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FOURIER COSINE–LAPLACE GENERALIZED CONVOLUTION INEQUALITIES AND APPLICATIONS

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Abstract. We introduce several weighted $L_p(\mathbb{R}_+)$-norm inequalities and integral transform related to the generalized convolution with a weight function for the Fourier cosine and Laplace transforms. Some applications of these inequalities to estimate the solutions of some partial differential equations are considered. We also obtained solutions of a class of the Toeplitz plus Hankel integro-differential equations in closed form.

1. Introduction

For the Fourier convolution (see [4])

$$\left(f \underset{F}{*} k\right)(x) = \int_{-\infty}^{\infty} f(y)k(x-y)dy, \ \ x \in \mathbb{R},$$

Young's theorem (see [2])

$$\left|\int_{-\infty}^{\infty} \left(f \underset{F}{*} k\right)(x).h(x)dx\right| \leqslant \|f\|_{L_p(\mathbb{R})}\, \|k\|_{L_q(\mathbb{R})}\, \|h\|_{L_r(\mathbb{R})}, \tag{1}$$

here $f \in L_p(\mathbb{R}),\ k \in L_q(\mathbb{R}),\ h \in L_r(\mathbb{R}),\ 1/p+1/q+1/r=2$, is fundamental. An important corollary of this theorem is the so-called Young's inequality for the Fourier convolution

$$\left\|f \underset{F}{*} k\right\|_{L_r(\mathbb{R})} \leqslant \|f\|_{L_p(\mathbb{R})}\, \|k\|_{L_q(\mathbb{R})}, \quad \frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}. \tag{2}$$

Note, however, that for the typical case $f,k \in L_2(\mathbb{R})$, the inequalities (1) and (2) do not hold. In [8], Saitoh introduced a weighted $L_p(\mathbb{R},|\rho_j|)$ $(j=1,2,\ p>1)$ inequality for the Fourier convolution

$$\left\|\left((F_1\rho_1) \underset{F}{*} (F_2\rho_2)\right)\left(\rho_1 \underset{F}{*} \rho_2\right)^{\frac{1}{p}-1}\right\|_{L_p(\mathbb{R})} \leqslant ||F_1||_{L_p(\mathbb{R}_+,|\rho_1|)}||F_2||_{L_p(\mathbb{R},|\rho_2|)},$$

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where $F_j \in L_p(\mathbb{R}, |\rho_j|)$. The reverse weighted L_p-norm inequality for the Fourier convolution has also been studied in [9].

For the Laplace convolution (see [4])

$$\left(f \underset{L}{*} k\right)(x) = \int_0^x f(y)k(x-y)dy, \;\; x \in \mathbb{R}_+.$$

In [6], the authors have built the Saitoh's type inequality for this convolution

$$\left\| \left((F_1\rho_1) \underset{L}{*} (F_2\rho_2)\right)\left(\rho_1 \underset{L}{*} \rho_2\right)^{\frac{1}{p}-1} \right\|_{L_p(\mathbb{R}_+)} \leqslant ||F_1||_{L_p(\mathbb{R}_+,|\rho_1|)} ||F_2||_{L_p(\mathbb{R}_+,|\rho_2|)},$$

where $F_j \in L_p(\mathbb{R}_+, |\rho_j|)$ $(j = 1,2,\, p > 1)$. The reverse weighted L_p-norm inequality for the Laplace convolution has also been studied and applications to inverse heat source problems (see [10]).

In this paper we are interested in the Fourier cosine-Laplace generalized convolution. It is the generalized convolution with a weight function $\gamma(y) = e^{-\mu y}(\mu > 0)$ of two functions f and g for the Fourier cosine and Laplace transforms (see [7])

$$\left(f \overset{\gamma}{*} k\right)(x) = \frac{1}{\pi} \int_{\mathbb{R}_+^2} \theta(x,u,v) f(u) k(v) du dv, \;\; x > 0, \tag{3}$$

where

$$\theta(x,u,v) = \frac{v+\mu}{(v+\mu)^2 + (x-u)^2} + \frac{v+\mu}{(v+\mu)^2 + (x+u)^2}. \tag{4}$$

For f and k in $L_1(\mathbb{R}_+)$, the following factorization property holds

$$F_c\left(f \overset{\gamma}{*} k\right)(y) = e^{-\mu y}\left(F_c f\right)(y)\left(\mathscr{L} k\right)(y), \; \forall y > 0, \tag{5}$$

here, let $F_c, \mathscr{L}$ denote the Fourier cosine and the Laplace transforms

$$\left(F_c f\right)(y) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos xy dx,$$
$$\left(\mathscr{L} k\right)(y) = \int_0^\infty k(x) e^{-xy} dx, \;\; y > 0.$$

We obtain several inequalities related to the Fourier cosine-Laplace generalized convolution (3) and apply them to estimate the solutions of some partial differential equations. However, we are interested in integral transform related to this convolution and apply solve a class of the Toeplitz plus Hankel integro-differential equations.

2. Fourier cosine-Laplace generalized convolution inequalities

In this section, we will study the Fourier cosine-Laplace generalized convolution (3) and related inequalities.

THEOREM 1. *Suppose that* $f \in L_2(\mathbb{R}_+)$ *and* $k \in L_1(\mathbb{R}_+)$. *Then, the generalized convolution* $f \overset{\gamma}{*} k \in L_2(\mathbb{R}_+)$ *satisfy the Parseval's type identity*

$$\big(f \overset{\gamma}{*} k\big)(x) = F_c\Big(e^{-\mu y}\big(F_c f\big)\big(\mathscr{L}k\big)\Big)(x),\ \forall x > 0, \tag{6}$$

and factorization identity (5).

