Mathematical
Inequalities
& Applications
Volume 22, Number 1 (2019), 227–247 doi:10.7153/mia-2019-22-18

ON GENERALIZED CESÀRO STABLE FUNCTIONS

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(*Communicated by J. Pečarić*)

Abstract. The notion of Cesàro stable function is generalized by introducing Cesàro mean of type $(b-1;c)$ which give rise to a new concept of generalized Cesàro stable function. As an application of generalized Cesàro stable functions we also prove for a convex function of order $\lambda \in [1/2,1)$, its Cesàro mean of type $(b-1;c)$ is close-to-convex of order λ. Further two conjectures are also posed in the direction of generalized Cesàro stable function. Some particular cases of these conjectures are also discussed.

1. Preliminaries

Let $b+1>c>0$ and $0<\mu<1$. Define the sequence $\{c_k\}$ as

$$c_{2k}=c_{2k+1}=d_k=\frac{B_{n-k}}{B_n}\frac{(\mu)_k}{k!}, \quad k=0,1,2,\ldots \tag{1}$$

where $B_0=1$ and $B_k=\frac{(b)_k}{(c)_k}\frac{1+b-c}{b}$ for $k \geqslant 1$.

This sequence was used in [21] to obtain the positivity of the trigonometric cosine sums.

THEOREM 1. [21] *Let the coefficient* $\{c_k\}$ *be given as in* (1). *Then for* $b \geqslant c>0$ *and* $n \in \mathbb{N}$

$$\sum_{k=0}^{n} c_k \cos k\theta > 0 \quad \textit{for} \quad \mu \leqslant \mu_0' \ \textit{and} \ 0<\theta<\pi,$$

where μ_0' *is the solution of*

$$\int_0^{3\pi/2} \frac{\cos t}{t^{1-\mu}}\left(1-\frac{2t}{3\pi}\right)^{b-c} dt = 0.$$

The positivity of sine sums analogous to Theorem 1 is also given in [21].

Mathematics subject classification (2010): 42A05, 40G05, 30C45.
Keywords and phrases: Trigonometric sums, Cesàro means, starlike functions, close-to-convex functions.
This research is supported by "Council of Scientific and Industrial Research, India" (grant code: 09/143(0827)/2013-EMR-1).

THEOREM 2. [21] *Let the coefficient* $\{c_k\}$ *be given as in* (1). *Then for* $b \geqslant c > 0$, $n \in \mathbb{N}$ *and* $0 < \theta < \pi$ *the following inequalities hold.*

$$\sum_{k=1}^{2n+1} c_k \sin k\theta > 0 \quad \textit{for} \quad \mu \leqslant \mu_0',$$
$$\sum_{k=1}^{2n} c_k \sin k\theta > 0 \quad \textit{for} \quad \mu \leqslant \left(\frac{1+b}{2c}\right) - \frac{1}{2}.$$

Note that for $b = 1$ and $c = 1$, c_k given in (1) reduces to γ_k given by Vietoris [23] which are defined as follows.

$$\gamma_0 = \gamma_1 = 1 \quad \text{and} \quad \gamma_{2k} = \gamma_{2k+1} = \frac{(1/2)_k}{k!}, \quad k \geqslant 1.$$

Clearly Theorem 1 and Theorem 2 are further development of the following theorem given by Vietoris [23], by choosing $a_k = \gamma_k$.

THEOREM 3. [23] *Let* $\{a_k\}_{k=0}^{\infty}$ *be a non-increasing sequence of non-negative real numbers such that* $a_0 > 0$ *and satisfying*

$$2ka_{2k} \leqslant (2k-1)a_{2k-1}, \quad k \geqslant 1,$$

then for all positive integers n *and* $\theta \in (0,\pi)$, *we have*

$$\sum_{k=1}^{n} a_k \sin k\theta > 0 \textit{ and } \sum_{k=0}^{n} a_k \cos k\theta > 0.$$

Vietoris [23] observed that these two inequalities for the special case in which $a_k = \gamma_k$ where the sequence γ_k is defined as above.

