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# EXTENSIONS OF FEFFERMAN–STEIN MAXIMAL INEQUALITIES

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*Abstract.* Let $\beta_1,\dots,\beta_m\in[0,\infty)$ and $\mathscr{M}_{L(\log L)^{\vec{\beta}}}$ be the maximal operator defined by

$$\mathscr{M}_{L(\log L)^{\vec{\beta}}}(f_1,\dots,f_m)(x)=\sup_{Q\ni x}\prod_{j=1}^{m}\|f_j\|_{L(\log L)^{\beta_j},Q}.$$

In this paper, we establish the weighted bounds in terms of the $A_{\vec{P}}(\mathbb{R}^{mn})$ constant for $\mathscr{M}_{L(\log L)^{\vec{\beta}}}$ from $L^{p_1}(l^{q_1};\mathbb{R}^n,w_1)\times\dots\times L^{p_m}(l^{q_m};\mathbb{R}^n,w_m)$ to $L^p(l^q;\mathbb{R}^n,\nu_{\vec{w}})$, where $p_1,\dots,p_m,q_1,\dots,q_m\in(1,\infty)$, $1/p=1/p_1+\dots+1/p_m$, $1/q=1/q_1+\dots+1/q_m$ and $\vec{w}=(w_1,\dots,w_m)$ a multiple $A_{\vec{P}}$ weights. A weak type endpoint inequality for vector-valued operator $\mathscr{M}_{L(\log L)^{\vec{\beta}}}$ is also given.

## 1. Introduction

We will work on $\mathbb{R}^n$, $n\in\mathbb{N}$. Let $M$ be the Hardy-Littlewood maximal operator. The well known Fefferman-Stein maximal inequalities (see [5, Theorem 1]) tell us that for all $p,q\in(1,\infty)$,

$$\|\{Mf_k\}\|_{L^p(l^q;\mathbb{R}^n)}\lesssim\|\{f_k\}\|_{L^p(l^q;\mathbb{R}^n)},$$

and for each $\lambda>0$,

$$|\{x\in\mathbb{R}^n:\|\{Mf_k(x)\}\|_{l^q}>\lambda\}|\lesssim\lambda^{-1}\|\{f_k\}\|_{L^1(l^q;\mathbb{R}^n)},$$

where and in the following, for $q\in(0,\infty)$ and numbers $\{a_k\}_{k=1}^{\infty}$, we denote $\|\{a_k\}\|_{l^q}=\big(\sum_k|a_k|^q\big)^{1/q}$; and for a weight $w$ and $p\in(1,\infty)$, $L^p(l^q;\mathbb{R}^n,w)$ is the space defined as

$$L^p(l^q;\mathbb{R}^n,w)=\big\{\{f_k\}_{k=1}^{\infty}:\|\{f_k\}\|_{L^p(l^q;\mathbb{R}^n,w)}<\infty\big\},$$

where

$$\|\{f_k\}\|_{L^p(l^q;\mathbb{R}^n,w)}=\Big(\int_{\mathbb{R}^n}\|\{f_k(x)\}\|_{l^q}^{p}w(x)\,\mathrm{d}x\Big)^{1/p}.$$

We denote $\|\{f_k\}\|_{L^p(l^q;\mathbb{R}^n,1)}$ by $\|\{f_k\}\|_{L^p(l^q;\mathbb{R}^n)}$ for simplicity. Anderson and John [1] considered weighted version of Fefferman-Stein maximal inequalities. For $p\in(1,\infty)$, let $A_p(\mathbb{R}^n)$ be the weight functions class of Muckenhoupt, that is,

$$A_p(\mathbb{R}^n)=\{w:w\text{ is nonnegative and locally integrable in }\mathbb{R}^n\text{ and }[w]_{A_p}<\infty\},$$

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with

$$[w]_{A_p} := \sup_Q \Big(\frac{1}{|Q|}\int_Q w(x)\mathrm{d}x\Big)\Big(\frac{1}{|Q|}\int_Q w^{-\frac{1}{p-1}}(x)\mathrm{d}x\Big)^{p-1},$$

if $p\in(1,\infty)$, and

$$[w]_{A_1} := \sup_{x\in\mathbb{R}^n}\frac{Mw(x)}{w(x)};$$

$[w]_{A_p}$ is called the $A_p$ constant of $w$ (for details of $A_p(\mathbb{R}^n)$, see [6]). Anderson and John [1] proved that for $p,q\in(1,\infty)$ and $w\in A_p(\mathbb{R}^n)$,

$$\|\{Mf_k\}\|_{L^p(l^q;\mathbb{R}^n,w)} \lesssim_{p,q,w} \|\{f_k\}\|_{L^p(l^q;\mathbb{R}^n,w)}, \tag{1.1}$$

and for each $\lambda>0$ and $w\in A_1(\mathbb{R}^n)$,

$$w(\{x\in\mathbb{R}^n : \|\{Mf_k(x)\}\|_{l^q}>\lambda\}) \lesssim_w \lambda^{-1}\|\{f_k\}\|_{L^1(l^q;\mathbb{R}^n,w)}.$$

It should be pointed out that (1.1) can be obtained from the boundedness of $M$ on $L^p(\mathbb{R}^n,w)$, see [7]. Cruz-Uribe, SFO, Martell and Pérez [3] considered the sharp weighted bounds for vector-valued Hardy-Littlewood maximal operator, and proved that for $p,q\in(1,\infty)$ and $w\in A_p(\mathbb{R}^n)$,

$$\|\{Mf_k\}\|_{L^p(\mathbb{R}^n,w)} \lesssim [w]_{A_p}^{\max\{\frac{1}{q},\frac{1}{p-1}\}}\|\{f_k\}\|_{L^p(\mathbb{R}^n,w)}.$$

For $\beta\in[0,\infty)$, a cube $Q\subset\mathbb{R}^n$ and a function $f$ with $\int_Q|f(t)|\log^\beta(1+|f(t)|)\mathrm{d}t<\infty$, define $\|f\|_{L(\log L)^\beta,Q}$ by

$$\|f\|_{L(\log L)^\beta,Q} = \inf\Big\{\lambda>0: \frac{1}{|Q|}\int_Q\frac{|f(y)|}{\lambda}\log^\beta\Big(1+\frac{|f(y)|}{\lambda}\Big)\mathrm{d}y\leqslant 1\Big\}.$$

Let $m\in\mathbb{N}$, $\beta_1,\ldots,\beta_m\in[0,\infty)$, set $\vec{\beta}=(\beta_1,\ldots,\beta_m)$. Define the maximal operator $\mathscr{M}_{L(\log L)^{\vec{\beta}}}$ by

$$\mathscr{M}_{L(\log L)^{\vec{\beta}}}(f_1,\ldots,f_m)(x) = \sup_{Q\ni x}\prod_{j=1}^m\|f_j\|_{L(\log L)^{\beta_j},Q}.$$

Operators of this type came from the study of the commutators of multilinear Calderón-Zygmund operators, see [13, 15]. For the case of $\beta_1=\ldots=\beta_m=0$, we denote $\mathscr{M}_{L(\log L)^{\vec{\beta}}}$ by $\mathscr{M}$; for the case of $m=1$, we denote $\mathscr{M}_{L(\log L)^{\vec{\beta}}}$ by $M_{L(\log L)^\beta}$ for simplicity. The operator $\mathscr{M}$ was introduced by Lerner, Ombrosi, Pérez, Torres and Trujillo-González [13] and plays an important role in the study of weighted estimates for the multilinear Calderón-Zygmund operators.

