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ON THEOREMS OF MORGAN AND COWLING–PRICE FOR SELECTED NILPOTENT LIE GROUPS

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Abstract. Let G be a connected, simply connected nilpotent Lie group. For $p,q \in [1,+\infty]$, the $L^p - L^q$ analogue of Morgan's theorem was proved only for two step nilpotent Lie groups. In order to study this problem in larger subclasses, we formulate and prove a version of $L^p - L^q$ Morgan's theorem on nilpotent Lie groups whose Lie algebra admits an ideal which is a polarization for a dense subset of generic linear forms on the Lie algebra. A proof of an analogue of Cowling-Price Theorem is also provided in the same context.

1. Introduction

A classical aspect of an uncertainty principle affirms that a non-zero integrable function f on the real line and its Fourier transform $\hat{f}$, defined by

$$\hat{f}(y) = \int_{\mathbb{R}} f(x)\mathrm{e}^{2i\pi xy}dx,\ y \in \mathbb{R},$$

cannot both be sharply localized.

An important result making this precise is Hardy's Theorem (see [11]):

THEOREM 1. *Let a,b,c be positive real numbers and f a measurable function on $\mathbb{R}$ such that:*

(i) $|f(x)| \leqslant ce^{-a\pi x^2},\ \ x \in \mathbb{R}$,

(ii) $|\hat{f}(y)| \leqslant ce^{-b\pi y^2},\ \ y \in \mathbb{R}$.

If $ab > 1$, then $f = 0$ almost everywhere. If $ab = 1$ then $f(x) = ke^{-a\pi x^2}$, for some constant k. If $ab < 1$, then there are infinitely many linearly independent functions satisfying (i) and (ii).

Cowling and Price (see [9]) generalized this theorem by replacing point wise Gaussian bounds for f by Gaussian bounds in L^p sense and in L^q sense for $\hat{f}$ as well. More precisely, they proved the following:

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THEOREM 2. *Let a,b be positive real numbers and f a measurable function on $\mathbb{R}$ such that:*

(i) $\|e^{a\pi x^2} f(x)\|_p < +\infty$,

(ii) $\|e^{b\pi y^2} \hat{f}(y)\|_q < +\infty$,

where $1 \leqslant p,q \leqslant +\infty$ such that $\min(p,q)$ is finite. If $ab \geqslant 1$, then $f = 0$ almost everywhere. If $ab < 1$, then there are infinitely many linearly independent functions satisfying (i) and (ii).

With the purpose of obtaining an even more general variant of Hardy's uncertainty principle, Morgan's Theorem (see [13]) involved the generalized Gaussians:

THEOREM 3. *Let a,b be positive real numbers and f a measurable function on $\mathbb{R}$. Let $p' > 2$ and q' be the conjugate of p', that is $\dfrac{1}{p'} + \dfrac{1}{q'} = 1$. Suppose that,*

$$e^{a\pi |x|^{p'}} f \in L^\infty(\mathbb{R}) \text{ and } e^{b\pi |y|^{q'}} \hat{f} \in L^\infty(\mathbb{R}).$$

Then $f = 0$ almost everywhere, if $(ap')^{\frac{1}{p'}} (bq')^{\frac{1}{q'}} > 2\left(\sin\left(\frac{\pi}{2}(q'-1)\right)\right)^{\frac{1}{q'}}$.

In [7], Ben Farah and K. Mokni provided a version referred to as the $L^p - L^q$ Morgan uncertainty principle and has the advantage to unify the last two principles:

THEOREM 4. *Let f be a measurable function on $\mathbb{R}$. Suppose for some $a,b > 0$, $p,q \in [1,+\infty]$, $p' > 2$, and q' such that $\dfrac{1}{p'} + \dfrac{1}{q'} = 1$, f satisfies*

$$e^{\pi a |x|^{p'}} f \in L^p(\mathbb{R}) \text{ and } e^{\pi b |y|^{q'}} \hat{f} \in L^q(\mathbb{R}).$$

If moreover $(ap')^{\frac{1}{p'}} (bq')^{\frac{1}{q'}} > 2\left(\sin\left(\frac{\pi}{2}(q'-1)\right)\right)^{\frac{1}{q'}}$ *then $f = 0$ almost everywhere.*

Several analogues for these aforementioned results have been obtained for some kinds of non-commutative Lie groups (see [2,4,8,16] etc.). Our attention in the first part of this paper is focused on the study of $L^p - L^q$ Morgan's theorem in the setup of connected, simply connected nilpotent Lie groups. Recently, F. Abdelmoula and A. Baklouti produced in [1] an analogue of $L^p - L^q$ Morgan's theorem for connected nilpotent Lie groups. However, their upshots hold only with an unnatural restriction on the hypothesis. They imposed that the real numbers p,q belong to $[2,+\infty]$. The problem when $1 \leqslant p,q \leqslant +\infty$ is only solved for Heisenberg group (see [7]) and two step nilpotent Lie groups (see [14]). For general nilpotent Lie groups the problem seems to be subtle and delicate and the difficulties involved in this problem are considerable.

One of the aims of this paper is to formulate and prove an analogue of $L^p - L^q$ Morgan's theorem when $1 \leqslant p,q \leqslant +\infty$ for a large subclass of nilpotent Lie groups. It concerns Lie groups admitting an ideal $\mathfrak{b}$ which polarizes all generic orbits. Our proof make use of the orbit method and the Plancherel theory.

The second part of this paper is devoted to Cowling-Price Theorem. In [15], Ray proved an analogue of this theorem for two step nilpotent Lie groups with the assumption $1 \leqslant p \leqslant +\infty$, $q \geqslant 2$ and $ab > 1$. Baklouti and Ben Salah [5] treated the situation when G is an arbitrary connected, simply connected nilpotent Lie group in the case where $p,q \in [2,+\infty]$ and $ab > 1$. To prove the sharpness of the constant 1, Baklouti and Thangavelu [6] provided a variant of Cowling-Price Theorem in the same context. However, their result holds for $p,q \in [2,+\infty]$. The second part of the paper aims to investigate the sharpness of the constant 1 in Cowling-Price Theorem with the original condition $p,q \in [1,+\infty]$, for nilpotent Lie groups admitting a common polarization ideal for all generic orbits.

2. Backgrounds

We begin this section by reviewing some useful facts and notations for nilpotent Lie group. This material is quite standard, we refer the reader to [10] for details. Throughout, $\mathfrak{g}$ will be a $n-$dimensional real nilpotent Lie algebra, G will be the associated connected, and simply connected nilpotent Lie group. The exponential map $\exp : \mathfrak{g} \to G$ is a global C^∞-diffeomorphism from $\mathfrak{g}$ into G.