Proof. From (4), we have

$$|\theta(x,u,v)| \leqslant \frac{2}{v+\mu} \leqslant \frac{2}{\mu}, \tag{7}$$

and

$$\begin{aligned}\int_0^\infty |\theta(x,u,v)|\,du &= \int_{-\infty}^x \frac{v+\mu}{(v+\mu)^2+t^2}dt + \int_x^\infty \frac{v+\mu}{(v+\mu)^2+t^2}dt\\ &= \int_{-\infty}^\infty \frac{v+\mu}{(v+\mu)^2+t^2}dt = \pi.\end{aligned} \tag{8}$$

From (7), (8) and using the Hölder theorem, we have

$$\begin{aligned}\Big|\big(f \overset{\gamma}{*} k\big)(x)\Big| &\leqslant \frac{1}{\pi}\Big[\int_{\mathbb{R}_+^2} |f(u)|^2\big|\theta(x,u,v)\big||k(v)|dudv\Big]^{1/2}\Big[\int_{\mathbb{R}_+^2} |k(v)|\big|\theta(x,u,v)\big|dudv\Big]^{1/2}\\ &\leqslant \frac{1}{\pi}\Big[\int_{\mathbb{R}_+^2} |f(u)|^2|k(v)|\frac{2}{\mu}dudv\Big]^{1/2}\Big[\int_0^\infty |k(v)|\pi dv\Big]^{1/2}\\ &= \Big(\frac{2}{\pi\mu}\Big)^{1/2}\|f\|_{L_2(\mathbb{R}_+)}\|k\|_{L_1(\mathbb{R}_+)} < \infty.\end{aligned}$$

Therefore convolution (3) exist and is continuous.

By using $\int_0^\infty e^{-vx}\cos xy\,dx = \dfrac{v}{v^2+y^2}$ $(v > 0)$ (formula (2.13.5), p. 91, [5]) and the Fubini theorem, we obtain that

$$\begin{aligned}\big(f \overset{\gamma}{*} k\big)(x) &= \frac{1}{\pi}\int_{\mathbb{R}_+^2}\Big[\int_0^\infty e^{-(v+\mu)y}\big(\cos(x-u)y+\cos(x+u)y\big)dy\Big]f(u)k(v)dudv\\ &= \frac{2}{\pi}\int_{\mathbb{R}_+^2}\Big[\int_0^\infty e^{-(v+\mu)y}(\cos yx.\cos yu)dy\Big]f(u)k(v)dudv\\ &= \frac{2}{\pi}\int_{\mathbb{R}_+^2}\Big[\int_0^\infty f(u)\cos yudu\int_0^\infty k(v)e^{-vy}dv\Big]\cos xydy\\ &= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-\mu y}\big(F_c f\big)(y)\big(\mathscr{L}k\big)(y)\cos xydy = F_c\Big(e^{-\mu y}\big(F_c f\big)\big(\mathscr{L}k\big)\Big)(x).\end{aligned}$$

On the other hand, from $f \in L_2(\mathbb{R}_+)$ we get $F_c f \in L_2(\mathbb{R}_+)$, and since $k \in L_1(\mathbb{R}_+)$ we get $\left|\big(\mathscr{L}k\big)(y)\right| \leqslant \int_0^\infty |e^{-vy}k(v)|\,dv \leqslant \int_0^\infty |k(v)|dv < \infty\,(y>0)$, that is, $\mathscr{L}k$ is bounded. Therefore $e^{-\mu y}\big(F_c f\big)\big(\mathscr{L}k\big) \in L_2(\mathbb{R}_+)$ and $F_c\Big(e^{-\mu y}\big(F_c f\big)\big(\mathscr{L}k\big)\Big) \in L_2(\mathbb{R}_+)$. Thus, the convolution $f \overset{\gamma}{*} k \in L_2(\mathbb{R}_+$, and the Parseval's type identity (6) holds. □

THEOREM 2. (Young's type theorem) *Let $p,q,r>1$ such that $1/p+1/q+1/r=2$, and $f \in L_p(\mathbb{R}_+)$, $k \in L_q(\mathbb{R}_+,(x+\mu)^{q-1})$ $(\mu>0)$, $h \in L_r(\mathbb{R}_+)$, then*

$$\left|\int_0^\infty \big(f \overset{\gamma}{*} k\big)(x)h(x)dx\right| \leqslant \mu^{\frac{1-q}{q}} \|f\|_{L_p(\mathbb{R}_+)} \|k\|_{L_q(\mathbb{R}_+,(x+\mu)^{q-1})} \|h\|_{L_r(\mathbb{R}_+)}.$$

Proof. From (4), we have

$$\int_0^\infty \frac{\big|\theta(x,u,v)\big|dv}{v+\mu} \leqslant 2\int_0^\infty \frac{dv}{(v+\mu)^2} \leqslant 2\int_0^\infty \frac{dv}{v^2+\mu^2} = \frac{\pi}{\mu}. \tag{9}$$

Let p_1,q_1,r_1 be the conjugate exponentials of p,q,r, respectively, it means

$$\frac{1}{p}+\frac{1}{p_1}=1, \quad \frac{1}{q}+\frac{1}{q_1}=1, \quad \frac{1}{r}+\frac{1}{r_1}=1.$$

Then it is obviously that $1/p_1+1/q_1+1/r_1=1$. Put

$$\begin{aligned}
U(x,u,v) &= |k(v)|^{\frac{q}{p_1}}\,|v+\mu|^{\frac{q-1}{p_1}}|h(x)|^{\frac{r}{p_1}}\,|\theta(x,u,v)|^{\frac{1}{p_1}},\\
V(x,u,v) &= |f(u)|^{\frac{p}{q_1}}|h(x)|^{\frac{r}{q_1}}\left|\frac{\theta(x,u,v)}{v+\mu}\right|^{\frac{1}{q_1}},\\
W(x,u,v) &= |f(u)|^{\frac{p}{r_1}}|k(v)|^{\frac{q}{r_1}}|v+\mu|^{\frac{q-1}{r_1}}\,|\theta(x,u,v)|^{\frac{1}{r_1}}.
\end{aligned}$$

