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ON RABIER'S RESULT AND NONBOUNDED MONTGOMERY'S IDENTITY

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Abstract. In this paper, we use generalized Montgomery's identity, given in [7], to give improvement of result from [9] for the class of n-convex functions.

1. Introduction

Steffensen [10] proved the following inequality: if $f,h:[\alpha,\beta]\to\mathbb{R}$, $0\leqslant h\leqslant 1$ and f is decreasing, then

$$\int_{\beta-\gamma}^{\beta} f(t)\,dt \leqslant \int_{\alpha}^{\beta} f(t)h(t)\,dt \leqslant \int_{\alpha}^{\alpha+\gamma} f(t)\,dt, \tag{1}$$

where $\gamma=\int_{\alpha}^{\beta} h(t)\,dt$. From (1) we see that integral $\int_{\alpha}^{\beta} f(t)h(t)\,dt$ is estimated from below and above. With similar inclinations toward Steffensen's inequality, but in much more general setting, Rabier in [9] gave lower and upper estimation of the weighted integral $\int\limits_{\mathbb{R}^n}|f(x)|\,\psi(|x|)\,dx$. The principal Rabier's result, see [9], is given in the next theorem.

THEOREM 1. *Let $\psi:[0,\infty)\to\mathbb{R}$ be non decreasing and locally integrable near 0. Then, $\Phi_{\psi,N}(r):=\int_0^r\psi(t^{\frac{1}{N}})\,dt$ is well defined and*

$$\omega_N\|f\|_\infty\Phi_{\psi,N}\left(\frac{\|f\|_1}{\omega_N\|f\|_\infty}\right)\leqslant\int_{\mathbb{R}^N}|f(x)|\psi(|x|)dx$$

for every $f\in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, $f\neq 0$, where ω_N is the measure of the unit ball of $\mathbb{R}^N$.

In this paper we gave an improvement of the inequality in Theorem 1 for the class of the n-convex functions. We use the following generalized Montgomery's identity given in [7].

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PROPOSITION 1. *Let* $\psi : [a,b] \to \mathbb{R}$ *be a differentiable function on* $[a,b]$, *such that* $\psi' \in L^1([a,b])$, *and* $w : [a,b] \to \mathbb{R}_+$ *such that* $\int_a^b w(s)ds = 1$.

Then,

$$\psi(x) = \int_a^b w(s)\psi(s)\,ds + \int_a^b p_w(x,s)\psi'(s)\,ds,$$

holds for the Peano Kernal p_w *defined as:*

$$p_w(x,s) = \begin{cases} W(s), \; a \leqslant s \leqslant x, \\ W(s)-1, \; x < s \leqslant b, \end{cases}$$

where

$$W(s) = \int_a^s w(\xi)d\xi \;\; for \;\; s \in [a,b].$$

REMARK 1. Observe that the function $x \mapsto p_w(x,s)$ is increasing function, for fixed s. Indeed, if $a \leqslant x_1 < x_2 \leqslant b$ then

$$p_w(x_2,s) - p_w(x_1,s) = \begin{cases} 0, \; a \leqslant s \leqslant x_1, \\ 1, \; x_1 < s \leqslant x_2 \\ 0, \; x_2 < s \leqslant b. \end{cases}$$

2. Main results

Before giving the main result we give the following simple lemma:

LEMMA 1. *Let* $\psi \in C^1([0,\infty))$, *such that* $\psi' \in L^1([0,\infty))$, *and* $w : [0,\infty) \to \mathbb{R}_+$, *such that* $\int_0^\infty w(s)ds = 1$. *Then*

$$\psi(x) = \int_0^\infty w(s)\psi(s)ds + \int_0^\infty p_w(x,s)\psi'(s)\,ds,$$

where

$$p_w(x,s) = \begin{cases} W(s), \; 0 \leqslant s \leqslant x, \\ W(s)-1, \; x < s < \infty, \end{cases}$$