2.1. The Kirillov theory

Let $\mathfrak{g}^*$ be the vector dual space of $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ acts on $\mathfrak{g}$ by the adjoint representation $\mathrm{ad}_{\mathfrak{g}}$, i.e.,

$$\mathrm{ad}_{\mathfrak{g}}(X)Y = \mathrm{ad}(X)Y = [X,Y],\ \forall\, X,\ Y \in \mathfrak{g}.$$

The group G acts on $\mathfrak{g}$ by the adjoint representation Ad_G, that is:

$$\mathrm{Ad}_G(g)Y = \mathrm{Ad}(g)Y = \mathrm{e}^{\mathrm{ad}(X)}Y,\ g = \exp X \in G,\ Y \in \mathfrak{g},$$

and on $\mathfrak{g}^*$ by the coadjoint representation Ad^*_G, i.e.

$$\langle \mathrm{Ad}^*_G(g)l, X\rangle = \langle g\cdot l, X\rangle = \langle l, \mathrm{Ad}_G(g^{-1})X\rangle,\ g \in G,\ \ l \in \mathfrak{g}^*,\ X \in \mathfrak{g}.$$

The coadjoint orbit of l is the set $G\cdot l = \{g\cdot l : g \in G\}$. The unitary dual $\hat{G}$ of G is parameterized via the Kirillov orbit method by the space of coadjoint orbits $\mathfrak{g}^*/G$. A subspace $\mathfrak{b} = \mathfrak{b}(l)$ of the Lie algebra $\mathfrak{g}$ is called a polarization for $l \in \mathfrak{g}^*$ if $\mathfrak{b}$ is a maximal dimension isotropic subalgebra with respect to the skew-symmetric bilinear form B_l defined by:

$$B_l(X,Y) = l([X,Y]),\ \ X,Y \in \mathfrak{g}.$$

So we can consider the unitary character χ_l of $B = \exp \mathfrak{b}$ associated to l defined by:

$$\chi_l(\exp \mathrm{X}) = \mathrm{e}^{2i\pi\langle l,X\rangle},\ X \in \mathfrak{b}.$$

The irreducible unitary representation $\pi_{l,\mathfrak{b}} = \mathrm{Ind}_B^G \chi_l$ is defined by letting G act on the right and its class $[\pi_{l,\mathfrak{b}}]$ depends only on the coadjoint orbit of l. Moreover, every irreducible unitary representation π is equivalent to an induced representation $\pi_{l,\mathfrak{b}}$ for some $l \in \mathfrak{g}^*$ and a polarization $\mathfrak{b}$ at l. The unitary dual $\hat{G}$ is homeomorphic to $\mathfrak{g}^*/G$ when these spaces are endowed with their usual topologies.

2.2. Plancherel formula

Let $\mathscr{B}=\{X_1,\ldots,X_n\}$ be a strong Malcev basis of $\mathfrak{g}$. Let $\mathfrak{g}_j=\mathbb{R}-span\{X_1,\ldots,X_j\}$, $j=1,\ldots,n$ and $\mathfrak{g}_0=\{0\}$. Let $\mathfrak{g}(l)=\{X\in\mathfrak{g}:\langle l,[X,\mathfrak{g}]\rangle=\{0\}\ \}$ be the stabilizer of $l\in\mathfrak{g}^*$ in $\mathfrak{g}$. An index $j\in\{1,\ldots,n\}$ is a jump index for l if

$$\mathfrak{g}(l)+\mathfrak{g}_j\neq\mathfrak{g}(l)+\mathfrak{g}_{j-1}.$$

We let

$$e(l)=\{j: j \text{ is a jump index for } l\}.$$

This set contains exactly $\dim(G\cdot l)$ indices, which is necessarily an even number. Even more, there are two disjoint sets of indices S,T with $S\cup T=\{1,\ldots,n\}$, and a G invariant Zariski open set $\mathscr{U}$ of $\mathfrak{g}^*$ (set of generic elements in the sense of Pukanszky) such that $e(l)=S$ for all $l\in\mathscr{U}$. Let $Pf(l)$ denote the Pfaffian of the skew-symmetric matrix $M_S(l)=(l([X_i,X_j]))_{i,j\in S}$. Then, one has that:

$$|Pf(l)|^2=\det M_S(l).$$

Let $\mathscr{B}^*=\{X_1^*,\ldots,X_n^*\}$ be the basis of $\mathfrak{g}^*$ dual to the basis $\mathscr{B}$. Let $V_T=\mathbb{R}-span\{X_i^*: i\in T\}, V_S=\mathbb{R}-span\{X_i^*: i\in S\}$ and dl be the Lebesgue measure on V_T such that the unit cube spanned by $\{X_i^*: i\in T\}$ has volume 1. Then, $\mathscr{W}=\mathscr{U}\cap V_T$ is a cross section of the generic orbits and $\mathscr{W}$ supports the Plancherel measure on $\hat{G}$. Furthermore, if dl is the Lebesgue measure on $\mathscr{W}$, then $d\mu=|Pf(l)|dl$ is a Plancherel measure for $\hat{G}$. Let dg be the Haar measure on G. For $\varphi\in L^1(G)\cap L^2(G)$, the Plancherel formula reads:

$$\|\varphi\|_2^2=\int_G|\varphi(g)|^2dg=\int_{\mathscr{W}}\|\pi_l(\varphi)\|_{HS}^2d\mu(l),$$

where $\pi_l(\varphi)=\displaystyle\int_G\varphi(g)\pi_l(g)dg$ and $\|\pi_l(\varphi)\|_{HS}$ denotes the Hilbert-Schmidt norm of the operator $\pi_l(\varphi)$.

2.3. Norm function on nilpotent Lie groups

Using the strong Malcev coordinates of the group G, we introduce a *norm function* on G by setting for $x=\exp x_1X_1\ldots\exp x_nX_n\in G,\ x_j\in\mathbb{R}$:

$$\mathscr{N}(x)=\sqrt{(x_1^2+\ldots+x_n^2)}.$$

The map:

$$\mathbb{R}^n\to G,\ (x_1,\ldots,x_n)\mapsto\prod_{j=1}^n\exp x_jX_j$$

is a diffeomorphism and maps the Lebesgue measure on $\mathbb{R}^n$ to the Haar measure on G. In this setup, we shall identify G as set with $\mathbb{R}^n$. We consider the Euclidean norm of $\mathfrak{g}^*$ with respect to the basis $\mathscr{B}^*$, that is,

$$\|\sum_{j=1}^n l_jX_j^*\|=\sqrt{(l_1^2+\ldots+l_n^2)}=\|l\|,\ \ l_j\in\mathbb{R}.$$

3. The $L^p - L^q$ analogue of Morgan's theorem

We start this section by the following definition:

DEFINITION 1. Let $G = \exp\mathfrak{g}$ be a connected, simply connected nilpotent Lie group. We say that G satisfies the ideal polarization condition if there exists an ideal $\mathfrak{b}$ of $\mathfrak{g}$ which is a polarization for every l in a G-invariant dense subset of $\mathfrak{g}^*$.