We have

$$\big(UVW\big)(x,u,v) = |f(u)||k(v)||h(x)|\,|\theta(x,u,v)|. \tag{10}$$

On the other hand, by using (8) we have

$$\begin{aligned}
\|U\|^{p_1}_{L_{p_1}(\mathbb{R}^3_+)} &= \int_{\mathbb{R}^3_+} |k(v)|^q|v+\mu|^{q-1}|h(x)|^r\,|\theta(x,u,v)|\,dudvdx \qquad (11)\\
&\leqslant \pi\int_0^\infty |k(v)|^q|v+\mu|^{q-1}dv\int_0^\infty |h(x)|^r dx\\
&= \pi\|k\|^q_{L_q(\mathbb{R}_+,(x+\mu)^{q-1})}\|h\|^r_{L_r(\mathbb{R}_+)},\\
\|W\|^{r_1}_{L_{r_1}(\mathbb{R}^3_+)} &= \int_{\mathbb{R}^3_+} |f(u)|^p|k(v)|^q|v+\mu|^{q-1}\,|\theta(x,u,v)|\,dudvdx \qquad (12)\\
&\leqslant \pi\|f\|^p_{L_p(\mathbb{R}_+)}\|k\|^q_{L_q(\mathbb{R}_+,(x+\mu)^{q-1})}.
\end{aligned}$$

By using (9), we have

$$\|V\|^{q_1}_{L_{q_1}(\mathbb{R}^3_+)} = \int_{\mathbb{R}^3_+} |f(u)|^p |h(x)|^r \left|\frac{\theta(x,u,v)}{v+\mu}\right| dudvdx \leqslant \frac{\pi}{\mu}\|f\|^p_{L_p(\mathbb{R}_+}\|h\|^r_{L_r(\mathbb{R}_+)}. \tag{13}$$

From (11), (12) and (13), we have

$$\|U\|_{L_{p_1}(\mathbb{R}^3_+)}\|V\|_{L_{q_1}(\mathbb{R}^3_+)}\|W\|_{L_{r_1}(\mathbb{R}^3_+)} \leqslant \pi\mu^{-\frac{1}{q_1}}\|f\|_{L_p(\mathbb{R}_+)}\|k\|_{L_q(\mathbb{R}_+,(x+\mu)^{q-1})}\|h\|_{L_r(\mathbb{R}_+)}. \tag{14}$$

From (10) and (14), by the three-function from of Hölder inequality we have

$$\begin{aligned}\left|\int_0^\infty \left(f \overset{\gamma}{*} k\right)(x)h(x)dx\right| &\leqslant \frac{1}{\pi}\int_{\mathbb{R}^3_+} |f(u)||k(v)|h(x)|\,|\theta(x,u,v)|\,dudvdx \\ &= \frac{1}{\pi}\int_{\mathbb{R}^3_+} U(x,u,v)V(x,u,v)W(x,u,v)dudvdx \\ &\leqslant \frac{1}{\pi}\|U\|_{L_{p_1}(\mathbb{R}^3_+)}\|V\|_{L_{q_1}(\mathbb{R}^3_+)}\|W\|_{L_{r_1}(\mathbb{R}^3_+)} \\ &\leqslant \mu^{-\frac{1}{q_1}}\|f\|_{L_p(\mathbb{R}_+)}\|k\|_{L_q(\mathbb{R}_+,(x+\mu)^{q-1})}\|h\|_{L_r(\mathbb{R}_+)} \\ &= \mu^{\frac{1-q}{q}}\|f\|_{L_p(\mathbb{R}_+)}\|k\|_{L_q(\mathbb{R}_+,(x+\mu)^{q-1})}\|h\|_{L_r(\mathbb{R}_+)}. \quad \square\end{aligned}$$

THEOREM 3. (Saitoh's type theorem) *For two positive functions* $\rho_j \ (j=1,2)$, *the following* $L_p(\mathbb{R}_+)$*-weighted inequality for the Fourier cosine-Laplace generalized convolution holds for any* $F_j \in L_p(\mathbb{R}_+,\rho_j) \ (p>1)$

$$\left\|\left((F_1\rho_1)\overset{\gamma}{*}(F_2\rho_2)\right)\left(\rho_1\overset{\gamma}{*}\rho_2\right)^{\frac{1}{p}-1}\right\|_{L_p(\mathbb{R}_+)} \leqslant ||F_1||_{L_p(\mathbb{R}_+,\rho_1)}||F_2||_{L_p(\mathbb{R}_+,\rho_2)}. \tag{15}$$

Proof. By raising the left-hand side of (15) to power p we obtain

$$\begin{aligned}&\left\|\left((F_1\rho_1)\overset{\gamma}{*}(F_2\rho_2)\right)\left(\rho_1\overset{\gamma}{*}\rho_2\right)^{\frac{1}{p}-1}\right\|^p_{L_p(\mathbb{R}_+)} \\ &= \frac{1}{\pi}\int_0^\infty \Big\{\Big|\int_{\mathbb{R}^2_+}\theta(x,u,v)(F_1\rho_1)(u)(F_2\rho_2)(v)dudv\Big|^p \\ &\quad\times\Big|\int_{\mathbb{R}^2_+}\theta(x,u,v)\rho_1(u)\rho_2(v)dudv\Big|^{1-p}\Big\}dx.\end{aligned} \tag{16}$$

On the other hand, ussing Hölder inequality for q is the exponential conjugate to p, we have

$$\left|\int_{\mathbb{R}_+^2} \theta(x,u,v)(F_1\rho_1)(u)(F_2\rho_2)(v)dudv\right| \tag{17}$$
$$\leqslant \left(\int_{\mathbb{R}_+^2} \left|\theta(x,u,v)\right| |F_1(u)|^p \rho_1(u) |F_2(v)|^p \rho_2(v)dudv\right)^{1/p}$$
$$\times \left(\int_{\mathbb{R}_+^2} \left|\theta(x,u,v)\right| \rho_1(u)\rho_2(v)dudv\right)^{1/q}.$$