Several generalizations of Theorem 3 can be found in the literature. For example, see [1, 5, 11, 21]. As an application of positive trigonometric sums, Ruscheweyh and Salinas [19] introduced the concept of stable functions. Due to its wide significance, the generalization of Theorem 3 is of much interest. For the recent development in this direction see [21] and the references therein.

In [21] the applications of Theorem 1 and Theorem 2 in finding the location of zeros of a class of trigonometric polynomials is discussed. Some new inequalities related to Gegenbauer polynomials are also given in [21]. It is of interest to interpret Theorem 1 and Theorem 2 in the context of geometric function theory. For this purpose, we recall some concepts and definitions.

The set of analytic functions in the unit disc $\mathbb{D} := \{z : |z| < 1\}$ is denoted by $\mathscr{A}$ and the set of all one-to-one (univalent) functions in $\mathbb{D}$ is denoted by $\mathscr{S}$. Let $\mathscr{A}_0$ and $\mathscr{A}_1$ are the subset of $\mathscr{A}$ with normalization $f(0) = 0, f'(0) = 1$ and $f(0) = 1$ respectively.

The following subclasses of $\mathscr{S}$ are useful for further discussion. Let $\mathscr{S}^*(\alpha)$, $0 \leqslant \alpha < 1$, be the class of starlike functions of order α, $f \in \mathscr{A}$ satisfying $\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha$ and $\mathscr{C}(\alpha)$, $0 \leqslant \alpha < 1$ be the class of convex function of order α, satisfying

$\operatorname{Re}\left(1+\frac{zf''(z)}{f(z)}\right) > \alpha$, for $z \in \mathbb{D}$. If we take $\alpha = 0$, these two subclasses reduce to starlike and convex class denoted by $\mathscr{S}^*$ and $\mathscr{C}$ respectively. The relation between these two subclasses is given by Alexander transformation i.e. $f \in \mathscr{C}(\alpha) \Longleftrightarrow zf' \in \mathscr{S}^*(\alpha)$. One another important subclass $\mathscr{K}(\alpha)$ be the class of all close-to-convex functions $f \in \mathscr{A}$ with respect to a starlike function $g(z) \in \mathscr{S}^*$ if $\operatorname{Re} e^{\iota\gamma}\left(\frac{zf'(z)}{g(z)}\right) > \alpha, \gamma \in \mathbb{R}$. For information regarding these classes we refer to [2, 3, 12]. There is a proper inclusion to hold among these classes.

$$\mathscr{C} \subsetneqq \mathscr{S}^* \subsetneqq \mathscr{K} \subsetneqq \mathscr{S}.$$

Further a function $f \in \mathscr{A}_0$ is called pre-starlike function of order α, $0 \leqslant \alpha < 1$ if $f(z) * k_\alpha(z) \in \mathscr{S}^*(\alpha)$. This class is denoted by $\mathscr{R}^*(\alpha)$, where $k_\alpha(z) = \frac{z}{(1-z)^{2-2\alpha}}$ plays the vital role as it is the extremal function of $\mathscr{S}^*(\alpha)$ and for a complete account of details on $\mathscr{R}^*(\alpha)$ see [15]. It is obvious that $\mathscr{R}^*(1/2) \equiv \mathscr{S}^*(1/2)$ and $\mathscr{R}^*(0) \equiv \mathscr{C}$. Here the Hadamard product or convolution denoted by $*$ is defined as follows:

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$, $z \in \mathbb{D}$. Then

$$(f * g)(z) := \sum_{k=0}^{\infty} a_k b_k z^k, \qquad \text{for all } z \in \mathbb{D}. \tag{2}$$

In the present context, the following lemma is of considerable interest, which plays important role in several problems in function theory involving duality technique.

LEMMA 1. [13, p. 54] *Let F be prestarlike of order $0 \leqslant \gamma < 1$, $G \in \mathscr{S}^*(\gamma)$ and H is any analytic function in $\mathbb{D}$. Then,*

$$\frac{F * (GH)}{F * G}(\mathbb{D}) \subset \overline{co}(H(\mathbb{D})),$$

where $co(A)$ is the convex hull of a set A.