DEFINITION 1.1. Let $m\in\mathbb{N}$, $w_1,\ldots,w_m$ be weights, $p_1,\ldots,p_m\in[1,\infty)$, $p\in(0,\infty)$ with $1/p=1/p_1+\ldots+1/p_m$. Set $\vec{w}=(w_1,\ldots,w_m)$, $\vec{P}=(p_1,\ldots,p_m)$ and $\nu_{\vec{w}}=\prod_{k=1}^m w_k^{p/p_k}$. We say that $\vec{w}\in A_{\vec{P}}(\mathbb{R}^{mn})$ if

$$[\vec{w}]_{A_{\vec{P}}} := \sup_{Q\subset\mathbb{R}^n}\Big(\frac{1}{|Q|}\int_Q\nu_{\vec{w}}(x)\,\mathrm{d}x\Big)^{1/p}\prod_{k=1}^m\Big(\frac{1}{|Q|}\int_Q w_k^{-\frac{1}{p_k-1}}(x)\,\mathrm{d}x\Big)^{1-1/p_k}<\infty,$$

when $p_k = 1$, $\left(\frac{1}{|Q|}\int_Q w_k^{-\frac{1}{p_k-1}}(x)\,\mathrm{d}x\right)^{1-1/p_k}$ is understood as $(\inf_Q w_k)^{-1}$.

Damián, Lerner and Pérez [4] considered the sharp weighted bound for $\mathscr{M}$ and proved that for $p_1,\dots,p_m \in (1,\infty)$, $p \in (\frac{1}{m},\infty)$ with $1/p = 1/p_1+\dots+1/p_m$, and $\vec{w} = (w_1,\dots,w_m) \in A_{\vec{P}}(\mathbb{R}^{mn})$,

$$\|\mathscr{M}(f_1,\dots,f_m)\|_{L^p(\mathbb{R}^n,v_{\vec{w}})} \lesssim [\vec{w}]_{A_p}^{\frac{1}{p}} \prod_{i=1}^m [w_i^{-\frac{1}{p_i-1}}]_{A_\infty}^{\frac{1}{p_i}} \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^n,w_j)}, \tag{1.2}$$

where and in the following, for a weight $u$, $[u]_{A_\infty}$ is defined by

$$[u]_{A_\infty} = \sup_{Q\subset\mathbb{R}^n} \frac{1}{u(Q)} \int_Q M(u\chi_Q)(x)\mathrm{d}x.$$

Li, Moen and Sun [14] established another sharp weighted bound for $\mathscr{M}$, which is independent of (1.2) each other. Li et al. [14] proved that for $p_1,\dots,p_m \in (1,\infty)$ and $\vec{w} = (w_1,\dots,w_m) \in A_{\vec{P}}(\mathbb{R}^{mn})$,

$$\|\mathscr{M}(f_1,\dots,f_m)\|_{L^p(\mathbb{R}^n,v_{\vec{w}})} \lesssim [\vec{w}]_{A_{\vec{P}}}^{\max\{1,\frac{p_1'}{p},\dots,\frac{p_m'}{p}\}} \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^n,w_j)}.$$

Our main purpose in this paper is to prove the following extension of Fefferman-Stein maximal inequalities.

THEOREM 1.2. *Let $m \in \mathbb{N}$, $\beta_1,\dots,\beta_m \in [0,\infty)$, $q_1,\dots,q_m \in (1,\infty)$ and $q \in (1/m,\infty)$ with $1/q = 1/q_1+\dots+1/q_m$, $\vec{w} = (w_1,\dots,w_m) \in A_{\vec{P}}(\mathbb{R}^{mn})$.*

(i) *If $p_1,\dots,p_m \in (1,\infty)$ and $p \in (1/m,\infty)$ with $1/p = 1/p_1+\dots+1/p_m$, then*

$$\begin{aligned}&\big\|\{\mathscr{M}_{L(\log L)^{\vec{\beta}}}(f_1^k,\dots,f_m^k)\}\big\|_{L^p(l^q;\mathbb{R}^n,v_{\vec{w}})}\\ &\quad\lesssim [\vec{w}]_{A_{\vec{P}}}^{\max\{\frac{1}{q},\frac{p_1'}{p},\dots,\frac{p_m'}{p}\}} \prod_{i=1}^m [w_i^{-\frac{1}{p-1}}]_{A_\infty}^{\beta_i} \prod_{j=1}^m \|\{f_j^k\}\|_{L^{p_j}(l^{q_j};\mathbb{R}^n,w_j)}.\end{aligned} \tag{1.3}$$

(ii) *If $\vec{w} = (w_1,\dots,w_m) \in A_{1,\dots,1}(\mathbb{R}^{mn})$, then for each fixed $\lambda > 0$,*

$$\begin{aligned}&v_{\vec{w}}(\{x\in\mathbb{R}^n : \big\|\{\mathscr{M}_{L(\log L)^{\vec{\beta}}}(f_1^k,\dots,f_m^k)(x)\}\big\|_{l^q} > \lambda\})\\ &\lesssim_{\vec{w}} \prod_{j=1}^m \Big(\int_{\mathbb{R}^n} \frac{\|\{f_j^k(y_j)\}\|_{l^{q_j}}}{\lambda^{\frac{1}{m}}} \log^{|\vec{\beta}|}\Big(1+\frac{\|\{f_j^k(y_j)\}\|_{l^{q_j}}}{\lambda^{\frac{1}{m}}}\Big) w_j(y_j)\mathrm{d}y_j\Big)^{\frac{1}{m}},\end{aligned} \tag{1.4}$$

*here and in the following, for $\vec{\beta} = (\beta_1,\dots,\beta_m)$, $|\vec{\beta}| = \beta_1+\dots+\beta_m$.*

REMARK 1.3. For the case $m = 1$, $\beta \in \mathbb{N}$ and $w \equiv 1$, the inequality (1.4) was proved by Hu [8]. However, the argument used in [8] does not apply to the case $\beta \in (0,\infty)\backslash\mathbb{N}$, and does not apply to the case $m \geqslant 1$.

In what follows, $C$ always denotes a positive constant that is independent of the main parameters involved but whose value may differ from line to line. We use the symbol $A \lesssim B$ to denote that there exists a positive constant $C$ such that $A \leqslant CB$. Specially, we use $A \lesssim_w B$ to denote that there exists a positive constant $C$ depending only on $w$ such that $A \leqslant CB$. Constant with subscript such as $C_1$, does not change in different occurrences. For any set $E \subset \mathbb{R}^n$, $\chi_E$ denotes its characteristic function. For a cube $Q \subset \mathbb{R}^n$ and $\lambda \in (0,\infty)$, we use $\ell(Q)$ $(\mathrm{diam}Q)$ to denote the side length (diamter) of $Q$, and $\lambda Q$ to denote the cube with the same center as $Q$ and whose side length is $\lambda$ times that of $Q$.