and

$$W(s) = \int_0^s w(\xi)d\xi \;\; for \;\; s \in [0,\infty).$$

Proof. From $\psi' \in L^1([0,\infty))$ we have $\psi(\infty)-\psi(0)=\int_0^\infty \psi'(s)ds$ i.e. $\psi(\infty)\in\mathbb{R}$. Now, the proof follows from the following lines

$$\begin{aligned}
\int_0^\infty p_w(x,s)\psi'(s)ds &= \int_0^x W(s)\psi'(s)ds + \int_x^\infty (W(s)-1)\psi'(s)ds \\
&= \int_0^x \left(\int_0^s w(t)dt\right)\psi'(s)ds + \int_x^\infty \left(\int_0^s w(t)dt\right)\psi'(s)ds - \int_x^\infty \psi'(s)ds \\
\text{(Fubini)} &= \int_0^x w(t)\left(\int_t^x \psi'(s)ds\right)dt + \int_0^x w(t)\left(\int_x^\infty \psi'(s)ds\right)dt \\
&\quad + \int_x^\infty w(t)\left(\int_t^\infty \psi'(s)ds\right)dt - \psi(\infty)+\psi(x) \\
&= \psi(x)W(x)-\psi(x)W(x)+\psi(\infty)W(\infty) \\
&\quad - \int_0^\infty w(t)\psi(t)dt - \psi(\infty)+\psi(x) \\
&= -\int_0^\infty w(t)\psi(t)dt + \psi(x). \qquad \square
\end{aligned}$$

The following theorem states our main result:

THEOREM 2. *Let $\psi\in C^n([0,\infty))$ be an n-convex real valued function and let $f\in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, $f\neq 0$. Then the following holds:*

$$\begin{aligned}
&\int_{\mathbb{R}^N} |f(x)|\psi(|x|)dx - \omega_N\|f\|_\infty \int_0^r \psi(t^{\frac{1}{N}})dt \\
&\quad \geqslant \sum_{k=0}^{n-2} \frac{\psi^{(k+1)}(0)}{k!} \int_0^\infty \left[\int_{\mathbb{R}^N} |f(x)|p_w(|x|,s)\,dx - \omega_N\|f\|_\infty \int_0^r p_w(t^{\frac{1}{N}},s)\,dt\right] s^k\,ds
\end{aligned}$$

where $r=\frac{\|f\|_1}{\omega_N\|f\|_\infty}$.

Proof. By using the Lemma 1 we have the following identity:

$$\begin{aligned}
&\int_{\mathbb{R}^N} |f(x)|\psi(|x|)dx - \omega_N\|f\|_\infty \int_0^r \psi(t^{\frac{1}{N}})dt \\
&= \int_{\mathbb{R}^N} |f(x)| \int_0^\infty w(s)\psi(s)\,dsdx + \int_{\mathbb{R}^N} |f(x)| \int_0^\infty p_w(|x|,s)\psi'(s)\,dsdx \\
&\quad - \omega_N\|f\|_\infty \int_0^r \int_0^\infty w(s)\psi(s)\,dsdt - \omega_N\|f\|_\infty \int_0^r \int_0^\infty p_w(t^{\frac{1}{N}},s)\psi'(s)\,dsdt.
\end{aligned}$$

By rearranging and using Fubini's theorem, we have:

$$\begin{aligned}
&\int_{\mathbb{R}^N} |f(x)|\psi(|x|)dx - \omega_N\|f\|_\infty \int_0^r \psi(t^{\frac{1}{N}})dt \\
&= \int_0^\infty w(s)\psi(s)\,ds \left[\int_{\mathbb{R}^N} |f(x)|dx - \omega_N\|f\|_\infty r\right] \\
&\quad + \int_0^\infty \left[\int_{\mathbb{R}^N} |f(x)|p_w(|x|,s)\,dx - \omega_N\|f\|_\infty \int_0^r p_w(t^{\frac{1}{N}},s)\,dt\right]\psi'(s)\,ds.
\end{aligned}$$