Another tool used in the sequel is the concept of subordination denoted by $\prec$. An analytic function f is subordinate to a univalent function g, written as $f(z) \prec g(z)$, if there exists a Schwarz function $\omega(z) : \mathbb{D} \to \mathbb{D}$, satisfying $|\omega(z)| \leqslant |z|$ such that $f(z) = g(\omega(z))$.

To apply Theorem 1 and Theorem 2 in context of geometric function theory, we generalize the concept of stable function by means of generalized Cesàro mean of type $(b-1;c)$. For $f \in \mathscr{A}_1$ and $b+1 > c > 0$, the nth Cesàro mean of type $(b-1;c)$ of $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathscr{A}_1$ is given by,

$$\sigma_n^{(b-1,c)}(f,z) := \frac{1}{B_n} \sum_{k=0}^{n} B_{n-k} a_k z^k = \sigma_n^{(b-1,c)}(z) * f(z), \quad n \in \mathbb{N}_0, \tag{3}$$

where B_k is defined as $B_0=1$ and $B_k=\frac{(b)_k}{(c)_k}\frac{1+b-c}{b}$ for $k\geqslant 1$. For $f\in\mathscr{A}$, we say $\sigma_n^{(b-1,c)}(f,z)$ is the nth Cesàro mean of type $(b-1;c)$ of f. Geometric properties of $\sigma_n^{(b-1,c)}(f,z)$ can be found in [22] and references therein. Further $s_n(f,z)=\sigma_n^{(1-1,1)}(f,z)$ was studied by Ruscheweyh with his collaborators, see [20] and references therein. Similarly $\sigma_n^{\alpha}(f,z)=\sigma_n^{(1+\alpha-1,1)}(f,z)$ was studied by Mondal and Swaminathan in [11].

2. Generalized Cesàro stable function

Using simple computation, (3) can be rewritten in the following form.

$$\begin{aligned}\sigma_n^{(b-1,c)}(f,z)=&\left(\frac{c+n-1}{b+n-1}\right)\sigma_{n-1}^{(b-1,c)}(f,z)+\frac{(b-c)}{B_n}\sum_{k=0}^{n-2}\frac{B_{n-k-1}}{(c+n-k-1)}a_kz^k\\&+\frac{1}{B_n}\left(\frac{1+b-2c}{c}\right)a_{n-1}z^{n-1}+\frac{B_0}{B_n}a_nz^n.\end{aligned}\tag{4}$$

In the sequel, we denote $f_\mu(z):=\dfrac{1}{(1-z)^\mu}$ which satisfies the following relations that are easy to verify.

$$\begin{aligned}\sigma_n^{(b-1,c)}(f_\mu,z)'&=\left(\frac{c+n-1}{b+n-1}\right)\sigma_{n-1}^{(b-1,c)}(f'_\mu,z),\\ z\sigma_n^{(b-1,c)}(f_\mu,z)'&=\sigma_n^{(b-1,c)}(zf'_\mu,z),\\ f_\mu-\frac{(1-z)}{\mu}f'_\mu&\equiv 0.\end{aligned}$$

Now we state the main result of this section. For the proof, we follow the procedure similar to the one given in [20, Theorem 1.1].

THEOREM 4. *For $b\geqslant\max\{c,2c-1\}>0$ and $\mu\in[-1,1]$, the following equation holds.*

$$(1-z)^\mu\sigma_n^{(b-1,c)}(f_\mu,z)\prec(1-z)^\mu.\tag{5}$$

Proof. The nth Cesàro mean of type $(b-1;c)$ of $f(z)=\sum_{k=0}^{\infty}a_kz^k\in\mathscr{A}_1$ is given in (3). Let $h(z):=1-(1-z)\sigma_n^{(b-1,c)}(f_\mu,z)^{\frac{1}{\mu}}$. In order to prove our result it is sufficient to prove $|h(z)|\leqslant 1$. Clearly, for $\mu=0$, $f_\mu=1$ and hence $|h(z)|\leqslant 1$. We consider the proof in two parts based on the range of μ. For the first part, let $\mu\in(0,1]$. Consider

$$\begin{aligned}(1-z)\sigma_n^{(b-1,c)}(f_\mu,z)'&=\sigma_n^{(b-1,c)}(f_\mu,z)'-z\sigma_n^{(b-1,c)}(f_\mu,z)'\\&=\left(\frac{c+n-1}{b+n-1}\right)\sigma_{n-1}^{(b-1,c)}(f'_\mu,z)-\sigma_n^{(b-1,c)}(zf'_\mu,z)\end{aligned}\tag{6}$$