## 2. Proof of Theorem 1.2

Recall that the standard dyadic grid in $\mathbb{R}^n$ consists of all cubes of the form

$$2^{-k}([0,1)^n + j), \;\; k \in \mathbb{Z}, \;\; j \in \mathbb{Z}^n.$$

Denote the standard dyadic grid by $\mathscr{D}_0$. For a fixed cube $Q$, denote by $\mathscr{D}_0(Q)$ the set of dyadic cubes with respect to $Q$, that is, the cubes from $\mathscr{D}_0(Q)$ are formed by repeating subdivision of $Q$ and each of descendants into $2^n$ congruent subcubes.

As usual, by a general dyadic grid $\mathscr{D}$, we mean a collection of cubes with the following properties: (i) for any cube $Q \in \mathscr{D}$, its side length $\ell(Q)$ is of the form $2^k$ for some $k \in \mathbb{Z}$; (ii) for any cubes $Q_1, Q_2 \in \mathscr{D}$, $Q_1 \cap Q_2 \in \{Q_1, Q_2, \emptyset\}$; (iii) for each $k \in \mathbb{Z}$, the cubes of side length $2^k$ form a partition of $\mathbb{R}^n$.

For a dyadic grid $\mathscr{D}$ and $\beta_1, \ldots, \beta_m \in [0,\infty)$, let $\mathscr{M}_{\mathscr{D}, L(\log L)^{\vec{\beta}}}$ be the operator defined by

$$\mathscr{M}_{\mathscr{D}, L(\log L)^{\vec{\beta}}}(f_1, \ldots, f_m)(x) = \sup_{Q \ni x, Q \in \mathscr{D}} \prod_{j=1}^{m} \|f_j\|_{L(\log L)^{\beta_j}, Q}.$$

For the case of $m = 1$, we denote $\mathscr{M}_{\mathscr{D}, L(\log L)^{\vec{\beta}}}$ by $M_{\mathscr{D}, L(\log L)^{\beta}}$.

LEMMA 2.1. *Let $\beta \in (0,\infty)$ and $q \in (1,\infty)$. Then for any cube $Q \subset \mathbb{R}^n$,*

$$\big\|\{\|f_k\|_{L(\log L)^{\beta}, Q}\}\big\|_{l^q} \lesssim \big\|\|\{f_k\}\|_{l^q}\big\|_{L(\log L)^{\beta}, Q}. \tag{2.1}$$

*Proof.* For $s \in (0,\infty)$, we define $\|h\|_{\exp L^s, Q}$ by

$$\|h\|_{\exp L^s, Q} = \inf\Big\{\lambda > 0 : \frac{1}{|Q|}\int_Q \exp\Big(\frac{|h(x)|}{\lambda}\Big)^s \mathrm{d}x \leqslant 2\Big\}.$$

We claim that if $s_1 \leqslant s_2$, then

$$\|h\|_{\exp L^{s_1}, Q} \lesssim \|h\|_{\exp L^{s_2}, Q}. \tag{2.2}$$

To see this, let $\lambda_0 > 0$ such that

$$\frac{1}{|Q|}\int_Q \exp\Big(\frac{|h(x)|}{\lambda_0}\Big)^{s_2} \mathrm{d}x \leqslant 2,$$

then

$$\frac{1}{|Q|}\int_Q \exp\Big(\frac{|h(x)|}{\lambda_0}\Big)^{s_1}\mathrm{d}x \leqslant \mathrm{e} + \frac{1}{|Q|}\int_{\{x\in Q:\,|h(x)|>\lambda_0\}} \exp\Big(\frac{|h(x)|}{\lambda_0}\Big)^{s_1}\mathrm{d}x \leqslant 8.$$

Thus by Hölder's inequality,

$$\frac{1}{|Q|}\int_Q \exp\Big(\frac{|h(x)|}{3^{\frac{1}{s_1}}\lambda_0}\Big)^{s_1}\mathrm{d}x \leqslant \Big(\frac{1}{|Q|}\int_Q \exp\Big(\frac{|h(x)|}{\lambda_0}\Big)^{s_1}\mathrm{d}x\Big)^{\frac{1}{3}} \leqslant 2.$$

This gives (2.2).

We now prove (2.1). By (2.2), it is easy to verify that

$$\big\|\|\{f_k\}\|_{l^{q'}}\big\|_{\exp L^{\frac{1}{\beta}},Q} = \Big\|\sum_k |f_k|^{q'}\Big\|^{\frac{1}{q'}}_{\exp L^{\frac{1}{q'\beta}},Q} \leqslant \Big\|\sum_k |f_k|^{q'}\Big\|^{\frac{1}{q'}}_{\exp L^{\frac{1}{\beta}},Q}.$$

On the other hand, by a standard duality argument (see [16, p. 20]), we deduce that

$$\begin{aligned}\Big\|\sum_k |f_k|^{q'}\Big\|_{\exp L^{\frac{1}{\beta}},Q} &\approx \sup_{\|g\|_{L(\log L)^\beta,Q}\leqslant 1} \frac{1}{|Q|}\Big|\int_Q \sum_k |f_k(y)|^{q'} g(y)\mathrm{d}y\Big| \\ &\lesssim \sup_{\|g\|_{L(\log L)^\beta,Q}\leqslant 1} \sum_k \big\||f_k|^{q'}\big\|_{\exp L^{\frac{1}{\beta}},Q}\|g\|_{L(\log L)^\beta,Q} \\ &\lesssim \sum_k \|f_k\|^{q'}_{\exp L^{\frac{1}{\beta}},Q}.\end{aligned}$$

Therefore,

$$\big\|\|\{f_k\}\|_{l^{q'}}\big\|_{\exp L^{\frac{1}{\beta}},Q} \lesssim \big\|\|\{f_k\}\|_{\exp L^{\frac{1}{\beta}},Q}\big\|_{l^{q'}}. \tag{2.3}$$

The inequality (2.1) is an easy consequence of inequality (2.3). In fact, by a duality argument, we have that

$$\begin{aligned}\Big\|\big\{\|f_k\|_{L(\log L)^\beta,Q}\big\}\Big\|_{l^q} &\lesssim \sup_{\big\|\|g_k\|_{\exp L^{\frac{1}{\beta}},Q}\big\|_{l^{q'}}\leqslant 1} \sum_k \Big|\frac{1}{|Q|}\int_Q f_k(y)g_k(y)\mathrm{d}y\Big| \\ &\lesssim \sup_{\big\|\|g_k\|_{\exp L^{\frac{1}{\beta}},Q}\big\|_{l^{q'}}\leqslant 1} \frac{1}{|Q|}\int_Q \|\{f_k(y)\}\|_{l^q}\|\{g_k(y)\}\|_{l^{q'}}\mathrm{d}y \\ &\lesssim \sup_{\big\|\|g_k\|_{\exp L^{\frac{1}{\beta}},Q}\big\|_{l^{q'}}\leqslant 1} \big\|\|\{f_k\}\|_{l^q}\big\|_{L(\log L)^\beta,Q}\big\|\|\{g_k\}\|_{l^{q'}}\big\|_{\exp L^{\frac{1}{\beta}},Q} \\ &\lesssim \big\|\|\{f_k\}\|_{l^q}\big\|_{L(\log L)^\beta,Q},\end{aligned}$$

where the second inequality follows from Minkowski's inequality, and the third inequality follows from the generalized Hölder's inequality (see [16, p. 64]). This completes the proof of Lemma 2.1. □