Also, $\int_{\mathbb{R}^N}|f(x)|dx-\omega_N\|f\|_\infty r=0$, since $r=\frac{\|f\|_1}{\omega_N\|f\|_\infty}$. Hence

$$\int_{\mathbb{R}^N}|f(x)|\psi(|x|)dx-\omega_N\|f\|_\infty\int_0^r\psi(t^{\frac{1}{N}})dt \tag{2}$$

$$=\int_0^\infty\left[\int_{\mathbb{R}^N}|f(x)|p_w(|x|,s)\,dx-\omega_N\|f\|_\infty\int_0^r p_w(t^{\frac{1}{N}},s)\,dt\right]\psi'(s)\,ds. \tag{3}$$

Using the $(n-2)$-th Taylor's expansion of ψ' we get:

$$\psi'(s)=\sum_{k=0}^{n-2}\psi^{(k+1)}(0)\frac{s^k}{k!}+\int_0^s\psi^{(n)}(\xi)\frac{(s-\xi)^{n-2}}{(n-2)!}\,d\xi.$$

After substitution in (2) and using Fubini's theorem, we get:

$$\begin{aligned}&\int_{\mathbb{R}^N}|f(x)|\psi(|x|)dx-\omega_N\|f\|_\infty\int_0^r\psi(t^{\frac{1}{N}})dt \qquad (4)\\ &=\sum_{k=0}^{n-2}\frac{\psi^{(k+1)}(0)}{k!}\int_0^\infty\left[\int_{\mathbb{R}^N}|f(x)|p_w(|x|,s)\,dx-\omega_N\|f\|_\infty\int_0^r p_w(t^{\frac{1}{N}},s)\,dt\right]s^k\,ds\\ &+\int_0^\infty\left[\int_\xi^\infty\left[\int_{\mathbb{R}^N}|f(x)|p_w(|x|,s)\,dx-\omega_N\|f\|_\infty\int_0^r p_w(t^{\frac{1}{N}},s)\,dt\right]\frac{(s-\xi)^{n-2}}{(n-2)!}\,ds\right]\\ &\times\psi^{(n)}(\xi)d\xi.\end{aligned}$$

Since ψ is n-convex, $\psi^{(n)}(\xi)\geqslant 0$, and since $p_w(\cdot,s)$ is non decreasing, from Theorem 1 we have $\int_{\mathbb{R}^N}|f(x)|p_w(|x|,s)\,dx-\omega_N\|f\|_\infty\int_0^r p_w(t^{\frac{1}{N}},s)\,dt\geqslant 0$, so the right hand side is non negative and we get the required result. □

REMARK 2. If $\psi^{(k)}(0)\geqslant 0$ for $k=1,2,\ldots,n-1$ then, by using the previous theorem, we get an improvement of the inequality given in Theorem 1 in the class of n-convex functions.

Using (4) we can make an estimate of the difference formed from Theorem 2.

COROLLARY 1. *Suppose that all the assumptions of Theorem 2 hold. Additionally, assume (p,q) is a pair of conjugate exponents, and $\psi^{(n)}\in L^p([0,\infty))$. Then*

$$\begin{aligned}&\Bigg|\int_{\mathbb{R}^N}|f(x)|\psi(|x|)dx-\omega_N\|f\|_\infty\int_0^r\psi(t^{\frac{1}{N}})dt \qquad (5)\\ &-\sum_{k=0}^{n-2}\frac{\psi^{(k+1)}(0)}{k!}\int_0^\infty\left[\int_{\mathbb{R}^N}|f(x)|p_w(|x|,s)\,dx-\omega_N\|f\|_\infty\int_0^r p_w(t^{\frac{1}{N}},s)\,dt\right]s^k\,ds\Bigg|\\ &\leqslant\|K\|_q\left\|\psi^{(n)}\right\|_p,\end{aligned}$$

where $K(\xi)=\int_\xi^\infty\left[\int_{\mathbb{R}^N}|f(x)|p_w(|x|,s)\,dx-\omega_N\|f\|_\infty\int_0^r p_w(t^{\frac{1}{N}},s)\,dt\right]\frac{(s-\xi)^{n-2}}{(n-2)!}\,ds.$

Inequality (5) is sharp for $1<p\leqslant\infty$ and for $p=1$ the constant

$$\int_0^\infty\left[\int_{\mathbb{R}^N}|f(x)|p_w(|x|,s)\,dx-\omega_N\|f\|_\infty\int_0^r p_w(t^{\frac{1}{N}},s)\,dt\right]\frac{s^{n-2}}{(n-2)!}\,ds$$

is the best possible.

