of Theorem 3 is a certain representation of $F_n(a)$. Here and in the following $z^{\overline{k}}$ denotes the rising factorial defined by $z^{\overline{k}} = z(z+1)\cdots(z+k-1)$, for $k \in \mathbb{N}$, and $z^{\overline{0}} = 1$. Furthermore, we use the falling factorial defined by $z^{\underline{k}} = z(z-1)\cdots(z-k+1)$, for $k \in \mathbb{N}$, and $z^{\underline{0}} = 1$.

LEMMA 2. *For $n \in \mathbb{N}$ and $x, y \in \mathbb{R}$, the function $F_n(a)$ can be written in the form*

$$F_n(a) = \sum_{k=0}^{n} \binom{n}{k} (x-y)^k y^{n-k} \frac{a^{\overline{k}}}{(2a)^{\overline{k}}}.$$

REMARK 6. A direct consequence is the limit

$$\lim_{a \to +\infty} F_n(a) = \sum_{k=0}^{n} \binom{n}{k} (x-y)^k y^{n-k} \frac{1}{2^k} = \left(\frac{x+y}{2}\right)^n. \tag{6}$$

Proof of Lemma 2. Applying the binomial formula to $x^k = ((x-y)+y)^k$, we obtain

$$\begin{aligned}\binom{n+2a-1}{n} F_n(a) &= \sum_{k=0}^{n} \binom{k+a-1}{k} \binom{n-k+a-1}{n-k} y^{n-k} \sum_{j=0}^{k} \binom{k}{j} (x-y)^j y^{k-j} \\ &= \sum_{j=0}^{n} (x-y)^j y^{n-j} \sum_{k=j}^{n} \binom{k}{j} \binom{k+a-1}{k} \binom{n-k+a-1}{n-k}.\end{aligned}$$

Using

$$\binom{k+a-1}{k} \binom{n-k+a-1}{n-k} = (-1)^n \binom{-a}{k} \binom{-a}{n-k},$$

we see that the inner sum is equal to

$$
\begin{aligned}
&(-1)^n \sum_{k=0}^{n-j} \binom{k+j}{j} \binom{-a}{k+j} \binom{-a}{n-j-k} \\
&= (-1)^n \binom{-a}{j} \sum_{k=0}^{n-j} \binom{-a-j}{k} \binom{-a}{n-j-k} \\
&= (-1)^n \binom{-a}{j} \binom{-2a-j}{n-j},
\end{aligned}
$$

where the latter equality follows by Vandermonde convolution. Using $\binom{n+2a-1}{n} = (-1)^n \binom{-2a}{n}$ we conclude that

$$
\begin{aligned}
F_n(a) &= \sum_{j=0}^{n} (x-y)^j y^{n-j} \binom{-a}{j} \binom{-2a-j}{n-j} \binom{-2a}{n}^{-1} \\
&= \sum_{j=0}^{n} \binom{n}{j} (x-y)^j y^{n-j} \frac{(-a)^{\underline{j}}}{(-2a)^{\underline{j}}}
\end{aligned}
$$

This implies the desired formula. $\square$

Proof of Theorem 3. Firstly, we prove that the functions $g_k(a) := \frac{a^{\overline{k}}}{(2a)^{\overline{k}}}$ $(k \in \mathbb{N})$ are completely monotonic, for $a > -1/2$. Writing

$$
g_k(a) = \prod_{\ell=0}^{k-1} \frac{a+\ell}{2a+\ell} = 2^{-k} \prod_{\ell=0}^{k-1} \left(1 + \frac{\ell}{2a+\ell}\right)
$$

we see that $g_k(a)$ is a product of finitely many completely monotonic functions $\left(1+\frac{\ell}{2a+\ell}\right)/2$, $\ell = 0, \ldots, k-1$. By Property (2), the function g_k is completely monotonic. Because of the symmetry $F_n(x,y;a) = F_n(y,x;a)$ in the parameters we can assume that $x \geqslant y \geqslant 0$ such that $\left(\frac{x-y}{2}\right)^k y^{n-k} \geqslant 0$, for $0 \leqslant k \leqslant n$. By Property (1), the proof is completed. $\square$

Proof of Theorem 4. We define

$$
\hat{F}_n(a) := F_n(a) - \lim_{a \to +\infty} F_n(a) = \sum_{k=2}^{n} \binom{n}{k} (x-y)^k y^{n-k} \left(\frac{a^{\overline{k}}}{(2a)^{\overline{k}}} - \frac{1}{2^k} \right). \tag{7}
$$

In particular, $\hat{F}_n(a) = 0$, for $n = 0$ and $n = 1$. Using partial fraction decomposition

$$
\frac{a^{\overline{k}}}{(2a)^{\overline{k}}} - \frac{1}{2^k} = \sum_{j=0}^{k-1} \frac{c_{k,j}}{2a+j}
$$