LEMMA 2.2. *Let $\beta \in [0,\infty)$ and $q \in (1,\infty)$. Then for each $\lambda > 0$,*

$$\begin{aligned} &\left|\left\{x\in\mathbb{R}^n : \left\|\left\{M_{L(\log L)^\beta} f_k(x)\right\}\right\|_{l^q} > \lambda\right\}\right| \\ &\quad\lesssim \int_{\mathbb{R}^n} \frac{\|\{f_k(x)\}\|_{l^q}}{\lambda} \log^\beta\left(1+\frac{\|\{f_k(x)\}\|_{l^q}}{\lambda}\right)\mathrm{d}x. \end{aligned} \tag{2.4}$$

*Proof.* We employ the ideas of Fefferman and Stein in [5], together with some other tricks. By the well known one-third trick (see [10, Lemma 2.5]), we know that there exists $3^n$ dyadic grid $\mathscr{D}_1,\ldots,\mathscr{D}_{3^n}$, such that for any $f$ and $x\in\mathbb{R}^n$,

$$M_{L(\log L)^\beta} f(x) \lesssim \sum_{j=1}^{3^n} M_{\mathscr{D}_j, L(\log L)^\beta} f(x).$$

Thus, it suffices to prove that for each dyadic grid $\mathscr{D}$,

$$\begin{aligned} &\left|\left\{x\in\mathbb{R}^n : \left\|\left\{M_{\mathscr{D},L(\log L)^\beta} f_k(x)\right\}\right\|_{l^q} > 1\right\}\right| \\ &\quad\lesssim \int_{\mathbb{R}^n} \|\{f_k(x)\}\|_{l^q} \log^\beta\left(1+\|\{f_k(x)\}\|_{l^q}\right)\mathrm{d}x. \end{aligned} \tag{2.5}$$

Write

$$\{x\in\mathbb{R}^n : M_{\mathscr{D},L(\log L)^\beta}(\|\{f_k\}\|_{l^q})(x) > 1\} = \cup_j Q_j,$$

with $\{Q_j\}\subset\mathscr{D}$ the maximal cubes such that $\big\|\|\{f_k\}\|_{l^q}\big\|_{L(\log L)^\beta, Q_j} > 1$. Obviously,

$$1 < \frac{1}{|Q_j|}\int_{Q_j} \|\{f_k(x)\}\|_{l^q} \log^\beta\left(1+\|\{f_k(x)\}\|_{l^q}\right)\mathrm{d}x \leqslant 2^n. \tag{2.6}$$

Set

$$f_k^1(x) = f_k(x)\chi_{\mathbb{R}^n\setminus\cup_j Q_j}(x),\ f_k^2(x) = f_k(x)\chi_{\cup_j Q_j}(x).$$

Since $\|\{f_k^1\}\|_{L^\infty(l^q;\mathbb{R}^n)} \lesssim 1$, it follows that

$$\begin{aligned} \left|\left\{x\in\mathbb{R}^n : \left\|\left\{M_{L(\log L)^\beta} f_k^1(x)\right\}\right\|_{l^q} > 1\right\}\right| &\lesssim \left\|\left\{M_{L(\log L)^\beta} f_k^1\right\}\right\|^q_{L^q(l^q;\mathbb{R}^n)} \\ &\lesssim \|\{f_k\}\|_{L^1(l^q;\mathbb{R}^n)}. \end{aligned}$$

Now let $E = \cup_j 4Q_j$. It is easy to verify that

$$\begin{aligned} |E| &\lesssim \sum_j \int_{Q_j} \|\{f_k(x)\}\|_{l^q} \log^\beta\left(1+\|\{f_k(x)\}\|_{l^q}\right)\mathrm{d}x \\ &\lesssim \int_{\mathbb{R}^n} \{f_k(x)\}\|_{l^q} \log^\beta\left(1+\|\{f_k(x)\}\|_{l^q}\right)\mathrm{d}x. \end{aligned}$$

For each fixed $k$, set

$$f_k^3(y) = \sum_j \|f_k\|_{L(\log L)^\beta, Q_j}\chi_{Q_j}(y).$$

An application of Lemma 2.1 leads to that

$$\|\{f_k^3(y)\}\|_{l^q} \lesssim \sum_j \big\|\|\{f_k\}\|_{l^q}\big\|_{L(\log L)^\beta, Q_j}\chi_{Q_j}(y).$$

It then follows from (2.6) that

$$\|\{f_k^3\}\|_{L^\infty(l^q;\mathbb{R}^n)} \lesssim 1,$$

and

$$\begin{aligned}\|\{f_k^3\}\|_{L^1(l^q,\mathbb{R}^n)} &\lesssim \sum_j |Q_j|\big\|\|\{f_k\}\|_{l^q}\big\|_{L(\log L)^\beta, Q_j} \\ &\lesssim \int_{\mathbb{R}^n} \|\{f_k(y)\}\|_{l^q} \log^\beta\big(1+\|\{f_k(y)\}\|_{l^q}\big)\mathrm{d}y,\end{aligned} \tag{2.7}$$

where in the last inequality, we have invoked the fact that $\big\|\|\{f_k\}\|_{l^q}\big\|_{L(\log L)^\beta, Q_j} \approx 1$. If we can prove that for $x\in\mathbb{R}^n\backslash E$,

$$M_{\mathscr{D}, L(\log L)^\beta} f_k^2(x) \leqslant C_1 M_{L(\log L)^\beta} f_k^3(x), \tag{2.8}$$

with $C_1$ a positive constant, then by the inequality (2.7),

$$\begin{aligned}&\Big|\Big\{x\in\mathbb{R}^n\backslash E : \big\|\{M_{\mathscr{D}, L(\log L)^\beta} f_k^2(x)\}\big\|_{l^q} > 1\Big\}\Big| \\ &\quad\leqslant \Big|\Big\{x\in\mathbb{R}^n\backslash E : \big\|\{M_{L(\log L)^\beta} f_k^3(x)\}\big\|_{l^q} > C_1\Big\}\Big| \\ &\quad\lesssim \big\|\{M_{L(\log L)^\beta} f_k^3\}\big\|^q_{L^q(l^q,\mathbb{R}^n)} \lesssim \big\|\{f_k^3\}\big\|^q_{L^q(l^q,\mathbb{R}^n)} \\ &\quad\lesssim \int_{\mathbb{R}^n} \|\{f_k(y)\}\|_{l^q} \log^\beta\big(1+\|\{f_k(y)\|_{l^q}\big)\mathrm{d}y.\end{aligned}$$

Our desired conclusion (2.5) follows directly.