Let's take all the conventions and notations of section 2. From now on, G denotes a nilpotent Lie group satisfying the ideal polarization condition. We assume that the fixed ideal $\mathfrak{b} = \mathrm{Lie}(B)$ is a polarization for every l in the set of generic elements $\mathscr{U}$. In particular, the ideal $\mathfrak{b}$ has to be abelian, as $[\mathfrak{b},\mathfrak{b}]$ is annihilated by a dense subset of $\mathfrak{g}^*$. We choose a strong Malcev basis $\mathscr{B} = \{X_1,\dots,X_n\}$ such that $\mathfrak{b} = \mathbb{R}-\mathrm{span}\{X_1,\dots,X_m\}$, for some $m < n$. With respect to the basis $\mathscr{B}$, we will let $S = \{j_1 < \dots < j_{2d}\}$, $T = \{t_1 < \dots < t_r\}$ denote the collection of jump and non-jump indices respectively. Then, the index set T is included in $\{1,\dots,m\}$ and $\{m+1,\dots,n\} = \{j_{d+1} < \dots < j_{2d}\}$. Let $S_{\mathfrak{b}} = \{j_1 < \dots < j_d\}$ and $V_{S_{\mathfrak{b}}} = \mathbb{R}-\mathrm{span}\{X_i^* : i \in S_{\mathfrak{b}}\}$. Let $\mathfrak{b}^*$ be the vector dual space of $\mathfrak{b}$. Hence, $\mathfrak{b}^* = V_T \oplus V_{S_{\mathfrak{b}}}$. As $\mathfrak{b}$ is a polarization for l, the stabilizer $\mathfrak{g}(l) \subset \mathfrak{b}$. This means that the coadjoint orbit of l is saturated in the directions $X_{m+1}^*,\dots,X_n^*$, i. e.

$$G\cdot l \equiv G\cdot l_{|\mathfrak{b}} + \mathfrak{b}^{\perp},$$

where $\mathfrak{b}^{\perp} = \{\xi \in \mathfrak{g}^* : \langle \xi, \mathfrak{b}\rangle \equiv 0\}$. Let's make the following conventions: For any $\xi \in \mathfrak{g}^*$ we write $\xi_{|\mathfrak{b}}$ for the element of $\mathfrak{g}^*$ such that:

$$\begin{cases} \langle \xi_{|\mathfrak{b}}, X_j\rangle = \langle \xi, X_j\rangle & \text{if } \quad 1 \leqslant j \leqslant m, \\ \langle \xi_{|\mathfrak{b}}, X_j\rangle = 0 & \text{if } m+1 \leqslant j \leqslant n. \end{cases}$$

Similarly, we identify $G\cdot l_{|\mathfrak{b}}$ with a subset of $\mathfrak{g}^*$. Considering the coadjoint action of G on $\mathfrak{g}^*$, we get parametrization of generic orbits in $\mathscr{U}$. From Theorem 3.1.9 of [10], there is a diffeomorphism $\psi : \mathscr{U}\cap V_T \times V_S \longrightarrow \mathscr{U}$ such that the Jacobian determinant is identically 1. If we identify $(u,\lambda) = \left(\sum_{k=1}^r u_k X_{t_k}^*, \sum_{h=1}^{2d} \lambda_h X_{j_h}^*\right)$ with $(u_1,\dots,u_r,\lambda_1,\dots,\lambda_{2d}) \in \mathbb{R}^r \times \mathbb{R}^{2d}$, we have:

$$\psi(u,\lambda) = \sum_{j=1}^{n} P_j(u,\lambda) X_j^*,$$

where:

(i) The P_j are rational, non singular on $\mathscr{U}\cap V_T \times \mathbb{R}^{2d}$.

(ii) If $j = t_k$,

$$P_j(u,\lambda) = u_k + R_{t_k}(u_1,\dots,u_{k-1},\lambda_1,\dots,\lambda_c), \tag{1}$$

where c is the largest index such that $j_c < t_k$. Moreover, $P_1(u,\lambda) = u_1$.

(iii) $P_{j_h}(u,\lambda) = \lambda_h$, $1 \leqslant h \leqslant 2d$.

Let $\mathscr{U}_{\mathfrak{b}} = \{u_{|\mathfrak{b}} : u \in \mathscr{U}\}$. Since $P_j(u,\lambda)$, $1 \leqslant j \leqslant m$, does not depend on $\lambda_{d+1}, \ldots, \lambda_{2d}$, the map $\mathscr{U} \cap V_T \times V_{S_{\mathfrak{b}}} \to \mathscr{U}_{\mathfrak{b}} : (u, \lambda_1, \ldots, \lambda_d) \mapsto \psi(u, \lambda_1, \ldots, \lambda_d, 0, \ldots, 0)$ is a diffeomorphism.

On the other hand, the map $\gamma : \mathbb{R}^d \mapsto G$ given by:

$$\gamma(s) = \gamma(s_1, \ldots, s_d) = \exp(s_1 X_{m+1}) \ldots \exp(s_d X_n)$$

is a cross-section for $B \setminus G$. Let φ be a complex valued function defined on G. For each fixed $\gamma(s)$, let $\varphi_{\gamma(s)}$ be the function defined on B by:

$$\varphi_{\gamma(s)}(z) = \varphi_{\gamma(s)}(z_1, \ldots, z_m) = \varphi\big(z\gamma(s)\big) = \varphi\Big(z \exp(s_1 X_{m+1}) \ldots \exp(s_d X_n)\Big),$$

where $z = \exp(z_1 X_1) \ldots \exp(z_m X_m) \in B$. Now for $\phi \in L^1(B)$ and $\mu = \sum_{j=1}^{m} \mu_j X_j^* \in \mathfrak{b}^*$, let

$$\widehat{\phi}(\mu) = \int_B \phi(z) \chi_\mu(z) dz = \int_{\mathbb{R}^m} \phi\Big(\exp(z_1 X_1) \ldots \exp(z_m X_m)\Big) e^{2i\pi \sum_{j=1}^m \mu_j z_j} dz_1 \ldots dz_m.$$

We now prove the following result, which is of major importance in the sequel.

LEMMA 1. *For $f \in L^1(G) \cap L^2(G)$ and $l \in \mathscr{W} = \mathscr{U} \cap V_T$,*

$$\|\pi_l(f)\|_{HS}^2 = \frac{1}{|Pf(l)|} \int_{\mathbb{R}^d} \int_{V_{S_{\mathfrak{b}}}} \left| \widehat{f_{\gamma(s)}}\big(\psi(l,\lambda)\big) \right|^2 d\lambda\, ds. \tag{2}$$

Proof. First of all, remark that the operator

$$\pi_l(f) = \int_G f(g) \pi_l(g) dg$$

is a kernel operator: It is of the form

$$\pi_l(f)\zeta(x) = \int_{B\setminus G} K(l,x,y) \zeta(y) d\dot{y},$$

where ζ belongs to the Hilbert space of the representation π_l and

$$K(l,x,y) = \int_B f(x^{-1} z y) \chi_l(z) dz.$$

The operator kernel K satisfies the covariance relation:

$$K(l, zx, z'y) = \overline{\chi_l(z)} \chi_l(z') K(l,x,y), \quad \forall z, z' \in B.$$