From (16) and (17), we have

$$\left\| \left((F_1\rho_1) \overset{\gamma}{*} (F_2\rho_2)\right)\left(\rho_1 \overset{\gamma}{*} \rho_2\right)^{\frac{1}{p}-1} \right\|^p_{L_p(\mathbb{R}_+)}$$
$$\leqslant \frac{1}{\pi}\int_0^\infty \Big[\left(\int_{\mathbb{R}_+^2} |\theta(x,u,v)|\, |F_1(u)|^p \rho_1(u)\, |F_2(v)|^p \rho_2(v)dudv\right)$$
$$\times \left(\int_{\mathbb{R}_+^2} |\theta(x,u,v)|\, \rho_1(u)\rho_2(v)dudv\right)^{\frac{p}{q}} \left(\int_{\mathbb{R}_+^2} |\theta(x,u,v)|\, \rho_1(u)\rho_2(v)dudv\right)^{1-p}\Big]dx$$
$$= \frac{1}{\pi}\int_{\mathbb{R}_+^3} |\theta(x,u,v)|\, |F_1(u)|^p \rho_1(u)\, |F_2(v)|^p \rho_2(v)dudvdx$$
$$\leqslant \frac{1}{\pi}\int_0^\infty |F_1(u)|^p \rho_1(u)du \int_0^\infty |F_2(v)|^p \rho_2(v)dv \int_0^\infty |\theta(x,u,v)|\, dx$$
$$\leqslant \|F_1\|^p_{L_p(\mathbb{R}_+,\rho_1)} \|F_2\|^p_{L_p(\mathbb{R}_+,\rho_2)}.$$

Therefore we obtain (15). $\square$

Note, in particular, for $\rho_1 = 1$ and $\rho_2 = \rho \in L_1(\mathbb{R}_+)$, the inequality (15) takes the form

$$\left\| F_1 \overset{\gamma}{*} (F_2\rho)\right\|_{L_p(\mathbb{R}_+)} \leqslant \|\rho\|^{1-\frac{1}{p}}_{L_1(\mathbb{R}_+)} \|F_1\|_{L_p(\mathbb{R}_+)} \|F_2\|_{L_p(\mathbb{R}_+,\rho)}. \tag{18}$$

THEOREM 4. *Let F_1 and F_2 be positive functions satisfying*

$$0 < m_1^{\frac{1}{p}} \leqslant F_1(x) \leqslant M_1^{\frac{1}{p}} < \infty, \quad 0 < m_2^{\frac{1}{p}} \leqslant F_2(x) \leqslant M_2^{\frac{1}{p}} < \infty, \quad p > 1, \quad x \in \mathbb{R}_+. \tag{19}$$

Then for any positive functions ρ_1 and ρ_2 we have the reverse $L_p(\mathbb{R}_+)$-weighted convolution inequality

$$\left\| \left((F_1\rho_1) \overset{\gamma}{*} (F_2\rho_2)\right)\left(\rho_1 \overset{\gamma}{*} \rho_2\right)^{\frac{1}{p}-1} \right\|_{L_p(\mathbb{R}_+)}$$
$$\geqslant \pi^{\frac{1}{p}}\left[A_{p,q}\left(\frac{m_1m_2}{M_1M_2}\right)\right]^{-1} \|F_1\|_{L_p(\mathbb{R}_+,\rho_1)} \|F_2\|_{L_p(\mathbb{R}_+,\rho_2)}, \tag{20}$$

here, $A_{p,q}(t) = p^{-\frac{1}{p}} q^{-\frac{1}{q}} t^{-\frac{1}{pq}} (1-t)(1-t^{\frac{1}{p}})^{-\frac{1}{p}} (1-t^{\frac{1}{q}})^{-\frac{1}{q}}$. *Inequality* (20) *and others should be understood in the sense that if the left hand side is finite, then so is the right hand side, and in this case the inequality holds.*

Proof. With θ is defined by (4), let

$$f(u,v) = \theta(x,u,v)F_1^p(u)\rho_1(u)F_2^p(v)\rho_2(v), \quad g(u,v) = \theta(x,u,v)\rho_1(u)\rho_2(v).$$

Then condition (19) implies

$$m_1 m_2 \leqslant \frac{f(u,v)}{g(u,v)} \leqslant M_1 M_2, \;\; u,v \in \mathbb{R}_+.$$

Hence, one can apply the reverse Hölder inequality for f and g to get

$$\Big(\int_{\mathbb{R}_+^2} \theta(x,u,v)F_1^p(u)\rho_1(u)F_2^p(v)\rho_2(v)dudv\Big)^{\frac{1}{p}} \Big(\int_{\mathbb{R}_+^2} \theta(x,u,v)\rho_1(u)\rho_2(v)dudv\Big)^{\frac{1}{q}}$$
$$\leqslant A_{p,q}\Big(\frac{m_1 m_2}{M_1 M_2}\Big) \int_{\mathbb{R}_+^2} \theta(x,u,v)F_1(u)F_2(v)\rho_1(u)\rho_2(v)dudv.$$

Hence,

$$\begin{aligned} &\int_{\mathbb{R}_+^2} \theta(x,u,v)F_1^p(u)\rho_1(u)F_2^p(v)\rho_2(v)dudv \\ &\leqslant \Big[A_{p,q}\Big(\frac{m_1 m_2}{M_1 M_2}\Big)\Big]^p \Big(\int_{\mathbb{R}_+^2} \theta(x,u,v)F_1(u)F_2(v)\rho_1(u)\rho_2(v)dudv\Big)^p \\ &\times \Big(\int_{\mathbb{R}_+^2} \theta(x,u,v)\rho_1(u)\rho_2(v)dudv\Big)^{p-1}. \end{aligned} \tag{21}$$

By using (8) and taking integration of both sides of (21) with respect to x from 0 to ∞ we obtain the inequality

$$\begin{aligned} &\pi \int_{\mathbb{R}_+^2} F_1^p(u)\rho_1(u)F_2^p(v)\rho_2(v)dudv \\ &\leqslant \Big[A_{p,q}\Big(\frac{m_1 m_2}{M_1 M_2}\Big)\Big]^p \int_0^\infty \Big[\Big(\int_{\mathbb{R}_+^2} \theta(x,u,v)F_1(u)F_2(v)\rho_1(u)\rho_2(v)dudv\Big)^p \\ &\times \Big(\int_{\mathbb{R}_+^2} \theta(x,u,v)\rho_1(u)\rho_2(v)dudv\Big)^{p-1}\Big]dx. \end{aligned} \tag{22}$$