Eq. (7) yields

$$
\hat{F}_n(a) = \sum_{k=2}^{n} \binom{n}{k} (x-y)^k y^{n-k} \sum_{j=0}^{k-1} \frac{c_{k,j}}{2a+j}.
$$

The coefficients $c_{k,j}$ are given by

$$c_{k,j} = (-1)^j \frac{(-j/2)^{\overline{k}}}{j!\,(k-1-j)!}.$$

Obviously, $c_{k,2i} = 0$, if $0 \leqslant 2i \leqslant k-1$, and

$$c_{k,2i+1} = -\frac{(-i-1/2)^{\overline{k}}}{(2i+1)!\,(k-2-2i)!}$$

if $0 \leqslant 2i \leqslant k-2$. Inserting

$$\begin{aligned}\left(-i-\frac{1}{2}\right)^{\overline{k}} &= (-1)^{i+1}\left(i+\frac{1}{2}\right)\left(i-\frac{1}{2}\right)\cdots\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdots\left(k-i-\frac{3}{2}\right)\\ &= \frac{(-1)^{i+1}}{2^k}(2i+1)(2i-1)\cdots 1\cdot 1\cdot 3\cdots(2k-2i-3)\\ &= \frac{(-1)^{i+1}}{2^k}\cdot\frac{(2i+1)!}{2^i i!}\cdot\frac{(2k-2i-2)!}{2^{k-i-1}(k-i-1)!}\end{aligned}$$

we obtain

$$c_{k,2i+1} = \frac{(-1)^i}{2^{2k-1}(k-1)!}\binom{k-1}{i}(2k-2i-2)^{\underline{k}}.$$

Noting that $(2k-2i-2)^{\underline{k}} = 0$, for $k-1 \leqslant 2i \leqslant 2k-2$, we have

$$\hat{F}_n(a) = \sum_{k=2}^{n}\binom{n}{k}(x-y)^k y^{n-k}\frac{1}{2^{2k-1}(k-1)!}\sum_{i=0}^{k-1}(-1)^i\binom{k-1}{i}\frac{(2k-2i-2)^{\underline{k}}}{2a+2i+1}.$$

Taking advantage of $(2a+2i+1)^{-1} = \int_0^\infty e^{-(2a+2i+1)t}dt$ we obtain

$$\hat{F}_n(a) = \int_0^\infty e^{-(2a+1)t} f_n(t)\,dt$$

with

$$f_n(t) = \sum_{k=2}^{n}\binom{n}{k}(x-y)^k y^{n-k}\frac{1}{2^{2k-1}(k-1)!}\sum_{i=0}^{k-1}(-1)^i\binom{k-1}{i}(2k-2i-2)^{\underline{k}}e^{-2it}.$$

Putting $w := e^t$ the inner sum is equal to

$$\begin{aligned}&\sum_{i=0}^{k-1}(-1)^i\binom{k-1}{i}(2k-2i-2)^{\underline{k}}w^{-2i}\\ &= w^{-k+2}\left(\frac{d}{dw}\right)^k\sum_{i=0}^{k-1}(-1)^i\binom{k-1}{i}w^{2k-2i-2} = w^{-k+2}\left(\frac{d}{dw}\right)^k\left(w^2-1\right)^{k-1}\\ &= 2^{k-1}(k-1)!w^{-k+2}P'_{k-1}(w),\end{aligned}$$

where the latter equality follows by the Rodrigues formula (3). This leads to the desired formula (5). The explicit form

$$P_k(w) = 2^{-k} \sum_{j=0}^{k} \binom{k}{j}^2 (w-1)^{k-j} (w+1)^j$$

of the Legendre polynomials implies that $P'_{k-1}(e^t) \geqslant 0$, for $t \geqslant 0$.
Now suppose that $x, y \geqslant 0$. For proving the non-negativity of f_n on $[0,+\infty)$ we can assume that $x \geqslant y \geqslant 0$. The symmetry $F_n(x,y;a) = F_n(y,x;a)$ in the parameters and the identity theorem for Laplace transform implies that f_n is invariant on the interchange of x and y. By Eq. (5), we conclude that $f_n(t) \geqslant 0$, for $t \geqslant 0$. $\square$

Proof of Remark 2. Since $f_n(t) \geqslant 0$, for $t \geqslant 0$, it follows by Lemma 1 that $\hat{F}_n$ is completely monotonic on $[0,+\infty)$. Eq. (4) shows that

$$\hat{F}_n^{(k)}(a) = (-1)^k \int_0^{\infty} e^{-(2a+1)t} (2t)^k f_n(t)\, dt \qquad (k = 0,1,2,\ldots).$$

If $n \geqslant 2$ and $x \neq y$, the function $f_n(t)$ is strictly positive, for $t > 0$. Therefore, $(-1)^k \hat{F}_n^{(k)}(a) > 0$. Noting that $F_n(a) = \hat{F}_n(a) + \left(\frac{x+y}{2}\right)^n$ and $F_n^{(k)}(a) = \hat{F}_n^{(k)}(a)$, for $k = 0,1,2,\ldots$, completes the proof of Remark 2. $\square$

3. A multidimensional version

In this section we give a multidimensional version of Lemma 2 which is interesting in itself.