It remains to prove (2.8). For each fixed $x\in\mathbb{R}^n\backslash E$ and each cube $I\in\mathscr{D}$ containing $x$, note that $I\cap Q_j\neq\emptyset$ implies that $Q_j\subset I$. Thus, for each $\lambda>0$, a straightforward computation tells us that

$$\begin{aligned}&\int_I \frac{|f_k^2(y)|}{\lambda}\log^\beta\Big(1+\frac{|f_k^2(y)|}{\lambda}\Big)\mathrm{d}y \\ &\quad= \sum_{j:Q_j\subset I}\int_{Q_j} \frac{|f_k^2(y)|}{\lambda}\log^\beta\Big(1+\frac{|f_k^2(y)|}{\lambda}\Big)\mathrm{d}y \\ &\quad\lesssim \sum_{j:Q_j\subset I}\int_{Q_j} \frac{\|f_k\|_{L(\log L)^\beta, Q_j}}{\lambda}\log^\beta\Big(1+\frac{\|f_k\|_{L(\log L)^\beta, Q_j}}{\lambda}\Big) \\ &\qquad\times\frac{|f_k(y)|}{\|f_k\|_{L(\log L)^\beta, Q_j}}\log^\beta\Big(1+\frac{|f_k(y)|}{\|f_k\|_{L(\log L)^\beta, Q_j}}\Big)\mathrm{d}y \\ &\quad\lesssim \sum_{j:Q_j\subset I}|Q_j|\frac{\|f_k\|_{L(\log L)^\beta, Q_j}}{\lambda}\log^\beta\Big(1+\frac{\|f_k\|_{L(\log L)^\beta, Q_j}}{\lambda}\Big),\end{aligned}$$

and

$$\int_I \frac{|f_k^3(y)|}{\lambda}\log^\beta\Big(1+\frac{|f_k^3(y)|}{\lambda}\Big)\mathrm{d}y$$
$$=\sum_{j:\,Q_j\subset I}|Q_j|\frac{\|f_k\|_{L(\log L)^\beta,Q_j}}{\lambda}\log^\beta\Big(1+\frac{\|f_k\|_{L(\log L)^\beta,Q_j}}{\lambda}\Big).$$

Therefore,

$$\|f_k^2\|_{L(\log L)^\beta,I}\lesssim\|f_k^3\|_{L(\log L)^\beta,I}.$$

This establishes (2.8) and completes the proof of Lemma 2.2. □

Let $Q\subset\mathbb{R}^n$ be a cube, and $f$ be a measurable real-valued function on $Q$. $m_f(Q)$, the median value of $f$ on $Q$, is one of the real numbers such that

$$\max\{|\{x\in Q:\,f(x)>m_f(Q)\}|,|\{x\in Q:\,f(x)<m_f(Q)\}|\}\leqslant|Q|/2.$$

The decreasing rearrangement of a measurable function $f$ is defined by

$$f^*(t)=\inf\{\lambda>0:\,|\{x\in\mathbb{R}^n:\,|f(x)|>\lambda\}|<t\},t\in(0,\infty).$$

For $\lambda\in(0,1)$ and cube $Q\subset\mathbb{R}^n$, the local mean oscillation of $f$ is defined by

$$\omega_\lambda(f;Q)=\inf_{c\in\mathbb{C}}\big((f-c)\chi_Q\big)^*(\lambda|Q|).$$

The following lemma was proved by Hytönen [9], which improves original Lerner's formula established in [12].

LEMMA 2.3. *Let $f$ be a measurable function on $\mathbb{R}^n$ and $Q_0\subset\mathbb{R}^n$ be a cube. Then there exists a sparse family $\mathscr{S}$ of cubes $Q\in\mathscr{D}_0(Q_0)$, such that for a. e. $x\in Q_0$,*

$$|f(x)-m_f(Q_0)|\leqslant 2\sum_{Q\in\mathscr{S}}\omega_{\frac{1}{2^{n+2}}}(f;Q)\chi_Q(x).$$

LEMMA 2.4. *Let $\rho\in[0,\infty)$ and $\delta\in(0,1)$, $T$ be a sublinear operator which satisfies the weak type estimate that*

$$|\{x\in\mathbb{R}^n:\,\|\{Tf^k\}(x)\}\|_{l^q}>\lambda\}|\lesssim\int_{\mathbb{R}^n}\frac{\|\{f^k(x)\}\|_{l^q}}{\lambda}\log^\rho\Big(1+\frac{\|\{f^k(x)\}\|_{l^q}}{\lambda}\Big)\mathrm{d}x.$$

*Then for any cube $I$ and appropriate functions $\{f^k\}$ with $\operatorname{supp} f^k\subset I$,*

$$\Big(\frac{1}{|I|}\int_I\|\{Tf^k(x)\}\|_{l^q}^\delta\mathrm{d}x\Big)^{\frac{1}{\delta}}\lesssim\big\|\|\{f^k\}\|_{l^q}\big\|_{L(\log L)^\rho,I}.$$

Lemma 2.4 can be proved by mimicking the proof of Kolmogrov's inequality, we omit the details for brevity.

Let $\eta\in(0,1)$ and $\mathscr{S}$ be a family of cubes. We say that $\mathscr{S}$ is $\eta$-sparse, if for each fixed $Q\in\mathscr{S}$, there exists a measurable subset $E_Q\subset Q$, such that $|E_Q|\geqslant\eta|Q|$ and $\{E_Q\}$ are pairwise disjoint. Associated with the sparse family $\mathscr{S}$ and nonnegative constants $\beta_1,\dots,\beta_m\in[0,\infty)$, we define the sparse operator $\mathscr{A}^q_{m,\mathscr{S},L(\log L)^{\vec\beta}}$ by

$$\mathscr{A}^q_{m,\mathscr{S},L(\log L)^{\vec\beta}}f(x)=\Big(\sum_{Q\in\mathscr{S}}\prod_{j=1}^m\|f_j\|^q_{L(\log L)^{\beta_j},Q}\chi_Q(x)\Big)^{\frac1q}.$$

THEOREM 2.5. *Let $p_1,\dots,p_m\in(1,\infty)$, $p\in(0,\infty)$ such that $1/p=1/p_1+\dots+1/p_m$, and $\vec w=(w_1,\dots,w_m)\in A_{\vec P}(\mathbb{R}^{nm})$. Set $\sigma_i=w_i^{-1/(p_i-1)}$. Let $\mathscr{D}$ be a dyadic grid and $\mathscr{S}\subset\mathscr{D}$ be a sparse family. Then for $\beta_1,\dots,\beta_m\in[0,\infty)$,*

$$\|\mathscr{A}^q_{m,\mathscr{S},L(\log L)^{\vec\beta}}(f_1,\dots,f_m)\|_{L^p(\mathbb{R}^n,\nu_{\vec w})}\lesssim[\vec w]_{A_p}^{\max\{\frac1q,\frac{p_1'}{p},\dots,\frac{p_m'}{p}\}}\prod_{i=1}^m[\sigma_i]_{A_\infty}^{\beta_i}\prod_{j=1}^m\|f_j\|_{L^{p_j}(\mathbb{R}^n,w_j)}.$$