The Hilbert-Schmidt norm of $\pi_l(f)$ is given by:

$$\begin{aligned}\|\pi_l(f)\|_{HS}^2 &= \int_{B\backslash G\times B\backslash G} |K(l,x,y)|^2 d\dot{x}d\dot{y}\\ &= \int_{B\backslash G\times B\backslash G} \left|\int_B f(x^{-1}zy)\chi_l(z)dz\right|^2 d\dot{x}d\dot{y}\\ &= \int_{B\backslash G\times B\backslash G} \left|\int_B f(zx^{-1}y)\chi_l(xzx^{-1})dz\right|^2 d\dot{x}d\dot{y}\\ &= \int_{B\backslash G\times B\backslash G} \left|\int_B f(zx^{-1}y)\chi_{x^{-1}\cdot l_{|\mathfrak{b}}}(z)dz\right|^2 d\dot{x}d\dot{y}\\ &= \int_{B\backslash G\times B\backslash G} \left|\int_B f(zy)\chi_{x^{-1}\cdot l_{|\mathfrak{b}}}(z)dz\right|^2 d\dot{x}d\dot{y}\end{aligned}$$

(by substituting $x^{-1}y$ for y). Hence, we get that:

$$\|\pi_l(f)\|_{HS}^2 = \int_{\mathbb{R}^d}\int_{B\backslash G} \left|\widehat{f_{\gamma(s)}}\left(x^{-1}\cdot l_{|\mathfrak{b}}\right)\right|^2 d\dot{x}ds. \tag{3}$$

Let $\mathfrak{b}^l = \{X\in\mathfrak{g} : \langle l,[X,\mathfrak{b}]\rangle \equiv \{0\}\}$ and $S' = \{1\leqslant j\leqslant n : \mathfrak{g}_{j-1}+\mathfrak{b}^l \neq \mathfrak{g}_j+\mathfrak{b}^l\}$. The maximality of $\mathfrak{b}$, implies that $\mathfrak{b}^l = \mathfrak{b}$, and so the set $S' = \{j_{d+1},\ldots,j_{2d}\}$. Moreover, the jacobian determinant of the map $B\setminus G \to \{l\}\times V_{S_\mathfrak{b}} : \dot{x}\mapsto \psi^{-1}(x^{-1}\cdot l_{|\mathfrak{b}}))$ is

$$\left|\det\left(x^{-1}\cdot l_{|\mathfrak{b}}([X_i,X_j])_{(i,j)\in S_\mathfrak{b}\times S'}\right)\right| = \left|\det\left(l([X_i,X_j])_{(i,j)\in S_\mathfrak{b}\times S'}\right)\right|.$$

As $\mathfrak{b}$ is abelian,

$$\begin{aligned}\mathrm{Pf}(l)^2 &= \det\left(l([X_i,X_j])_{i,j\in S}\right)\\ &= \det\begin{pmatrix} (0) & l([X_j,X_i])_{j_{d+1}\leqslant j\leqslant j_{2d},\ j_1\leqslant i\leqslant j_d}\\ l([X_i,X_j])_{j_1\leqslant i\leqslant j_d,\ j_{d+1}\leqslant j\leqslant j_{2d}} & (*)\end{pmatrix}\\ &= \left(\det\left(l([X_i,X_j])_{i\in S_\mathfrak{b},j\in S'}\right)\right)^2.\end{aligned}$$

Thus,

$$\int_{V_{S_\mathfrak{b}}} \left|\widehat{f_{\gamma(s)}}(\psi(l,\lambda))\right|^2 d\lambda = |\mathrm{Pf}(l)|\int_{B\backslash G}\left|\widehat{f_{\gamma(s)}}\left(x^{-1}\cdot l_{|\mathfrak{b}}\right)\right|^2 d\dot{x}, \tag{4}$$

where $d\lambda$ is the Lebesgue measure on $V_{S_\mathfrak{b}}$. In view of equations (3) and (4),

$$\|\pi_l(f)\|_{HS}^2 = \frac{1}{|\mathrm{Pf}(l)|}\int_{\mathbb{R}^d}\int_{V_{S_\mathfrak{b}}} \left|\widehat{f_{\gamma(s)}}(\psi(l,\lambda))\right|^2 d\lambda ds,$$

which is the desired formula. $\square$

We shall now prove an analogue of $L^p - L^q$ Morgan's theorem for a large subclass of nilpotent Lie groups.

THEOREM 5. *Let G be a connected, simply connected nilpotent Lie group. Let's assume that G satisfies the ideal polarization condition. Let f be a square integrable function on G satisfying the following decay conditions:*

$$\int_G |f(x)|^p e^{\pi ap\mathcal{N}(x)^{p'}} dx < +\infty, \tag{5}$$

$$\int_{\mathscr{W}} \|\pi_l(f)\|_{HS}^q e^{\pi bq\|l\|^{q'}} |Pf(l)|^{\frac{q}{2}} dl < +\infty, \tag{6}$$

where $a,b > 0$, $p,q \in [1,+\infty]$, $p' \geqslant 2$, and q' such that $\frac{1}{p'} + \frac{1}{q'} = 1$. Then f vanishes almost everywhere whenever $(ap')^{\frac{1}{p'}}(bq')^{\frac{1}{q'}} > 2\Big(\sin\big(\frac{\pi}{2}(q'-1)\big)\Big)^{\frac{1}{q'}}$.

Proof. First of all, we mention that it is sufficient to consider the case where p, q are both finite and therefore the case $p = q = 1$. In fact, we can choose $a' < a$, $b' < b$ such that $(a'p')^{\frac{1}{p'}}(b'q')^{\frac{1}{q'}} > 2\Big(\sin\big(\frac{\pi}{2}(q'-1)\big)\Big)^{\frac{1}{q'}}$ and use Hölder's inequality to show that

$$\int_G |f(x)| \mathrm{e}^{\pi a'\mathcal{N}(x)^{p'}} dx < +\infty, \tag{7}$$

$$\int_{\mathscr{W}} \|\pi_l(f)\|_{HS} \mathrm{e}^{\pi b'\|l\|^{q'}} |\mathrm{Pf}(l)|^{\frac{1}{2}} dl < +\infty. \tag{8}$$

The mechanism of our proof basically consists in bringing the study of the function f defined on the group G to the study of new function defined on $\mathbb{R}$ satisfying equivalent conditions.