Using (4), $\sigma_n^{(b-1,c)}(zf'_\mu,z)$ can be rewritten as,

$$\sigma_n^{(b-1,c)}(zf'_\mu,z)=\left(\frac{c+n-1}{b+n-1}\right)\sigma_{n-1}^{(b-1,c)}(zf'_\mu,z)+\frac{(b-c)}{B_n}\sum_{k=0}^{n-2}\frac{B_{n-k-1}}{(c+n-k-1)}ka_kz^k$$
$$+\left(\frac{1+b-2c}{c}\right)\frac{(n-1)a_{n-1}}{B_n}z^{n-1}+\frac{B_0}{B_n}na_nz^n. \tag{7}$$

After substituting the value of $a_k=\frac{(\mu)_k}{k!}$, from (6) and (7) we obtain,

$$\begin{aligned}&(1-z)\sigma_n^{(b-1,c)}(f_\mu,z)'\\&=\left(\frac{c+n-1}{b+n-1}\right)\sigma_{n-1}^{(b-1,c)}((1-z)f'_\mu,z)-\frac{(b-c)}{B_n}\sum_{k=0}^{n-2}\frac{B_{n-k-1}}{(c+n-k-1)}\frac{k(\mu)_k}{k!}z^k\\&\quad-\left(\frac{1+b-2c}{c}\right)\frac{(n-1)(\mu)_{n-1}}{B_n(n-1)!}z^{n-1}-\frac{B_0}{B_n}\frac{n(\mu)_n}{n!}z^n.\end{aligned}$$

Therefore,

$$\begin{aligned}&\sigma_n^{(b-1,c)}(f_\mu,z)-\frac{(1-z)}{\mu}\sigma_n^{(b-1,c)}(f_\mu,z)'\\&=\left(\frac{c+n-1}{b+n-1}\right)\sigma_{n-1}^{(b-1,c)}\left(f_\mu-\frac{1-z}{\mu}f'_\mu,z\right)\\&\quad+\frac{(b-c)}{B_n}\sum_{k=0}^{n-2}\frac{B_{n-k-1}}{(c+n-k-1)}\left(\frac{(\mu)_k}{k!}+\frac{k(\mu)_k}{\mu k!}\right)z^k\\&\quad+\left(\frac{1+b-2c}{c}\right)\left(\frac{(\mu)_{n-1}}{(n-1)!}+\frac{(n-1)(\mu)_{n-1}}{\mu(n-1)!}\right)\frac{z^{n-1}}{B_n}+\left(\frac{(\mu)_n}{n!}+\frac{n(\mu)_n}{\mu n!}\right)\frac{B_0}{B_n}z^n\\&=\frac{(b-c)}{B_n}\sum_{k=0}^{n-2}\frac{B_{n-k-1}}{(c+n-k-1)}\frac{(\mu+1)_k}{k!}z^k\\&\quad+\left(\frac{1+b-2c}{c}\right)\frac{1}{B_n}\frac{(\mu+1)_{n-1}}{(n-1)!}z^{n-1}+\frac{(\mu+1)_n}{n!}\frac{B_0}{B_n}z^n\end{aligned}$$