Raising both sides of the inequality (22) to power $\frac{1}{p}$, we have

$$\begin{aligned} &\pi^{\frac{1}{p}} \Big(\int_0^\infty F_1^p(u)\rho_1(u)du\Big)^{\frac{1}{p}} \Big(\int_0^\infty F_2^p(v)\rho_2(v)dv\Big)^{\frac{1}{p}} \\ &\leqslant A_{p,q}\Big(\frac{m_1 m_2}{M_1 M_2}\Big) \Big\{\int_0^\infty \Big[\Big(\int_{\mathbb{R}_+^2} \theta(x,u,v)(F_1\rho_1)(u)(F_2\rho_2)(v)dudv\Big)^p \\ &\times \Big(\int_{\mathbb{R}_+^2} \theta(x,u,v)\rho_1(u)\rho_2(v)dudv\Big)^{p-1}\Big]dx\Big\}^{\frac{1}{p}}. \end{aligned}$$

Therefore the inequality (20). $\square$

3. Fourier cosine-Laplace generalized convolution transform

In this section, we will study the integral transform which related Fourier cosine-Laplace generalized convolution (3), namely, the transform of the form

$$f(x) \mapsto g(x) = \big(T_{k_1,k_2}f\big)(x) = \Big(1 - \frac{d^2}{dx^2}\Big)\Big\{\big(f \overset{\gamma}{*} k_1\big)(x) + \big(f \underset{F_c}{*} k_2\big)(x)\Big\}. \tag{23}$$

Where $f \underset{F_c}{*} k_2$ is the Fourier cosine convolution of two functions f and k_2 (see [4])

$$\big(f \underset{F_c}{*} k_2\big)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(y)\big[k_2(|x-y|) + k_2(x+y)\big]dy, \;\; x > 0,$$

this convolution satisfy the following Parseval's type identity (see [11])

$$\big(f \underset{F_c}{*} k_2\big)(x) = F_c\Big(\big(F_c f\big)\big(F_c k_2\big)\Big)(x), \; \forall x > 0, \;\; f, k_2 \in L_2(\mathbb{R}_+). \tag{24}$$

THEOREM 5. (Watson's type theorem) *Suppose that $k_1 \in L_1(\mathbb{R}_+)$ and $k_2 \in L_2(\mathbb{R}_+)$, then necessary and sufficient condition to ensure that the transform (23) is unitary on $L_2(\mathbb{R}_+)$ is that*

$$\big|e^{-\mu y}\big(\mathscr{L}k_1\big)(y) + \big(F_c k_2\big)(y)\big| = \frac{1}{1+y^2}. \tag{25}$$

Moreover, the inverse transform has the form

$$f(x) = \Big(1 - \frac{d^2}{dx^2}\Big)\Big\{\big(g \overset{\gamma}{*} \overline{k_1}\big)(x) + \big(g \underset{F_c}{*} \overline{k_2}\big)(x)\Big\}. \tag{26}$$

Proof. Necessity. Assume that k_1 and k_2 satisfy condition (25). We known that $h(y)$, $yh(y)$, $y^2h(y) \in L_2(\mathbb{R})$ if and only if $\big(Fh\big)(x)$, $\frac{d}{dx}\big(Fh\big)(x)$, $\frac{d^2}{dx^2}\big(Fh\big)(x) \in L_2(\mathbb{R})$ (Theorem 68, p. 92, [1]). Moreover,

$$\frac{d^2}{dx^2}\big(Fh\big)(x) = \frac{1}{\sqrt{2\pi}} \frac{d^2}{dx^2} \int_{-\infty}^{\infty} h(y)e^{-ixy}dy = F\Big((-iy)^2 h(y)\Big)(x).$$

Specially, if h is an even or odd function such that $h(y), y^2h(y) \in L_2(\mathbb{R}_+)$, then the following equality holds

$$\Big(1 - \frac{d^2}{dx^2}\Big)\big(F_c h\big)(x) = F_c\Big((1+y^2)h(y)\Big)(x). \tag{27}$$

From condition (25), therefore $e^{-\mu y}\big(\mathscr{L}k_1\big)(y) + \big(F_c k_2\big)(y)$ is bounded, combining with $f \in L_2(\mathbb{R}_+)$, hence $(1+y^2)\big[e^{-\mu y}\big(\mathscr{L}k_1\big)(y) + \big(F_c k_2\big)(y)\big]\big(F_{\left\{\substack{c\\s}\right\}} f\big)(y) \in L_2(\mathbb{R}_+)$. Using Parseval's type properties (6), (24) and formula (27), we have

$$\begin{aligned} g(x) &= \Big(1 - \frac{d^2}{dx^2}\Big)F_c\Big[e^{-\mu y}\big(F_c f\big)(y)\big(\mathscr{L}k_1\big)(y) + \big(F_c f\big)(y)\big(F_c k_2\big)(y)\Big](x) \\ &= F_c\Big[(1+y^2)\Big(e^{-\mu y}\big(\mathscr{L}k_1\big)(y) + (F_c k_2)(y)\Big)\big(F_c f\big)(y)\Big](x). \end{aligned} \tag{28}$$

Therefore the Parseval identity $\|f\|_{L_2(\mathbb{R}_+)} = \|F_c f\|_{L_2(\mathbb{R}_+)}$ and condition (25) gives

$$\begin{aligned}\|g\|_{L_2(\mathbb{R}_+)} &= \left\|(1+y^2)\left[e^{-\mu y}\big(\mathscr{L}k_1\big)(y) + \big(F_c k_2\big)(y)\right]\big(F_c f\big)(y)\right\|_{L_2(\mathbb{R}_+)}\\ &= \left\|\big(F_c f\big)(y)\right\|_{L_2(\mathbb{R}_+)} = \|f\|_{L_2(\mathbb{R}_+)}.\end{aligned}$$

It shows that the transform (23) is isometric.