Let $r \in \mathbb{N}$. For $\mathbf{x} = (x_1,\ldots,x_r) \in \mathbb{R}^r$, put $|\mathbf{x}| = x_1 + \cdots + x_r$. We define

$$F_n(\mathbf{x};\mathbf{a}) = \binom{n+|\mathbf{a}|-1}{n}^{-1} \sum_{|\mathbf{k}|=n} \binom{k_1+a_1-1}{k_1} \cdots \binom{k_r+a_r-1}{k_r} x_1^{k_1} \cdots x_r^{k_r},$$

where the sum runs over all $\mathbf{k} \in \mathbb{Z}^r_{\geqslant 0}$ such that $k_1 + \cdots + k_r = n$. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^r$ we write $\mathbf{x} \leqslant \mathbf{y}$ if $x_i \leqslant y_i$, for all $i \in \{1,\ldots,r\}$. Finally, for $\mathbf{k} \in \mathbb{Z}^r_{\geqslant 0}$ with $|\mathbf{k}| \leqslant n$, the quantity $\binom{n}{\mathbf{k}} = \frac{n!}{k_1!\cdots k_r!(n-|\mathbf{k}|)!}$ denotes the multinomial coefficient.

THEOREM 5. *Let $r \in \mathbb{N}$. For $n = 0,1,2,\ldots$ and $\mathbf{x}, \mathbf{a} \in \mathbb{R}^r$,*

$$F_n(\mathbf{x};\mathbf{a}) = \sum_{|\mathbf{j}|=n} \binom{n}{\mathbf{j}} \left(\prod_{\nu=1}^{r-1} (x_\nu - x_r)^{j_\nu} \right) x_r^{j_r} \frac{\prod_{\nu=1}^{r-1} a_\nu^{\overline{j_\nu}}}{|\mathbf{a}|^{\overline{n-j_r}}},$$

provided that $|\mathbf{a}| \notin \{0,1,\ldots,n-1\}$.

REMARK 7. For $r=2$ and reals a_1,a_2 such that $a_1+a_2 \notin \{0,1,\ldots,n-1\}$, we have

$$\binom{n+a_1+a_2-1}{n}^{-1} \sum_{k=0}^{n} \binom{k+a_1-1}{k} \binom{n-k+a_2-1}{n-k} x^k y^{n-k}$$
$$= \sum_{j=0}^{n} \binom{n}{j} (x-y)^j \, y^{n-j} \frac{a_1^{\overline{j}}}{(a_1+a_2)^{\overline{j}}},$$

which is a generalization of Lemma 2.

In the special case $a_1 = \cdots = a_r =: a$, we define

$$F_{n,r}(a) = \binom{n+ra-1}{n}^{-1} \sum_{|\mathbf{k}|=n} \binom{k_1+a-1}{k_1} \cdots \binom{k_r+a-1}{k_r} x_1^{k_1} \cdots x_r^{k_r}.$$

It is a consequence of Theorem 5 that

$$F_{n,r}(a) = \sum_{|\mathbf{j}|=n} \binom{n}{\mathbf{j}} \left(\prod_{\nu=1}^{r-1} (x_\nu - x_r)^{j_\nu} \right) x_r^{j_r} \frac{\prod_{\nu=1}^{r-1} a^{\overline{j_\nu}}}{(ra)^{\overline{n-j_r}}}.$$

Proof of Theorem 5. We have

$$\binom{n+|\mathbf{a}|-1}{n} F_n(\mathbf{x};\mathbf{a})$$
$$= (-1)^n \sum_{|\mathbf{k}|=n} \left[\prod_{\nu=1}^{r} \binom{-a_\nu}{k_\nu} x_\nu^{k_\nu} \right]$$
$$= (-1)^n \sum_{|\mathbf{k}|=n} \prod_{\nu=1}^{r-1} \left[\binom{-a_\nu}{k_\nu} \sum_{j_\nu=0}^{k_\nu} \binom{k_\nu}{j_\nu} (x_\nu - x_r)^{k_\nu - j_\nu} x_r^{j_\nu} \right] \binom{-a_r}{k_r} x_r^{k_r}.$$