By condition (8) of our hypothesis and lemma 1, we have

$$\begin{aligned} +\infty &> \int_{\mathscr{W}} \|\pi_l(f)\|_{HS} \mathrm{e}^{\pi b'\|l\|^{q'}} |\mathrm{Pf}(l)|^{\frac{1}{2}} dl \\ &\geqslant \int_{\mathscr{W}} \|\pi_l(f)\|_{HS} \mathrm{e}^{\pi b'|l_1|^{q'}} |\mathrm{Pf}(l)|^{\frac{1}{2}} dl \\ &= \int_{\mathscr{W}} \Big(\int_{\mathbb{R}^d}\int_{V_{S_{\mathfrak{b}}}} |\widehat{f_{\gamma(s)}}(\psi(l,\lambda))|^2 d\lambda ds\Big)^{\frac{1}{2}} \mathrm{e}^{\pi b'|l_1|^{q'}} dl \\ &\geqslant \Big(\int_{\mathbb{R}^d}\int_{V_{S_{\mathfrak{b}}}} \Big(\int_{\mathscr{W}} |\widehat{f_{\gamma(s)}}(\psi(l,\lambda))| \mathrm{e}^{\pi b'|l_1|^{q'}} dl\Big)^2 d\lambda ds\Big)^{\frac{1}{2}} \end{aligned}$$

(using the generalized Minkowsky inequality). Then, by substituting $l_k + R_{t_k}(l,\lambda)$ for l_k, by means of equation (1), $k = 2,\dots,r$, we get that

$$\Big(\int_{\mathbb{R}^d}\int_{V_{S_{\mathfrak{b}}}} \Big(\int_{\mathscr{W}} |\widehat{f_{\gamma(s)}}(l+\lambda)| \mathrm{e}^{\pi b'|l_1|^{q'}} dl\Big)^2 d\lambda ds\Big)^{\frac{1}{2}} < +\infty,$$

where $l+\lambda = \sum_{k=1}^r l_k X_{t_k}^* + \sum_{h=1}^{2d} \lambda_h X_{j_h}^*$. It results that,

$$\int_{\mathbb{R}} |\widehat{f_{\gamma(s)}}(l+\lambda)| \mathrm{e}^{\pi b'|l_1|^{q'}} dl_1 < +\infty,$$

for almost all $l_2,\dots,l_r \in \mathbb{R}$, $\lambda \in V_{S_\mathfrak{b}}$ and $s_1,\dots,s_d \in \mathbb{R}$. Let $F_{\gamma(s)}$ be the function defined on $\mathbb{R}$ by:

$$F_{\gamma(s)}(z_1) = \int_{\mathbb{R}^{m-1}} f_{\gamma(s)}(z_1,\dots,z_m) \mathrm{e}^{2i\pi\left(\sum_{k=2}^{r} z_{t_k} l_k + \sum_{h=1}^{d} z_{j_h}\lambda_h\right)} dz_2 \dots dz_m,$$

for fixed $l_k \in \mathbb{R}$, $k = 2,\dots,r$, $\lambda \in V_{S_\mathfrak{b}}$ and $s_1,\dots,s_d \in \mathbb{R}$. The function $F_{\gamma(s)}$ is integrable and $\widehat{F}_{\gamma(s)}(l_1) = \widehat{f_{\gamma(s)}}(l+\lambda)$. Then obviously,

$$\int_{\mathbb{R}} |\widehat{F}_{\gamma(s)}(l_1)| \mathrm{e}^{\pi b'|l_1|^{q'}} dl_1 < +\infty.$$

On the other hand,

$$\begin{aligned}
\int_{\mathbb{R}} |F_{\gamma(s)}(z_1)| \mathrm{e}^{\pi a'|z_1|^{p'}} dz_1 &\leqslant \int_{\mathbb{R}^m} |f_{\gamma(s)}(z_1,\dots,z_m)| \mathrm{e}^{\pi a'|z_1|^{p'}} dz_1 \dots dz_m \\
&= \int_{\mathbb{R}^m} \left| f\Big(\exp(z_1X_1)\dots\exp(z_mX_m)\gamma(s)\Big)\right| \mathrm{e}^{\pi a'|x_1|^{p'}} dz_1 \dots dz_m \\
&< +\infty,
\end{aligned}$$

by condition (7). Hence $F_{\gamma(s)}$ vanishes almost everywhere, using Theorem 4 as $(a'p')^{\frac{1}{p'}}$ $(b'q')^{\frac{1}{q'}} > 2\Big(\sin\big(\frac{\pi}{2}(q'-1)\big)\Big)^{\frac{1}{q'}}$. In the case where $p' = 2$, we use Theorem 2. This implies that, $\widehat{f_{\gamma(s)}} = 0$ almost everywhere, and then $\|\pi_l(f)\|_{HS} = 0$. It follows, using the Plancherel formula, that $f = 0$ almost everywhere. $\square$

4. On the Cowling-Price Theorem

As we mention before, considerable attention have been devoted to give an independent proof of Cowling-Price Theorem in context of nilpotent Lie groups. Nevertheless none of these works solve the sharpness problem in this theorem. For general nilpotent Lie groups Baklouti and Thangavelu proved an analogue of Cowling-Price Theorem with the sharpness of the constant 1 as a consequence of Miyachi's theorem (Theorem 2.4 in [6]). Their result covers only the case where $p,q \in [2,+\infty]$:

THEOREM 6. *Let G be a connected, simply connected nilpotent Lie group and $p,q \in [2,+\infty]$ such that $\min(p,q) < +\infty$. Let f be a measurable function on G such that, for some positive numbers a and b,*

$$\int_G |f(x)|^p e^{\pi ap\|x\|^2} dx < +\infty, \tag{9}$$

$$\int_{\mathscr{W}} \|\pi_l(f)\|_{HS}^q e^{\pi bq\|l\|^2} |Pf(l)|^{\frac{q}{2}} dl < +\infty. \tag{10}$$

Then f is zero almost everywhere on G whenever $ab \geqslant 1$.

Here the function norm $x \mapsto \|x\|$ is defined as follows: let $\{X_1, \ldots, X_n\}$ be a strong Malcev basis of $\mathfrak{g}$ and use it to define exponential coordinates on G. For $x = \exp\left(x_1X_1 + \ldots + x_nX_n\right) \in G$, $\|x\| = \sqrt{x_1^2 + \ldots . + x_n^2}$. Now, we are going to give an independent proof of Cowling-Price theorem for a large subclass of nilpotent Lie groups which also shows that the constant 1 is sharp for $q \in [1,2]$. For technical reasons, we will use the norm function $\mathscr{N}(\cdot)$ instead of $\|\cdot\|$.

THEOREM 7. *Let G be a connected, simply connected nilpotent Lie group. Let's assume that G satisfies the ideal polarization condition. Let $p,q \in [1,+\infty]$ such that $\min(p,q) < +\infty$. Let f be a square integrable function on G satisfying the following decay conditions:*

$$\int_G |f(x)|^p e^{\pi ap\mathscr{N}(x)^2} dx < +\infty, \tag{11}$$

$$\int_{\mathscr{W}} \|\pi_l(f)\|_{HS}^q e^{\pi bq\|l\|^2} |Pf(l)|^{\frac{q}{2}} dl < +\infty. \tag{12}$$

Then:

(i) f vanishes almost everywhere if $ab \geqslant 1$ and $1 \leqslant q \leqslant 2$,
(2i) f vanishes almost everywhere if $ab \geqslant 1$ and $2 \leqslant p,q \leqslant +\infty$,
(3i) f vanishes almost everywhere if $ab > 1$, $1 \leqslant p < 2$ and $2 \leqslant q \leqslant +\infty$.