*Proof.* We employ the ideas used in the proof of Theorem 3.2 in [14]. As it is well known, $\vec w\in A_{\vec P}(\mathbb{R}^{mn})$ implies $\sigma_j=w_j^{-1/(p_j-1)}\in A_{mp_j'}(\mathbb{R}^n)$, and for $r_{\sigma_j}=1+\frac{1}{2^{11+n}[\sigma_j]_{A_\infty}}$,

$$\Big(\frac{1}{|Q|}\int_Q\sigma_j^{r_{\sigma_j}}(x)\mathrm{d}x\Big)^{\frac{1}{r_{\sigma_j}}}\leqslant2\frac{1}{|Q|}\int_Q\sigma_j(x)\mathrm{d}x,$$

see [11]. Let $\rho_j=(1+p_j)/2$. Recalling that for $\delta>1$

$$\|h\|_{L(\log L)^\rho,Q}\lesssim\max\Big\{1,\frac{1}{(\delta-1)^\rho}\Big\}\Big(\frac{1}{|Q|}\int_Q|h(y)|^\delta\mathrm{d}y\Big)^{\frac1\delta},$$

we then have by the generalization of Hölder's inequality that

$$\begin{aligned}\|f_j\sigma_j\|_{L(\log L)^{\beta_j},Q}&\lesssim\Big(\frac{1}{|Q|}\int_Q|f_j|^{\rho_j}\sigma_j\Big)^{\frac{1}{\rho_j}}\|\sigma_j^{\frac{1}{\rho_j'}}\|_{L^{\rho_j'}(\log L)^{\rho_j'\beta_j},Q}\\&\lesssim[\sigma_j]_{A_\infty}^{\beta_j}\Big(\frac{1}{|Q|}\int_Q|f_j|^{\rho_j}\sigma_j\Big)^{\frac{1}{\rho_j}}\Big(\frac{1}{|Q|}\int_Q\sigma_j\Big)^{\frac{1}{\rho_j'}}\\&\lesssim[\sigma_j]_{A_\infty}^{\beta_j}\inf_{y\in Q}M_{\sigma_j,\rho_j}f_j(y)\langle\sigma_j\rangle_Q,\end{aligned}\tag{2.9}$$

with

$$M_{\sigma_j,\rho_j}f_j(x)=\sup_{I\ni x,I\in\mathscr{D}}\Big(\frac{1}{\sigma_j(I)}\int_I|f_j(y)|^{\rho_j}\sigma_j(y)\mathrm{d}y\Big)^{\frac{1}{\rho_j}}.$$

Let

$$\langle h\rangle_Q^{\sigma_j}=\frac{1}{\sigma_j(Q)}\int_Q h(y)\sigma_j(y)\mathrm{d}y,$$

and define the operator $\widetilde{\mathscr{A}^q_{m,\mathscr{S}}}$ by

$$\widetilde{\mathscr{A}^q_{m,\mathscr{S}}}(f_1,\dots,f_m)(x)=\Big(\sum_{Q\in\mathscr{S}}\prod_{j=1}^m(\langle f_j\rangle_Q^{\sigma_j}\langle\sigma_j\rangle_Q)^q\chi_Q(x)\Big)^{\frac{1}{q}}.$$

Recall that $M_{\sigma_j,\rho_j}$ is bounded on $L^{p_j}(\mathbb{R}^n,\sigma_j)$ with bound depending only on $p_j$. Our proof is now reduced to proving that

$$\|\widetilde{\mathscr{A}^q_{m,\mathscr{S}}}(f_1,\dots,f_m)\|_{L^p(\mathbb{R}^n,\nu_{\vec{w}})}\lesssim[\vec{w}]_{A_p}^{\max\{\frac{1}{q},\frac{p_1'}{p},\dots,\frac{p_m'}{p}\}}\prod_{j=1}^m\|f_j\|_{L^{p_j}(\mathbb{R}^n,\sigma_j)}. \tag{2.10}$$

When $p\leqslant q$, the proof of (2.10) follows from the argument in [14, p. 757–759]. In fact, as in [14], we obtain that

$$\begin{aligned}\|\widetilde{\mathscr{A}^q_{m,\mathscr{S}}}(f_1,\dots,f_m)\|^p_{L^p(\mathbb{R}^n,\nu_{\vec{w}})}&\leqslant\sum_{Q\in\mathscr{S}}\prod_{j=1}^m(\langle f_j\rangle_Q^{\sigma_j}\langle\sigma_j\rangle_Q)^p\nu_{\vec{w}}(Q)\\&\lesssim[\vec{w}]_{A_p}^{\max\{p_1',\dots,p_m'\}}\prod_{j=1}^m\|f_j\|^p_{L^{p_j}(\mathbb{R}^n,\sigma_j)}.\end{aligned}$$

Now let $p>q$ and $\tau=\max\{1,q\frac{p_1'}{p},\dots,q\frac{p_m'}{p}\}$. It is obvious that $\tau p/p_j'\geqslant q$ and so

$$\{\sigma_j(Q)\}^{\tau\frac{p}{p_j'}-q}\geqslant\{\sigma_j(E_Q)\}^{\tau\frac{p}{p_j'}-q}.$$

By the fact that

$$|Q|\lesssim\nu_{\vec{w}}(E_Q)^{\frac{1}{mp}}\prod_{j=1}^m\{\sigma_j(E_Q)\}^{\frac{1}{mp_j'}}$$

(see [14, p. 758] for details), as in the proof of Theorem B in [2], a straightforward computation gives us that

$$\begin{aligned}&\sum_{Q\in\mathscr{S}}\prod_{j=1}^m(\langle f_j\rangle_Q^{\sigma_j}\langle\sigma_j\rangle_Q)^q\int_Q g(x)\nu_{\vec{w}}(x)\mathrm{d}x\\&\lesssim[\vec{w}]^\tau_{A_{\vec{P}}}\sum_{Q\in\mathscr{S}}\frac{|Q|^{m(\tau p-q)}}{(\nu_{\vec{w}}(Q))^\tau\prod_{i=1}^m\sigma_i(Q)^{\tau\frac{p}{p_i'}}}\int_Q g(x)\nu_{\vec{w}}(x)\mathrm{d}x\prod_{j=1}^m\Big(\int_Q f_j(y_j)\sigma_j(y_j)\mathrm{d}y_j\Big)^q\\&\lesssim[\vec{w}]^\tau_{A_{\vec{P}}}\sum_{Q\in\mathscr{S}}\Big(\frac{1}{\nu_{\vec{w}}(Q)}\int_Q g\nu_{\vec{w}}(x)\mathrm{d}x\Big)\big(\nu_{\vec{w}}(E_Q)\big)^{\frac{1}{(\frac{p}{q})'}}\\&\quad\times\prod_{j=1}^m\Big(\frac{1}{\sigma_j(Q)}\int_Q f_j(y_j)\sigma_j(y_j)\mathrm{d}y_j\Big)^q(\sigma_j(E_Q))^{\frac{q}{p_j}}\end{aligned}$$

$$\lesssim [\vec{w}]_{A_{\vec{P}}}^{\tau}\Big\{\sum_{Q\in\mathscr{S}}\Big(\frac{1}{\nu_{\vec{w}}(Q)}\int_Q g(x)\nu_{\vec{w}}(x)\mathrm{d}x\Big)^{(\frac{p}{q})'}\nu_{\vec{w}}(E_Q)\Big\}^{\frac{1}{(\frac{p}{q})'}}$$
$$\times\prod_{j=1}^{m}\Big\{\sum_{Q\in\mathscr{S}}\Big(\frac{1}{\sigma_j(Q)}\int_Q f_j(y_j)\sigma_j(y_j)\mathrm{d}y_j\Big)^{p_j}\sigma_j(E_Q)\Big\}^{\frac{q}{p_j}}$$
$$\lesssim [\vec{w}]_{A_{\vec{P}}}^{\tau}\prod_{j=1}^{m}\|f_j\|_{L^{p_j}(\mathbb{R}^n,\sigma_j)}^{q}\|g\|_{L^{(\frac{p}{q})'}(\mathbb{R}^n,\nu_{\vec{w}})}.$$