Taking advantage of the identity $\binom{-a_\nu}{k_\nu}\binom{k_\nu}{j_\nu} = \binom{-a_\nu}{j_\nu}\binom{-a_\nu - j_\nu}{k_\nu - j_\nu}$ we obtain

$$\binom{n+|\mathbf{a}|-1}{n} F_n(\mathbf{x};\mathbf{a})$$
$$= (-1)^n \sum_{|\mathbf{k}|=n} \prod_{\nu=1}^{r-1} \left[\sum_{j_\nu=0}^{k_\nu} \binom{-a_\nu}{j_\nu} \binom{-a_\nu - j_\nu}{k_\nu - j_\nu} (x_\nu - x_r)^{j_\nu} x_r^{k_\nu - j_\nu} \right] \binom{-a_r}{k_r} x_r^{k_r}$$
$$= (-1)^n \sum_{\substack{|\mathbf{j}| \leqslant n, \\ j_r = 0}} \sum_{\substack{|\mathbf{k}|=n, \\ \mathbf{k} \geqslant \mathbf{j}}} \left[\prod_{\nu=1}^{r-1} \binom{-a_\nu}{j_\nu} \binom{-a_\nu - j_\nu}{k_\nu - j_\nu} (x_\nu - x_r)^{j_\nu} x_r^{k_\nu - j_\nu} \right] \binom{-a_r}{k_r} x_r^{k_r}.$$

Since $j_r = 0$ we have

$$\binom{n+|\mathbf{a}|-1}{n} F_n(\mathbf{x};\mathbf{a})$$
$$= (-1)^n \sum_{\substack{|\mathbf{j}|\leqslant n,\\ j_r=0}} \left[\prod_{\nu=1}^{r-1} \binom{-a_\nu}{j_\nu} (x_\nu - x_r)^{j_\nu}\right] x_r^{n-|\mathbf{j}|} \sum_{|\mathbf{k}|=n-|\mathbf{j}|} \left[\prod_{\nu=1}^{r-1} \binom{-a_\nu - j_\nu}{k_\nu}\right] \binom{-a_r}{k_r}.$$

The Vandermonde formula tells us that the inner sum is equal to $\binom{-|\mathbf{a}|-|j|}{n-|\mathbf{j}|}$. Furthermore, we have

$$(-1)^n \binom{-|\mathbf{a}|-|j|}{n-|\mathbf{j}|} \binom{n+|\mathbf{a}|-1}{n}^{-1} = \binom{-|\mathbf{a}|-|\mathbf{j}|}{n-|\mathbf{j}|} \binom{-|\mathbf{a}|}{n}^{-1} = \frac{n!}{(n-|\mathbf{j}|)!} \cdot \frac{1}{|\mathbf{a}|^{\overline{n-|\mathbf{j}|}}}.$$

Therefore,

$$F_n(\mathbf{x};\mathbf{a}) = \sum_{\substack{|\mathbf{j}|\leqslant n,\\ j_r=0}} \left[\prod_{\nu=1}^{r-1} \binom{-a_\nu}{j_\nu} (x_\nu - x_r)^{j_\nu}\right] x_r^{n-|\mathbf{j}|} \frac{n!}{(n-|\mathbf{j}|)!} \cdot \frac{1}{|\mathbf{a}|^{\overline{n-|\mathbf{j}|}}}$$
$$= \sum_{|\mathbf{j}|=n} \binom{n}{\mathbf{j}} \left[\prod_{\nu=1}^{r-1} (x_\nu - x_r)^{j_\nu}\right] x_r^{j_r} \frac{\prod_{\nu=1}^{r-1} a_\nu^{\overline{j_\nu}}}{|\mathbf{a}|^{\overline{n-j_r}}}$$

which completes the proof of Theorem 5. □

REFERENCES

[1] G. BENNETT, *Hausdorff means and moment sequences*, Positivity **15** (2011), 17–48.

[2] G. H. HARDY, J. E. LITTLEWOOD, G. PÓLYA, *Inequalities*, Cambridge University Press, Cambridge, 1952.

[3] H. Y. L. LIANG, L. MU, Y. WANG, *Catalan-like numbers and Stieltjes moment sequences*, Discrete Math. **339** (2016), 484–488.

[4] J. A. SHOHAT, J. D. TAMARKIN, *The Problem of Moments*, Amer. Math. Soc., New York, 1943.

[5] X.-T. SU, *Proof of a monotonicity conjecture*, Math. Inequal. Appl. **21** (2018), 91–98.

[6] Y. WANG, B.-X. ZHU, *Log-convex and Stieltjes moment sequences*, Adv. in Appl. Math. **81** (2016), 115–127.

[7] D. V. WIDDER, *The Laplace Transform*, Princeton University Press, Princeton, 1946.

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