Further,

$$\begin{aligned}h'(z)&=\left[\sigma_n^{(b-1,c)}(f_\mu,z)\right]^{\frac{1}{\mu}}-\frac{(1-z)}{\mu}\left[\sigma_n^{(b-1,c)}(f_\mu,z)\right]^{\frac{1}{\mu}-1}\cdot\left[\sigma_n^{(b-1,c)}(f_\mu,z)\right]'\\&=\left[\sigma_n^{(b-1,c)}(f_\mu,z)\right]^{\frac{1}{\mu}-1}\left[\sigma_n^{(b-1,c)}(f_\mu,z)-\frac{(1-z)}{\mu}\sigma_n^{(b-1,c)}(f_\mu,z)'\right]\\&=\left[\sigma_n^{(b-1,c)}(f_\mu,z)\right]^{\frac{1}{\mu}-1}\times\left[\frac{(b-c)}{B_n}\sum_{k=0}^{n-2}\frac{B_{n-k-1}}{(c+n-k-1)}\frac{(\mu+1)_k}{k!}z^k\right.\\&\qquad\left.+\left(\frac{1+b-2c}{c}\right)\frac{1}{B_n}\frac{(\mu+1)_{n-1}}{(n-1)!}z^{n-1}+\frac{(\mu+1)_n}{n!}\frac{B_0}{B_n}z^n\right].\end{aligned}$$

Clearly, $f_\mu(z) = (1-z)^{-\mu} = 1 + \mu z + \frac{\mu(\mu+1)}{2!}z^2 + \cdots + \frac{(\mu)_k}{k!}z^k + \cdots$. Since $0 < \mu \leqslant 1$, the Taylor coefficients of f_μ are positive. Thus,

$$\left|\sigma_n^{(b-1,c)}(f_\mu, z)\right| \leqslant \sum_{k=0}^{n} \frac{B_{n-k}}{B_n} \frac{(\mu)_k}{k!} |z|^k = \sigma_n^{(b-1,c)}(f_\mu, |z|)$$

We obtained that the Taylor coefficients of $h'(z)$ are positive and from the definition of $h(z)$, we have $h(0) = 0$ and $h(1) = 1$. Hence,

$$|h(z)| = \left|\int_0^z h'(t)dt\right| \leqslant \int_0^1 |h'(tz)|dt \leqslant \int_0^1 h'(t)dt = 1, \quad z \in \mathbb{D}.$$

Now for the second case $-1 \leqslant \mu < 0$, the coefficients of $(1-z)^{-\mu} = 1 + \mu z + \frac{\mu(\mu+1)}{2!}z^2 + \cdots + \frac{(\mu)_k}{k!}z^k + \cdots$ are negative except 1 and $\sigma_n^{(b-1,c)}(f_\mu, z) = 1 + \sum_{k=1}^{n} \frac{B_{n-k}}{B_n} \frac{(\mu)_k}{k!} z^k = 1 - b(z)$, where $b(z)$ has positive Taylor series coefficients. Therefore,

$$\sigma_n^{(b-1,c)}(f_\mu, z)^{\frac{1}{\mu}-1} = (1-b(z))^{\frac{1}{\mu}-1} = 1 + \sum_{k=1}^{\infty} \frac{(1-\frac{1}{\mu})_k}{k!} (b(z))^k.$$

This implies, $\sigma_n^{(b-1,c)}(f_\mu, z)^{\frac{1}{\mu}-1}$ has non-negative Taylor series coefficients and following the same steps as in part one, we obtain the result. $\square$

If we choose $b = c = 1$, then Theorem 4 reduces to the following corollary given in [20].

COROLLARY 1. [20] *Let $s_n(z, f)$ denote the nth partial sum of $f(z)$. Then for $n \in \mathbb{N} \cup \{0\}$ and for $\mu \in [-1, 1]$,*

$$(1-z)^\mu s_n(z, f_\mu) \prec (1-z)^\mu.$$

Important member of $\mathscr{S}^*(\lambda)$ are $zf_{2-2\lambda} = \frac{z}{(1-z)^{2-2\lambda}}$ that plays the role of extremal function while studying several properties such as growth, distortion etc. Clearly, an equivalent form of Theorem 4 for $\lambda \in [1/2, 1)$, is given as

$$(1-z)^{2-2\lambda} \sigma_n^{(b-1,c)}\left(\frac{1}{(1-z)^{2-2\lambda}}, z\right) \prec (1-z)^{2-2\lambda}, \quad \text{for all } z \in \mathbb{D}. \tag{8}$$

It seems that starlike function of order λ, $\lambda \in [1/2, 1)$ is comparably a much narrow class but on the other side it has several interesting properties. For example, our next theorem exhibits that (8) remains valid while in the left hand side of (8), $f_{2-2\lambda}$ is replaced by any $f \in \mathscr{S}^*(\lambda)$ for $\lambda \in [1/2, 1)$.