We then deduce (2.10) for $p>q$. This completes the proof of Theorem 2.5. □

LEMMA 2.6. *Let $\beta_1,\ldots,\beta_m\in[0,\infty)$ and $\vec{w}=(w_1,\ldots,w_m)\in A_{1,\ldots,1}(\mathbb{R}^{mn})$. Then for each $\lambda>0$,*

$$\nu_{\vec{w}}\big(\{x\in\mathbb{R}^n:\mathscr{M}_{L(\log L)^{\vec{\beta}}}(f_1,\ldots,f_m)(x)>\lambda\}\big)$$
$$\lesssim\prod_{j=1}^{m}\Big(\int_{\mathbb{R}^n}\frac{|f_j(x)|}{\lambda^{\frac{1}{m}}}\log^{|\beta|}\Big(1+\frac{|f_j(x)|}{\lambda^{\frac{1}{m}}}\Big)w_j(x)\mathrm{d}x\Big)^{\frac{1}{m}}.$$

This Lemma can be proved by repeating the argument in the proof of Theorem 3.17 in [13], see also [15]. We omit the details for brevity.

*Proof of Theorem* 1.2. Let $\mathscr{D}$ be a dyadic grid. We claim that for each $Q\subset\mathscr{D}$ and each $\lambda\in(0,1)$,

$$\omega_{\lambda}\big(\|\{\mathscr{M}_{\mathscr{D},L(\log L)^{\vec{\beta}}}(f_1^k,\ldots,f_m^k)\}\|_{l^q}^q,Q\big)\lesssim\prod_{j=1}^{m}\big\|\|\{f_j^k\}\|_{l^{q_j}}\big\|_{L(\log L)^{\beta_j},Q}^q. \tag{2.11}$$

In fact, we know from Lemma 2.2 and Lemma 2.4 that for each $\delta\in(0,\frac{1}{mq})$,

$$\Big(\frac{1}{|Q|}\int_Q\big\|\{M_{\mathscr{D},L(\log L)^{\beta_j}}(f_j^k\chi_Q)(x)\}\big\|_{l^{q_j}}^{m\delta}\mathrm{d}x\Big)^{\frac{1}{m\delta}}\lesssim\big\|\|\{f_j^k\}\|_{l^{q_j}}\big\|_{L(\log L)^{\beta_j},Q}. \tag{2.12}$$

Let $c_k=\sup_{I\ni x,\mathscr{D}\ni I\supset Q}\prod_{j=1}^m\|f_j^k\|_{L(\log L)^{\beta_j},I}$ and $D=\|\{c_k\}\|_{l^q}$. For $\delta\in(0,\frac{1}{mq})$, it follows from Hölder's inequality that

$$\Big(\frac{1}{|Q|}\int_Q\Big|\|\{\mathscr{M}_{\mathscr{D},L(\log L)^{\vec{\beta}}}(f_1^k,\ldots,f_m^k)(x)\}\|_{l^q}^q-D^q\Big|^{\delta}\mathrm{d}x\Big)^{\frac{1}{\delta}}$$
$$\lesssim\Big(\frac{1}{|Q|}\int_Q\|\{\mathscr{M}_{\mathscr{D},L(\log L)^{\vec{\beta}}}(f_1^k\chi_Q,\ldots,f_m^k\chi_Q)(x)\}\|_{l^q}^{q\delta}\mathrm{d}x\Big)^{\frac{1}{\delta}}$$
$$\lesssim\prod_{j=1}^{m}\Big(\frac{1}{|Q|}\int_Q\|\{M_{L(\log L)^{\beta_j}}(f_j^k\chi_Q)(x)\}\|_{l^{q_j}}^{mq\delta}\mathrm{d}x\Big)^{\frac{1}{m\delta}}.$$

This, via (2.12), yields (2.11).

We now prove (1.3). For each cube $Q \subset \mathscr{D}$, we deduce from Lemma 2.3, inequality (2.11) and Theorem 2.5 that

$$\int_Q \Big| \|\{\mathscr{M}_{\mathscr{D},L(\log L)^{\vec{\beta}}}(f_1^k,\cdots,f_m^k)(x)\}\|_{l^q}^q - m_{\|\{\mathscr{M}_{\mathscr{D},L(\log L)^{\vec{\beta}}}(f_1^k,\cdots,f_m^k)\}\|_{l^q}^q}(Q)\Big|^{\frac{p}{q}} \nu_{\vec{w}}(x)\mathrm{d}x$$
$$\lesssim \int_{\mathbb{R}^n} \Big(\mathscr{A}^q_{m;\mathscr{S},L(\log L)^{\vec{\beta}}}(\|f_1^k\|_{l^q},\cdots,\|f_m^k\|_{l^q})(x)\Big)^p \nu_{\vec{w}}(x)\mathrm{d}x$$
$$\lesssim \Big([\vec{w}]_{A_{\vec{p}}}^{\max\{\frac{1}{q},\frac{p_1'}{p},\cdots,\frac{p_m'}{p}\}} \prod_{i=1}^m [\sigma_i]_{A_\infty}^{\beta_i} \prod_{j=1}^m \|\{f_j^k\}\|_{L^{p_j}(l^q;\mathbb{R}^n,w_j)}\Big)^p,$$

with $\mathscr{S} \subset \mathscr{D}_0(Q)$ being a sparse family. As in the proof of Theorem 5.1 in [3], we then obtain from the last inequality that

$$\Big(\int_{\mathbb{R}^n} \|\{\mathscr{M}_{\mathscr{D},L(\log L)^{\vec{\beta}}}(f_1^k,\ldots,f_m^k)(x)\}\|_{l^q}^p \nu_{\vec{w}}(x)\mathrm{d}x\Big)^{\frac{1}{p}}$$
$$\lesssim [\vec{w}]_{A_{\vec{p}}}^{\max\{\frac{1}{q},\frac{p_1'}{p},\ldots,\frac{p_m'}{p}\}} \prod_{i=1}^m [\sigma_i]_{A_\infty}^{\beta_i} \prod_{j=1}^m \|\{f_j^k\}\|_{L^{p_j}(l^q;\mathbb{R}^n,w_j)}.$$

This, along with the one-third trick (see [10, Lemma 2.5]), leads to (1.3) for the case of $q \in (1,\infty)$.