Before starting the proof, we give more notation and then prove a preliminary lemma. We need a localized version of the Plancherel measure (see [3]). Let $Z = \exp\mathfrak{z}$ be the center of G and fix a non zero vector X_1 of $\mathfrak{z}$. Let $A = \exp\mathfrak{a} = \exp\mathbb{R}X_1$ be the closed connected subgroup of Z and $\chi = \chi_\psi$, $\psi \in \mathbb{R}X_1^*$, be a unitary character of A. Let $A \setminus G$ be the left quotient of G with A and $d\dot{g}$ be a Haar measure on $A \setminus G$. Let $1 \leqslant p' < +\infty$ and $L^{p'}(A \setminus G, \psi)$ be a space of all measurable functions $\varphi : G \to \mathbb{C}$ such that $\varphi(zg) = \overline{\chi(z)}\varphi(g)$ for almost all $g \in G$ and $z \in A$ and

$$\|\varphi\|_{L^{p'}(A\setminus G)}^{p'} := \int_{A\setminus G} |\varphi(g)|^{p'} d\dot{g} < +\infty.$$

Let

$$\hat{G}_\chi = \{\pi \in \hat{G} : \pi_{|A} = \chi \cdot I\}.$$

Then $\hat{G}_\chi$ is a closed subset of $\hat{G}$, in fact it is the dual space of the convolution algebra $L^1(A \setminus G, \psi)$. The convolution here is defined for φ and φ' in $L^1(A \setminus G, \psi)$ by

$$\varphi * \varphi'(\dot{x}) = \int_{A\setminus G} \varphi(g)\varphi'(g^{-1}\dot{x}) d\dot{g},\ \dot{x} \in A \setminus G.$$

So, in this case the Plancherel formula reads: if

$$\pi(\varphi) = \int_{A\setminus G} \varphi(g)\pi(g) d\dot{g},\ \pi \in \hat{G}_\chi,$$

then

$$\int_{A\setminus G} |\varphi(g)|^2 d\dot{g} = \int_{\hat{G}_\chi} \operatorname{tr} \pi(\varphi^* * \varphi) d\mu_\chi(\pi)$$

where the measure $d\mu_\chi$ is obtained in the following way. Let $\mathfrak{g}_\psi^* = \psi + \mathfrak{a}^\perp$ and $\mathscr{W}_\psi = \mathscr{W} \cap \mathfrak{g}_\psi^*$. For $\varphi \in L^1(A \setminus G, \psi) \cap L^2(A \setminus G, \psi)$ we have:

$$\|\varphi\|_{L^2(A\setminus G)} = \Big(\int_{\mathscr{W}_\psi} \|\pi_l(\varphi)\|_{HS}^2 |Pf(l)|dl\Big)^{\frac{1}{2}}. \tag{13}$$

If $2d$ is the maximal dimension of coadjoint orbits in $\mathfrak{g}^*$, then T has $n-2d$ elements and thus V_T can be identified with $\mathbb{R}^{n-2d}$. We can identify V_T with $\mathbb{R}X_1^* \oplus \mathbb{R}^{n-2d-1}$. We denote by

$$p_* : V_T \to \mathbb{R}X_1^*, \quad l \mapsto l_1 X_1^*$$

the canonical projection, then it will be convenient to write elements $l \in \mathscr{W}$, as (l_1, l') where $l_1 \in p_*(\mathscr{W})$ and $l' \in \mathscr{W}_{l_1} = \{l' \in \mathbb{R}^{n-2d-1} : (l_1, l') \in \mathscr{W}\}$. Define the function f^y on $\mathbb{R}$ by

$$f^y(x_1) = f\Big(\exp x_1 X_1 \prod_{j=2}^{n} \exp y_j X_j\Big), \text{ for all } x_1 \in \mathbb{R} \text{ and } y = (y_2, \ldots, y_n) \in \mathbb{R}^{n-1}.$$

Then we have the following lemma.

LEMMA 2. *For $p \geqslant 2$, let f meet the condition (11) of Theorem 7. Then,*

$$\int_{\mathbb{R}^{n-1}} \big|\widehat{f^y}(l_1)\big|^2 dy = \int_{\mathscr{W}_{l_1}} \|\pi_l(f)\|_{HS}^2 |Pf(l)|dl'.$$

Proof. Let $l_1 \in p_*(\mathscr{W})$ and $V_\alpha(l_1) = \Big[l_1 - \dfrac{1}{2\alpha}, l_1 + \dfrac{1}{2\alpha}\Big]$, $\alpha \in \mathbb{N}^*$. From [12, p. 491], one has that:

$$\int_{\mathbb{R}^{n-1}} \big|\widehat{f_y}(l_1)\big|^2 dy = \lim_{\alpha \to +\infty} \alpha \int_{V_\alpha(l_1)} \int_{\mathscr{W}_{\eta_1}} |Pf(\eta)| \|\pi_\eta(f)\|_{HS}^2 d\eta' d\eta_1,$$

where $\eta = (\eta_1, \eta')$ and

$$f_y(x_1) = f\Big(\exp\big(x_1 X_1 + \sum_{j=2}^{n} y_j X_j\big)\Big) = f\Big(\exp x_1 X_1 \exp\big(\sum_{j=2}^{n} y_j X_j\big)\Big).$$

Then using the localized Plancherel formula for $A \setminus G$, we get

$$\int_{\mathbb{R}^{n-1}} \big|\widehat{f_y}(l_1)\big|^2 dy = \lim_{\alpha \to +\infty} \int_{A\setminus G} \alpha \int_{V_\alpha(l_1)} |f_{\eta_1}(g)|^2 d\eta_1 d\dot{g},$$

where $f_{\eta_1}(g) = \int_{\mathbb{R}} f(\exp(x_1 X_1) g) \mathrm{e}^{2i\pi x_1 \eta_1} dx_1$.

We are going to use the dominated convergence theorem. We remak that

$$\alpha \int_{V_\alpha(l_1)} |f_{\eta_1}(g)|^2 d\eta_1 = |f_{c_\alpha}(g)|^2,$$

for some $c_\alpha \in V_\alpha(l_1)$. In addition,

$$|f_{c_\alpha}(g)|^2 = \Big| \int_{\mathbb{R}} f(\exp(x_1X_1)g)\mathrm{e}^{2i\pi x_1 c_\alpha} dx_1 \Big|^2 \leqslant \Big(\int_{\mathbb{R}} |f(\exp(x_1X_1)g)| dx_1 \Big)^2.$$

It is therefore enough to prove that the integral

$$E = \int_{A\backslash G} \Big(\int_{\mathbb{R}} |f(\exp(x_1X_1)g)| dx_1 \Big)^2 d\dot{g}$$

converges. Using the generalized Minkowski inequality, one gets:

$$\begin{aligned} E &\leqslant \Big(\int_{\mathbb{R}} \Big(\int_{A\backslash G} |f(\exp(x_1X_1)g)|^2 d\dot{g} \Big)^{\frac{1}{2}} dx_1 \Big)^2 \\ &= \Big(\int_{\mathbb{R}} \Big(\int_{\mathbb{R}^{n-1}} \Big| f\Big(\exp x_1X_1 \prod_{j=2}^{n} \exp y_jX_j \Big) \Big|^2 dy \Big)^{\frac{1}{2}} dx_1 \Big)^2. \end{aligned}$$