On the other hand, since

$$(1+y^2)\left[e^{-\mu y}\big(\mathscr{L}k_1\big)(y) + \big(F_c k_2\big)(y)\right]\big(F_c f\big)(y) \in L_2(\mathbb{R}_+),$$

we have

$$\big(F_c g\big)(y) = (1+y^2)\left[e^{-\mu y}\big(\mathscr{L}k_1\big)(y) + \big(F_c k_2\big)(y)\right]\big(F_c f\big)(y).$$

Using condition (25), we have

$$\big(F_c f\big)(y) = (1+y^2)\left[e^{-\mu y}\big(\mathscr{L}\overline{k_1}\big)(y) + \big(F_c \overline{k_2}\big)(y)\right]\big(F_c g\big)(y).$$

Again, condition (25) shows that

$$(1+y^2)\left[e^{-\mu y}\big(\mathscr{L}\overline{k_1}\big)(y) + \big(F_c \overline{k_2}\big)(y)\right]\big(F_c g\big)(y) \in L_2(\mathbb{R}_+).$$

By using (27), we have

$$\begin{aligned}f(x) &= F_c\left[(1+y^2)\left(e^{-\mu y}\big(\mathscr{L}\overline{k_1}\big)(y) + \big(F_c \overline{k_2}\big)(y)\right)\big(F_c g\big)(y)\right](x)\\ &= \left(1-\frac{d^2}{dx^2}\right)F_c\left[e^{-\mu y}\big(F_c g\big)(y)\big(\mathscr{L}\overline{k_1}\big)(y) + \big(F_c g\big)(y)\big(F_c \overline{k_2}\big)(y)\right](x)\\ &= \left(1-\frac{d^2}{dx^2}\right)\left[\big(g \overset{\gamma}{*} \overline{k_1}\big)c(x) + \big(g \underset{F_c}{*} \overline{k_2}\big)(x)\right].\end{aligned}$$

Thus, the transform (23) is unitary on $L_2(\mathbb{R}_+)$ and the inverse transform have the form (26).

Sufficiency. Assume that, the transform (23) is unitary on $L_2(\mathbb{R}_+)$. Then the Parseval identity for Fourier cosine transform yield

$$\begin{aligned}\|g\|_{L_2(\mathbb{R}_+)} &= \left\|(1+y^2)\left[e^{-\mu y}\big(\mathscr{L}k_1\big)(y) + \big(F_c k_2\big)(y)\right]\big(F_c f\big)(y)\right\|_{L_2(\mathbb{R}_+)}\\ &= \left\|\big(F_c f\big)(y)\right\|_{L_2(\mathbb{R}_+)} = \|f\|_{L_2(\mathbb{R}_+)}.\end{aligned}$$

Therefore the operator $M_\theta[f](y) = \theta(y)f(y)$, here $\theta(y) = (1+y^2)\left[e^{-\mu y}\big(\mathscr{L}k_1\big)(y) + \big(F_c k_2\big)(y)\right]$ is unitary on $L_2(\mathbb{R}_+)$, or equivalent, the condition (25) holds. □

REMARK 1. Suppose that $k_1 \in L_1(\mathbb{R}_+)$ and $k_2 \in L_2(\mathbb{R}_+)$ such that

$$0 < C_1 \leqslant \left|(1+y^2)\left[e^{-\mu y}\big(\mathscr{L}k_1\big)(y) + \big(F_c k_2\big)(y)\right]\right| \leqslant C_2 < \infty, \tag{29}$$

then T_{k_1,k_2} defines a isomophirm on $L_2(\mathbb{R}_+)$, and the following estimation hold

$$C_1\|f\|_{L_2(\mathbb{R}_+)} \leqslant \|g\|_{L_2(\mathbb{R}_+)} \leqslant C_2\|f\|_{L_2(\mathbb{R}_+)}. \tag{30}$$

Moreover, the inverse transform has the form

$$f(x) = \Big(1-\frac{d^2}{dx^2}\Big)\big(g \underset{F_c}{*} k\big)(x), \tag{31}$$

here $k \in L_2(\mathbb{R}_+)$ such that

$$\big(F_c k\big)(y) = \frac{1}{(1+y^2)^2\big[e^{-\mu y}\big(\mathscr{L}k_1\big)(y) + \big(F_c k_2\big)(y)\big]}. \tag{32}$$

Proof. From (28) and (29), we have

$$\begin{aligned} C_1\left\|\big(F_c f\big)(y)\right\|_{L_2(\mathbb{R}_+)} &\leqslant \left\|(1+y^2)\big[e^{-\mu y}\big(\mathscr{L}k_1\big)(y) + \big(F_c k_2\big)(y)\big]\big(F_c f\big)(y)\right\|_{L_2(\mathbb{R}_+)} \\ &\leqslant C_2\left\|\big(F_c f\big)(y)\right\|_{L_2(\mathbb{R}_+)}, \end{aligned}$$

therefore estimation (30) holds.

Besides, from condition (29), we get

$$\begin{aligned} \frac{1}{C_2(1+y^2)} &\leqslant \frac{1}{(1+y^2)^2\big[e^{-\mu y}\big(\mathscr{L}k_1\big)(y) + \big(F_c k_2\big)(y)\big]} \\ &\leqslant \frac{1}{C_1(1+y^2)}. \end{aligned}$$

Therefore

$$\frac{1}{(1+y^2)^2\big[e^{-\mu y}\big(\mathscr{L}k_1\big)(y) + \big(F_c k_2\big)(y)\big]} \in L_2(\mathbb{R}_+),$$

there exists $k \in L_2(\mathbb{R}_+)$ satisfy the condition (32). From (28) and (32) we have

$$\begin{aligned} \big(F_c f\big)(y) &= \frac{1}{(1+y^2)\big[e^{-\mu y}\big(\mathscr{L}k_1\big)(y) + \big(F_c k_2\big)(y)\big]}\big(F_c g\big)(y) \\ &= (1+y^2)\frac{1}{(1+y^2)^2\big[e^{-\mu y}\big(\mathscr{L}k_1\big)(y) + \big(F_c k_2\big)(y)\big]}\big(F_c g\big)(y) \\ &= (1+y^2)\big(F_c k\big)(y)\big(F_c g\big)(y) \\ &= (1+y^2)F_c\big(g \underset{F_c}{*} k\big)(y). \end{aligned}$$

Thus, the inverse transform (31) holds. $\square$

4. Applications

Let us consider the Laplace equation in the first quadrant

$$u_{xx} + u_{tt} = 0,\ \ 0 < x, t < \infty, \tag{33}$$

with the boundary conditions

$$u(x,0) = \Big(\frac{a}{a^2+\tau^2} \overset{\gamma}{*} (h\rho)(\tau)\Big)(x),\ \ 0 < x < \infty, \tag{34}$$

$$u_x(0,t) = 0,\ \ \forall t > 0, \tag{35}$$

$$u_x(x,t) \to 0\ \ as\ x \to \infty, t \to \infty, \tag{36}$$

here h and ρ are given functions such that $h \in L_1(\mathbb{R}_+,\rho) \cap L_p(\mathbb{R}_+,\rho)$.