It remains to prove (1.4). Again by the one-third trick, it suffices to prove that for each dyadic $\mathscr{D}$,

$$\nu_{\vec{w}}(\{x \in \mathbb{R}^n : \big\|\{\mathscr{M}_{\mathscr{D},L(\log L)^{\vec{\beta}}}(f_1^k,\ldots,f_m^k)(x)\}\big\|_{l^q} > 1\})$$
$$\lesssim_{\vec{w}} \prod_{j=1}^m \Big(\int_{\mathbb{R}^n} \|\{f_j^k(y_j)\}\|_{l^{q_j}} \log^{\beta_j}\big(1+\|\{f_j^k(y_j)\}\|_{l^{q_j}}\big) w_j(y_j)\mathrm{d}y_j\Big)^{\frac{1}{m}}. \tag{2.13}$$

Associated with $\mathscr{D}$, define the sharp maximal function $M_{\mathscr{D}}^{\sharp}$ as

$$M_{\mathscr{D}}^{\sharp} f(x) = \sup_{\substack{Q \ni x\\ Q \in \mathscr{D}}} \inf_{c \in \mathbb{C}} \frac{1}{|Q|}\int_Q |f(y)-c|dy.$$

For $\delta \in (0,1)$, let $M_{\mathscr{D},\delta}^{\sharp} f(x) = \big[M_{\mathscr{D}}^{\sharp}(|f|^{\delta})(x)\big]^{1/\delta}$. Repeating the argument in [17, p. 153], we can verify that if $u \in A_\infty(\mathbb{R}^n)$ and $\Phi$ is a increasing function on $[0,\infty)$ which satisfies that

$$\Phi(2t) \leqslant C\Phi(t), t \in [0,\infty),$$

then

$$\sup_{\lambda>0} \Phi(\lambda) u\big(\{x \in \mathbb{R}^n : |h(x)| > \lambda\}\big) \lesssim_u \sup_{\lambda>0} \Phi(\lambda) u\big(\{x \in \mathbb{R}^n : M_{\mathscr{D},\delta}^{\sharp} h(x) > \lambda\}\big),$$

provided that $\sup_{\lambda>0}\Phi(\lambda)u\big(\{x \in \mathbb{R}^n : M_{\mathscr{D},\delta}h(x) > \lambda\}\big) < \infty$. On the other hand, it follows from (2.11) that for each fixed $\delta \in (0,\min\{1,1/q\})$,

$$M_{\mathscr{D},\delta}^{\sharp}\big(\big\|\{\mathscr{M}_{\mathscr{D},L(\log L)^{\vec{\beta}}}(f_1^k,\ldots,f_m^k)\}\big\|_{l^q}\big)(x) \lesssim \mathscr{M}_{L(\log L)^{\vec{\beta}}}(\|\{f_1^k\}\|_{l^{q_1}},\ldots,\|\{f_m^k\}\|_{l^{q_m}})(x).$$

This, together with Lemma 2.6, gives us that

$$\begin{aligned}
&\nu_{\vec{w}}(\{x \in \mathbb{R}^n : \big\|\{\mathscr{M}_{\mathscr{D},L(\log L)^{\vec{\beta}}}(f_1^k,\ldots,f_m^k)(x)\}\big\|_{l^q} > 1\}) \\
&\lesssim \sup_{t>0} \psi(t)\nu_{\vec{w}}\big(\{x \in \mathbb{R}^n : M^{\sharp}_{\mathscr{D},\delta}\big(\big\|\{\mathscr{M}_{\mathscr{D},L(\log L)^{\vec{\beta}}}(f_1^k,\ldots,f_m^k)\}\big\|_{l^q}\big)(x) > t\}\big) \\
&\lesssim \sup_{t>0} \psi(t)\nu_{\vec{w}}\big(\{x \in \mathbb{R}^n : \mathscr{M}_{L(\log L)^{\vec{\beta}}}(\|\{f_1^k\}\|_{l^{q_1}},\ldots,\|\{f_m^k\}\|_{l^{q_m}})(x) > t\}\big) \\
&\lesssim \prod_{j=1}^m \Big(\int_{\mathbb{R}^n} \|\{f_j^k(y_j)\}\|_{l^{q_j}} \log^{|\vec{\beta}|}\big(1+\|\{f_j^k(y_j)\}\|_{l^{q_j}}\big) w_j(y_j)\mathrm{d}y_j\Big)^{\frac{1}{m}},
\end{aligned}$$

here we take $\psi(t) = t^{1/m}\log^{-m|\beta|}(1+t^{-1/m})$. This leads to (2.13) and completes the proof of Theorem 1.2. $\square$

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## REFERENCES

[1] K. F. ANDERSON AND R. T. JOHN, *Weighted inequality for vector-valued maximal functions and singular integrals*, Studia Math. **69** (1980), 19–31.

[2] T. A. BUI, J. CONDE-ALONSO, X. T. DUONG AND M. HORMOZI, *A note on weighted bounds for singular integral operators with nonsmooth kernels*, Studia Math. **236** (2017), 245–269.

[3] D. CRUZ-URIBE, SFO, J. MARTELL AND C. PÉREZ, *Sharp weighted estimates for classical operators*, Adv. Math. **229** (2012), 408–441.

[4] W. DAMIÁN, A. K. LERNER AND C. PÉREZ, *Sharp weighted bouns for multilinear maximal functions and Calderón-Zygmund operators*, J. Fourier Anal. Appl. **21** (2012), 751–765.

[5] C. FEFFERMAN AND E. M. STEIN, *Some maximal inequalities*, Amer. J. Math. **93** (1971), 107–115.

[6] L. GRAFAKOS, *Modern Fourier analysis, GTM250*, 2nd Edition, Springer, New York, 2008.

[7] L. GRAFAKOS AND J. M. MARTELL, *Extrapolation of weighted norm inequalities for multivariable operators and applications*, J. Geom. Anal. **14** (2004), 19–46.

[8] G. HU, *Weighted vector-valued estimates for a non-standard Calderón-Zygmund operator*, Nonlinear Anal. **165** (2017), 143–162.

[9] T. HYTÖNEN, *The $A_2$ theorem: Remarks and complements*, Contemp. Math. 612, Amer. Math. Soc. Providence, RI, 2014, 91–106.

[10] T. HYTÖNEN, M. T. LACEY AND C. PÉREZ, *Sharp weighted bounds for the $q$-variation of singular integrals*, Bull. Lond. Math. Soc. **45** (2013), 529–540.

[11] T. HYTÖNEN AND C. PÉREZ, *Sharp weighted bounds involving $A_\infty$*, Anal. PDE. **6** (2013), 777–818.

[12] A. K. LERNER, *A poinwisewise estimate for the local sharp maximal function with applications to singular integrals*, Bull. London. Math. Soc. **42** (2010), 843–856.

[13] A. LERNER, S. OMBROSI, C. PÉREZ, R. H. TORRES AND R. TRUJILLO-GONZÁLEZ, *New maximal functions and multiple weights for the multilinear Calderón-Zygmund theorey*, Adv. Math. **220** (2009), 1222–1264.

[14] K. LI, K. MOEN AND W. SUN, *The sharp weighted bound for multilinear maximal functions and Calderón-Zygmund operators*, J. Four. Anal. Appl. **20** (2014), 751–765.

[15] C. PÉREZ, G. PRADOLINI, R. H. TORRES AND R. TRUJILLO-GONZÁLEZ, *End-point estimates for iterated commutators of multilinear singular integrals*, Bull. London Math. Soc. **46** (2010), 26–42.

[16] M. RAO AND Z. REN, *Theory of Orlicz spaces*, Monographs and Textbooks in Pure and Applied Mathematics, 146, Marcel Dekker Inc., New York, 1991.
[17] E. M. STEIN, *Harmonic Analysis, Real Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Univ. Press, Princeton, NJ. 1993.

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