Choose a positive r such that $r < a$. Cauchy-Schwarz inequality implies that the last integral is dominated by

$$C_1 \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \mathrm{e}^{2\pi r x_1^2} \Big| f\Big(\exp x_1X_1 \prod_{j=2}^{n} \exp y_jX_j \Big) \Big|^2 dy dx_1,$$

for some positive constant C_1. We then have

$$\begin{aligned} E &\leqslant C_1 \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \mathrm{e}^{2\pi \left((r-a)x_1^2 - a\sum_{i=2}^{n} y_i^2 \right)} \mathrm{e}^{2\pi a \left(x_1^2 + \sum_{i=2}^{n} y_i^2 \right)} \Big| f\Big(\exp x_1X_1 \prod_{j=2}^{n} \exp y_jX_j \Big) \Big|^2 dy dx_1 \\ &\leqslant C_1C_2 \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \mathrm{e}^{\pi a p (x_1^2 + \sum_{j=2}^{n} y_j^2)} \Big| f\Big(\exp x_1X_1 \prod_{j=2}^{n} \exp y_jX_j \Big) \Big|^p dy dx_1 < +\infty \end{aligned}$$

(by Hölder inequality), for some positive constant C_2. It follows that,

$$\int_{\mathbb{R}^{n-1}} \big| \widehat{f_y}(l_1) \big|^2 dy = \int_{\mathscr{W}_{l_1}} \|\pi_l(f)\|_{HS}^2 |Pf(l)| dl'.$$

On the other hand,

$$\int_{\mathbb{R}^{n-1}} \big| \widehat{f^y}(l_1) \big|^2 dy = \int_{\mathbb{R}^{n-1}} \Big| \int_{\mathbb{R}} f\Big(\exp x_1X_1 \prod_{j=2}^{n} \exp y_jX_j \Big) \mathrm{e}^{2i\pi x_1 l_1} dx_1 \Big|^2 dy.$$

Remark that

$$\exp x_1X_1 \prod_{j=2}^{n} \exp y_jX_j = \exp\Big((x_1 + Q_1(y))X_1 + \sum_{j=2}^{n} Q_j(y)X_j \Big),$$

where, for $1 \leqslant j \leqslant n$, Q_j is a polynomial function depending on $y_2, \ldots, y_n$. Furthermore, one can write

$$Q_j(y) = y_j + Q'_j(y_{j+1}, \ldots, y_n), \quad j = 2, \ldots, n. \tag{14}$$

It results that,

$$\int_{\mathbb{R}^{n-1}} \left|\widehat{f^y}(l_1)\right|^2 dy = \int_{\mathbb{R}^{n-1}} \left| \int_{\mathbb{R}} f\Big(\exp\Big((x_1+Q_1(y))X_1 + \sum_{j=2}^{n} Q_j(y)X_j\Big)\Big) e^{2i\pi x_1 l_1} dx_1 \right|^2 dy$$

$$= \int_{\mathbb{R}^{n-1}} \left| \int_{\mathbb{R}} f\Big(\exp\Big(x_1 X_1 + \sum_{j=2}^{n} Q_j(y)X_j\Big)\Big) e^{2i\pi x_1 l_1} dx_1 \right|^2 dy$$

(by substituting $x_1 + Q_1(y)$ for x_1)

$$= \int_{\mathbb{R}^{n-1}} \left| \int_{\mathbb{R}} f\Big(\exp\Big(x_1 X_1 + \sum_{j=2}^{n} y_j X_j\Big)\Big) e^{2i\pi x_1 l_1} dx_1 \right|^2 dy$$

$$= \int_{\mathbb{R}^{n-1}} \left|\widehat{f_y}(l_1)\right|^2 dy$$

(by substituting $Q_j(y)$ for y_j, $j = 2, \dots, n$, using equation (14)). $\square$

Proof of Theorem 7. We shall study the cases separately.

(i) We keep the same notations as in section 3. By lemma 1, the Hilbert-Schmidt norm of $\pi_l(f)$ is given by:

$$\|\pi_l(f)\|_{HS}^2 = \frac{1}{|\mathrm{Pf}(l)|} \int_{\mathbb{R}^d} \int_{V_{S_{\mathfrak{b}}}} \left|\widehat{f_{\gamma(s)}}(\psi(l,\lambda))\right|^2 d\lambda\, ds. \tag{15}$$

From condition (12),

$$+\infty > \int_{\mathscr{W}} \|\pi_l(f)\|_{HS}^q e^{\pi b q \|l\|^2} |\mathrm{Pf}(l)|^{\frac{q}{2}} dl$$

$$\geqslant \int_{\mathscr{W}} \|\pi_l(f)\|_{HS}^q e^{\pi b q l_1^2} |\mathrm{Pf}(l)|^{\frac{q}{2}} dl$$

$$= \int_{\mathscr{W}} \Big(\int_{\mathbb{R}^d} \int_{V_{S_{\mathfrak{b}}}} \left|\widehat{f_{\gamma(s)}}(\psi(l,\lambda))\right|^2 d\lambda\, ds\Big)^{\frac{q}{2}} e^{\pi b q l_1^2} dl$$

$$\geqslant \Big(\int_{\mathbb{R}^d} \int_{V_{S_{\mathfrak{b}}}} \Big(\int_{\mathscr{W}} \left|\widehat{f_{\gamma(s)}}(\psi(l,\lambda))\right|^q e^{\pi b q l_1^2} dl\Big)^{\frac{2}{q}} d\lambda\, ds\Big)^{\frac{q}{2}}$$

(using the generalized Minkowski inequality). Now by substituting $l_k + R_{t_k}(l,\lambda)$ for l_k, $k = 2, \dots, r$, we obtain

$$\Big(\int_{\mathbb{R}^d} \int_{V_{S_{\mathfrak{b}}}} \Big(\int_{\mathscr{W}} \left|\widehat{f_{\gamma(s)}}(l+\lambda)\right|^q e^{\pi b q l_1^2} dl\Big)^{\frac{2}{q}} d\lambda\, ds\Big)^{\frac{q}{2}} < +\infty.$$

It follows that,

$$\int_{\mathbb{R}} \left|\widehat{f_{\gamma(s)}}(l+\lambda)\right|^q e^{\pi b q l_1^2} dl_1 < +\infty,$$

for almost all $s_1, \dots, s_d \in \mathbb{R}$, $\lambda \in V_{S_{\mathfrak{b}}}$ and $l_k \in \mathbb{R}$, $k = 2, \dots, r$. For fixed $s_1, \dots, s_d \in \mathbb{R}$, $\lambda \in V_{S_{\mathfrak{b}}}$ and $l_k \in \mathbb{R}$, $k = 2, \dots, r$, let $F_{\gamma(s)}$ be the function defined in section 3. Then obviously,

$$\int_{\mathbb{R}} |\widehat{F}_{\gamma(s)}(l_1)|^q e^{\pi b q l_1^2} dl_1 < +\infty.$$