We introduce the Fourier cosine transform with respect to x of a function of two variables $u(x,t)$

$$(F_c u)(y,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x,t) \cos xy dx. \tag{37}$$

Applying the Fourier cosine transform (37) to both sides of (33), using conditions (34)–(36), we have

$$\frac{d^2}{dt^2}(F_c u)(y,t) - y^2 (F_c u)(y,t) = 0, \tag{38}$$

with the boundary condition

$$(F_c u)(y,0) = e^{-\mu y}\Big(\sqrt{\frac{\pi}{2}} e^{-ay}\Big) \mathscr{L}(h\rho)(y). \tag{39}$$

The solution of the equation (38) with condition (39) is of the form

$$(F_c u)(y,t) = (F_c u)(y,0) e^{-yt}.$$

Using formula (1.4.1) in [3] and the factorization property (5), we have

$$\begin{aligned}(F_c u)(y,t) &= e^{-\mu y}\Big(\sqrt{\frac{\pi}{2}} e^{-y(t+a)}\Big) \mathscr{L}(h\rho)(y) \\ &= e^{-\mu y} F_c\Big(\frac{t+a}{(t+a)^2+\tau^2}\Big)(y,t) \mathscr{L}(h\rho)(y) \\ &= F_c\Big(\frac{t+a}{(t+a)^2+\tau^2} \overset{\gamma}{*} (h\rho)(\tau)\Big)(y,t).\end{aligned}$$

Therefore

$$u(x,t) = \Big(\frac{t+a}{(t+a)^2+\tau^2} \overset{\gamma}{*} (h\rho)(\tau)\Big)(x,t).$$

For each $t>0$, using inequality (18) we obtain the following estimation

$$\|u\|_{L_p(\mathbb{R}_+)} \leqslant \|\rho\|^{1-\frac{1}{p}}_{L_1(\mathbb{R}_+)} \left\| \frac{t+a}{(t+a)^2+\tau^2} \right\|_{L_p(\mathbb{R}_+)} \|h\|_{L_p(\mathbb{R}_+,\rho)}$$
$$= \frac{\Gamma(p-\frac{1}{2})}{\Gamma(p)} \|\rho\|^{1-\frac{1}{p}}_{L_1(\mathbb{R}_+)} \|h\|_{L_p(\mathbb{R}_+,\rho)} (t+a)^{1-p}.$$

Here, $\Gamma(.)$ denotes the Gamma function $\Gamma(s)=\int_0^\infty t^{s-1}e^{-t}dt$.

Consider the initial value problem for the one-dimensional diffusion equation with no sources or sinks

$$u_t = ku_{xx}, \;\; 0<x<\infty, \; t>0. \tag{40}$$

with the boundary conditions

$$u_x(0,t)=0, \;\; \forall t>0, \tag{41}$$
$$u_x(x,t)\to 0 \;\text{ as } x\to\infty, \tag{42}$$
$$u(x,t)\to 0 \;\text{ as } x\to\infty, \tag{43}$$

and the initial condition

$$u(x,0) = \left(\frac{e^{-\frac{y^2}{4a}}}{\sqrt{a}} \overset{\gamma}{*} (h\rho)(\tau) \right)(x), \;\; 0<x<\infty, \tag{44}$$

where h,ρ are given functions such that $h\in L_1(\mathbb{R}_+,\rho)\cap L_p(\mathbb{R}_+,\rho)$, and $k>0$ is a diffusivity constant.

Again, by applying the Fourier cosine transform (37) with respect to x to both sides of equation (40) and using conditions (41)–(44) we obtain

$$\frac{d}{dt}(F_c u)(y,t) = -ky^2 (F_c u)(y,t), \tag{45}$$

with the initial condition

$$(F_c u)(y,0) = e^{-\mu y} \left(\sqrt{\frac{\pi}{2}} e^{-ay^2} \right) \mathscr{L}(h\rho)(y). \tag{46}$$

The solution of the equation (45) with condition (46) is of the form

$$(F_c u)(y,t) = (F_c u)(y,0) e^{-ky^2 t}.$$

Using formula (1.4.11) in [3] and the factorization property (5), we have

$$(F_c u)(y,t) = e^{-\mu y} \left(\sqrt{\frac{\pi}{2}} e^{-y^2(kt+a)} \right) \mathscr{L}(h\rho)(y)$$
$$= e^{-\mu y} F_c \left(\frac{e^{-\frac{\tau^2}{4(kt+a)}}}{\sqrt{kt+a}} \right)(y,t) \mathscr{L}(h\rho)(y)$$
$$= F_c \left(\frac{e^{-\frac{\tau^2}{4(kt+a)}}}{\sqrt{kt+a}} \overset{\gamma}{*} (h\rho)(\tau) \right)(y,t).$$

Therefore

$$u(x,t) = \left(\frac{e^{-\frac{\tau^2}{4(kt+a)}}}{\sqrt{kt+a}} \overset{\gamma}{*} (h\rho)(\tau) \right)(x,t).$$

For each $t > 0$, using inequality (18) we obtain the following estimation

$$\begin{aligned} \|u\|_{L_p(\mathbb{R}_+)} &\leqslant \|\rho\|_{L_1(\mathbb{R}_+)}^{1-\frac{1}{p}} \left\| \frac{e^{-\frac{\tau^2}{4(kt+a)}}}{\sqrt{kt+a}} \right\|_{L_p(\mathbb{R}_+)} \|h\|_{L_p(\mathbb{R}_+,\rho)} \\ &= \left(\frac{\pi}{\sqrt{p}(\sqrt{kt+a})^{p-1}} \right)^{\frac{1}{p}} \|\rho\|_{L_1(\mathbb{R}_+)}^{1-\frac{1}{p}} \|h\|_{L_p(\mathbb{R}_+,\rho)}. \end{aligned}$$