Proof. The first part follows from (4) and Hölder's inequality.

For the proof of the sharpness we will find a function ψ for which the equality in (5) is obtained.

For $1 < p < \infty$ take ψ to be such that

$$\psi^{(n)}(t) = K(t)^{\frac{1}{p-1}}.$$

For $p = \infty$ take

$$\psi^{(n)}(t) = K(t).$$

For $p = 1$ we take

$$\psi(t) = \frac{t^n}{\varepsilon n!} 1_{(0,\varepsilon)}(t). \quad \square$$

3. Further refinements

Theorem 2 can be refined further for some classes of functions, using exponential convexity (for details see [1, 2]). First, let us define a linear functional $\mathscr{L}$ by:

$$\begin{aligned} \mathscr{L}\psi = & \int_{\mathbb{R}^N} |f(x)|\psi(|x|)dx - \omega_N\|f\|_\infty \int_0^r \psi(t^{\frac{1}{N}})dt \\ & - \sum_{k=0}^{n-2} \frac{\psi^{(k+1)}(0)}{k!} \int_0^\infty \left[\int_{\mathbb{R}^N} |f(x)| p_w(|x|,s)\,dx - \omega_N\|f\|_\infty \int_0^r p_w(t^{\frac{1}{N}},s)\,dt \right] s^k\,ds. \end{aligned} \tag{6}$$

Under the assumptions of Theorem 2 we conclude that $\mathscr{L}$ acts non-negatively on the class of n-convex functions.

Further, let us introduce a family of n-convex functions on $[0,\infty)$ with

$$\varphi_t(x) = \frac{e^{-tx}}{(-t)^n} \tag{7}$$

This is, indeed, a family of n-convex functions since $\frac{d^n}{dx^n}\varphi_t(x) = e^{-tx} \geqslant 0$.

Since $t \mapsto e^{-tx}$ is exponentially convex function, the quadratic form

$$\sum_{i,j=1}^m \xi_i\xi_j \frac{d^n}{dx^n}\varphi_{\frac{p_i+p_j}{2}}(x) \tag{8}$$

is positively semi-definite. According to Theorem 2,

$$\sum_{i,j=1}^m \xi_i\xi_j \mathscr{L}\varphi_{\frac{p_i+p_j}{2}} \tag{9}$$

is also positively semi-definite, for any $m \in \mathbb{N}$, $\xi_i \in \mathbb{R}$ and $p_i \in \mathbb{R}$, concluding exponential convexity of the mapping $p \mapsto \mathscr{L}\varphi_p$. In particular, if we take $m = 2$ in (9) we have additionaly that $p \mapsto \mathscr{L}\varphi_p$ is also log-convex mapping, property that we will need in the next theorem.

THEOREM 3. *Under the assumptions of Theorem 2 the following statements hold.*

(i) The mapping $p \mapsto \mathscr{L}\varphi_p$ *is exponentially convex on* $\mathbb{R}$.

(ii) For $p,q,r \in \mathbb{R}$ such that $p < q < r$, we have

$$(\mathscr{L}\varphi_q)^{r-p} \leqslant (\mathscr{L}\varphi_p)^{r-q} (\mathscr{L}\varphi_r)^{q-p}. \tag{10}$$

REMARK 3. We have outlined proof of the theorem in lines above. Second part of Theorem 3 is known as Lyapunov inequality, it follows from log-convexity, and it refines lower (upper) bound for action of the functional on the class of functions given in (7). This conclusion is a simple consequence of the fact that exponentially convex mappings are non-negative and if exponentially convex mapping attains zero value at some point it is zero everywhere (see [5]).

Similar estimation technique can be applied for classes of n-convex functions given in the paper [5].

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