On the other hand, we have

$$\begin{aligned}
\int_{\mathbb{R}} \mathrm{e}^{\pi p a z_1^2} |F_{\gamma(s)}(z_1)|^p dz_1 &\leqslant \int_{\mathbb{R}} \mathrm{e}^{\pi p a z_1^2} \Big(\int_{\mathbb{R}^{m-1}} |f_{\gamma(s)}(z_1,\dots,z_m)| dz_2 \dots dz_m \Big)^p dz_1 \\
&\leqslant \Big(\int_{\mathbb{R}^{m-1}} \Big(\int_{\mathbb{R}} \mathrm{e}^{\pi p a z_1^2} |f_{\gamma(s)}(z_1,\dots,z_m)|^p dz_1 \Big)^{\frac{1}{p}} dz_2 \dots dz_m \Big)^p \\
&\quad \text{(using the generalized Minkowski inequality)} \\
&\leqslant C \int_{\mathbb{R}^m} \mathrm{e}^{\pi p a \mathscr{N}(z_1,\dots,z_m)^2} |f_{\gamma(s)}(z_1,\dots,z_m)|^p dz_1 \dots dz_m \\
&\quad \text{(using Hölder inequality)} \\
&= C \int_{\mathbb{R}^m} \mathrm{e}^{\pi p a \mathscr{N}(z_1,\dots,z_m)^2} \Big| f\Big(\exp\big(\prod_{k=1}^{m} z_k X_k \big) \gamma(s) \Big) \Big|^p dz_1 \dots dz_m \\
&< +\infty,
\end{aligned}$$

which is finite by the first condition of theorem 7 and where C is a positive constant. Hence $F_{\gamma(s)}$ vanishes almost everywhere, using Theorem 2, as $ab \geqslant 1$. We make use the previous arguments to achieve the proof of this case.

(2i) Suppose that $p,q \in [2,+\infty]$. For fixed Schwartz function φ on $\mathbb{R}^{n-1}$, define the function F on $\mathbb{R}$ by

$$F(x_1) = \int_{\mathbb{R}^{n-1}} f^y(x_1)\varphi(y)dy.$$

It is obvious that

$$\hat{F}(l_1) = \int_{\mathbb{R}^{n-1}} \widehat{f^y}(l_1)\varphi(y)dy,$$

and then there exists a positive constant M_1 such that

$$|\hat{F}(l_1)|^2 \leqslant M_1 \int_{\mathbb{R}^{n-1}} |\widehat{f^y}(l_1)|^2 dy = M_1 \int_{\mathscr{W}_{l_1}} \|\pi_l(f)\|_{HS}^2 |Pf(l)| dl'$$

(using lemma 2). Thus,

$$|\hat{F}(l_1)|^q \leqslant M_1^{\frac{q}{2}} \Big(\int_{\mathscr{W}_{l_1}} \|\pi_l(f)\|_{HS}^2 |Pf(l)| dl' \Big)^{\frac{q}{2}}.$$

It follows that,

$$\begin{aligned}
\int_{\mathbb{R}} \mathrm{e}^{\pi b q l_1^2} |\hat{F}(l_1)|^q dl_1 &\leqslant M_1^{\frac{q}{2}} \int_{\mathbb{R}} \Big(\int_{\mathscr{W}_{l_1}} \|\pi_l(f)\|_{HS}^2 |Pf(l)| dl' \Big)^{\frac{q}{2}} \mathrm{e}^{\pi b q l_1^2} dl_1 \\
&\leqslant M_1^{\frac{q}{2}} \Big(\int_{\mathscr{W}_{l_1}} \Big(\int_{\mathbb{R}} \|\pi_l(f)\|_{HS}^q |Pf(l)|^{\frac{q}{2}} \mathrm{e}^{\pi b q l_1^2} dl_1 \Big)^{\frac{2}{q}} dl' \Big)^{\frac{q}{2}} \\
&\quad \text{(using the generalized Minkowski inequality)} \\
&\leqslant M_1^{\frac{q}{2}} M_2 \int_{\mathscr{W}_{l_1}} \int_{\mathbb{R}} \|\pi_l(f)\|_{HS}^q |Pf(l)|^{\frac{q}{2}} \mathrm{e}^{\pi b q (l_1^2 + \|l'\|^2)} dl_1 dl' \\
&\quad \text{(using Hölder inequality)} \\
&= M_1^{\frac{q}{2}} M_2 \int_{\mathscr{W}} \|\pi_l(f)\|_{HS}^q |Pf(l)|^{\frac{q}{2}} \mathrm{e}^{\pi b q \|l\|^2} dl < +\infty,
\end{aligned}$$

for some positive constant M_2. In addition, we can see that

$$\begin{aligned}
\int_{\mathbb{R}} |F(x_1)|^p \mathrm{e}^{\pi a p x_1^2} dx_1 &\leqslant \int_{\mathbb{R}} \Big(\int_{\mathbb{R}^{n-1}} |f^y(x_1)||\varphi(y)|dy \Big)^p \mathrm{e}^{\pi a p x_1^2} dx_1 \\
&\leqslant \Big(\int_{\mathbb{R}^{n-1}} \Big(\int_{\mathbb{R}} |f^y(x_1)|^p \mathrm{e}^{\pi a p x_1^2} dx_1 \Big)^{\frac{1}{p}} |\varphi(y)|dy \Big)^p \\
&\quad \text{(using the generalized Minkowski inequality)} \\
&\leqslant M \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} |f^y(x_1)|^p \mathrm{e}^{\pi a p (x_1^2 + \Sigma_{j=2}^n y_j^2)} dx_1 dy \\
&\quad \text{(using Hölder inequality)} \\
&= M \int_G |f(x)|^p \mathrm{e}^{\pi a p \mathscr{N}^2(x)} dx_1 \dots dx_n < +\infty,
\end{aligned}$$

where $x = \exp x_1 X_1 \dots \exp x_n X_n$ and M is a positive constant. We have shown finally that F verifies the decay conditions of the Cowling-Price Theorem on $\mathbb{R}$. Then for $ab \geqslant 1$, $F = 0$ almost for every $x_1 \in \mathbb{R}$. Allowing φ to vary through the space of Schwartz functions on $\mathbb{R}^{n-1}$, we obtain that $f^y = 0$ almost everywhere. It follows that f is zero almost everywhere on the group G.

(3i) In this case we can show that $f = 0$ almost everywhere for every p, q in $[1, +\infty]$. Since $ab > 1$ it is sufficient to consider the case where $p = q = 1$. The technique of the proof of case (i) allows us to conclude. $\square$

REMARK 1. The arguments used in Theorem 6 could be easily adapted to prove the same result for the function norm $\mathscr{N}(\cdot)$, using the strong Malcev coordinates of the group G. In fact, a careful reading of proofs of Theorem 2.4 and Corollary 2.6 in [6] shows that they are valid after changing f_y by f^y. However, the proofs of Theorem 5 and Theorem 7 fail to be correct if we use the exponential coordinates of G and the function norm $x \mapsto \|\cdot\|$.

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