Consider the Toeplitz plus Hankel integro-differential equation

$$f(x) + f''(x) + \left(1 - \frac{d^2}{dx^2}\right) \int_0^\infty f(u)[k(x-u) + k(x+u)]du = h(x), \ \ x > 0, \tag{47}$$
$$f'(0) = f(0) = 0,$$

where

$$k(t) = \frac{1}{\pi} \int_0^\infty \frac{v+\mu}{(v+\mu)^2 + t^2} \varphi(v) dv + \frac{1}{\sqrt{2\pi}} \psi(|t|), \ \ \mu > 0,$$

and φ, ψ, h are given functions and f is unknown function.

THEOREM 6. *Suppose* φ, $\varphi" \in L_1(\mathbb{R}_+)$, $\varphi'(0) = \varphi(0) = 0$, $\psi, h \in L_2(\mathbb{R}_+)$ *and the following condition holds*

$$\sup_{y \in \mathbb{R}_+} \left| \left[1 + e^{-\mu y} \big(\mathscr{L}\varphi\big)(y) + \big(F_c\psi\big)(y) \right]^{-1} \right| < \infty. \tag{48}$$

Then equation (47) *has unique solution in* $L_2(\mathbb{R}_+)$. *Moreover, the solution can be presented in closed form as follows*

$$f(x) = \sqrt{\frac{\pi}{2}} \Big(h \underset{F_c}{*} e^{-t} \Big)(x) - \sqrt{\frac{\pi}{2}} \Big(\big(h \underset{F_c}{*} e^{-t} \big) \underset{F_c}{*} q \Big)(x), \tag{49}$$

where $q \in L_2(\mathbb{R}_+)$ *is defined by*

$$\big(F_c q\big)(y) = \frac{e^{-\mu y} \big(\mathscr{L}\varphi\big)(y) + \big(F_c\psi\big)(y)}{1 + e^{-\mu y} \big(\mathscr{L}\varphi\big)(y) + \big(F_c\psi\big)(y)}. \tag{50}$$

Proof. The equation (47) can be rewritten in the form related to the transform (23)

$$f(x) + f''(x) + \left(1 - \frac{d^2}{dx^2}\right) \Big[\big(f \overset{\gamma}{*} \varphi \big)(x) + \big(f \underset{F_c}{*} \psi \big)(x) \Big] = h(x). \tag{51}$$

By using Parseval's type identities (6) and (24) for the equations (51), we get

$$\begin{aligned}&\big(F_cf\big)(y)+y^2\big(F_cf\big)(y)\\&\qquad+(1+y^2)\big[e^{-\mu y}\big(F_cf\big)(y)\big(\mathscr{L}\varphi\big)(y)+\big(F_cf\big)(y)\big(F_c\psi\big)(y)\big]=\big(F_ch\big)(y),\end{aligned}$$

therefore

$$\big(F_cf\big)(y)\Big[1+y^2+(1+y^2)\Big(e^{-\mu y}\big(\mathscr{L}\varphi\big)(y)+\big(F_c\psi\big)(y)\Big)\Big]=\big(F_ch\big)(y). \tag{52}$$

From condition (48) and (52), we have

$$\big(F_cf\big)(y)=\frac{\big(F_ch\big)(y)}{1+y^2}\Big[1-\frac{e^{-\mu y}\big(\mathscr{L}\varphi\big)(y)+\big(F_c\psi\big)(y)}{1+e^{-\mu y}\big(\mathscr{L}\varphi\big)(y)+\big(F_c\psi\big)(y)}\Big]. \tag{53}$$

On the other hand, from the hypothesis of this theorem and using formula $(2.13.5)$ in [5], we have

$$\begin{aligned}e^{-\mu y}\big(\mathscr{L}\varphi\big)(y)&=e^{-\mu y}\frac{1}{1+y^2}\mathscr{L}\big(\varphi+\varphi\text{''}\big)(y)\\&=\sqrt{\frac{\pi}{2}}e^{-\mu y}\big(F_ce^{-t}\big)(y)\mathscr{L}\big(\varphi+\varphi\text{''}\big)(y)\\&=\sqrt{\frac{\pi}{2}}F_c\big(e^{-t}\overset{\gamma}{*}(\varphi+\varphi\text{''})\big)(y).\end{aligned}$$

Therefore

$$e^{-\mu y}\big(\mathscr{L}\varphi\big)(y)+\big(F_c\psi\big)(y)=F_c\Big[\sqrt{\frac{\pi}{2}}\big(e^{-t}\overset{\gamma}{*}(\varphi+\varphi\text{''})\big)+\psi\Big](y)\in L_2(\mathbb{R}_+). \tag{54}$$

From (54), therefore there esixts a function $q\in L_2(\mathbb{R}_+)$ defined by (50). Thus, from (53) and the hypothesis of theorem, we have

$$\begin{aligned}\big(F_cf\big)(y)&=\sqrt{\frac{\pi}{2}}\big(F_ce^{-t}\big)(y)\big(F_ch\big)(y)\big[1-(F_cq)(y)\big]\\&=\sqrt{\frac{\pi}{2}}F_c\big(h\underset{F_c}{*}e^{-t}\big)(y)-\sqrt{\frac{\pi}{2}}F_c\big(h\underset{F_c}{*}e^{-t}\big)(y)\big(F_cq\big)(y)\\&=\sqrt{\frac{\pi}{2}}F_c\big(h\underset{F_c}{*}e^{-t}\big)(y)-\sqrt{\frac{\pi}{2}}F_c\Big(\big(h\underset{F_c}{*}e^{-t}\big)\underset{F_c}{*}q\Big)(y)\in L_2(\mathbb{R}_+).\end{aligned}$$

Therefore, we obtain solution f in $L_2(\mathbb{R}_+)$ defined by (49). $\square$

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