THEOREM 5. *Let $f \in \mathscr{S}^*(\lambda)$, for $\lambda \in [\frac{1}{2}, 1)$, then*

$$\frac{z\sigma_n^{(b-1,c)}(f/z, z)}{f} \prec (1-z)^{2-2\lambda}, \text{ for all } z \in \mathbb{D}. \tag{9}$$

Proof. Let $f \in \mathscr{S}^*(\lambda)$, then $\exists$ a unique prestarlike function $F(z)$ of order λ such that $f(z) = zf_{2-2\lambda} * F(z)$. Then from Theorem 4,

$$\frac{\sigma_n^{(b-1,c)}(f_{2-2\lambda}, z)}{f_{2-2\lambda}} \prec \frac{1}{f_{2-2\lambda}} \quad \text{for } \lambda \in [\tfrac{1}{2}, 1), \;\; z \in \mathbb{D}.$$

Using Lemma 1,

$$\begin{aligned}
\frac{z\sigma_n^{(b-1,c)}(f/z,z)}{f} &= \frac{z(\sigma_n^{(b-1,c)}(z) * \frac{f(z)}{z})}{f} = \frac{z\sigma_n^{(b-1,c)}(z) * f(z)}{F(z) * zf_{2-2\lambda}} \\
&= \frac{z\sigma_n^{(b-1,c)}(z) * (F(z) * zf_{2-2\lambda})}{F(z) * zf_{2-2\lambda}} = \frac{F(z) * (z\sigma_n^{(b-1,c)}(z) * zf_{2-2\lambda})}{F(z) * zf_{2-2\lambda}} \\
&= \frac{F(z) * \left(zf_{2-2\lambda} \cdot \frac{\sigma_n^{(b-1,c)}(f_{2-2\lambda}, z)}{f_{2-2\lambda}}\right)}{F(z) * zf_{2-2\lambda}} \in \overline{co}\left(\frac{\sigma_n^{(b-1,c)}(f_{2-2\lambda}, z)}{f_{2-2\lambda}}(\mathbb{D})\right),
\end{aligned}$$

This means by Lemma 1, the range of $\frac{\sigma_n^{(b-1,c)}(f/z,z)}{f/z}$ lies in the closed convex hull of image of $\frac{\sigma_n^{(b-1,c)}(f_{2-2\lambda},z)}{f_{2-2\lambda}}$ under $\mathbb{D}$. From (8), for $\lambda \in [\frac{1}{2}, 1)$, we have $\frac{\sigma_n^{(b-1,c)}(f_{2-2\lambda},z)}{f_{2-2\lambda}} \prec \frac{1}{f_{2-2\lambda}}$. Therefore,

$$\frac{\sigma_n^{(b-1,c)}(f/z,z)}{f/z} \prec \frac{1}{f_{2-2\lambda}},$$

which is equivalent to (9) and the proof is complete. $\square$

Theorem 5 has several consequences with Kakeya Eneström theorem, that will be discussed in Section 5. Taking $b = c = 1$, it reduces to the following result given by Ruscheweyh [20].

COROLLARY 2. [20] *Let $f \in \mathscr{S}^*(\lambda)$, $\lambda \in [1/2, 1)$. Then for $n \in \mathbb{N} \cup \{0\}$,*

$$\frac{zs_n(z, f/z)}{f} \prec \frac{1}{f_{2-2\lambda}}.$$

REMARK 1. If we take $b = 1 + \beta$ and $c = 1$, then it was proved in [11] that for $\beta \geqslant 0$,

$$\frac{\sigma_n^{\beta}(f_\mu, z)}{f_\mu} \prec \begin{cases} \frac{1}{f_{\mu-\beta}}, & \mu \in [-1, 0]; \\ \frac{1}{f_{\mu+\beta}}, & \mu \in (0, 1] \text{ such that } \mu + \beta \leqslant 1. \end{cases}$$

The condition $\mu + \beta \leqslant 1$ restricts β to lie in $[0, 1]$ where as Theorem 4 does not impose an upper bound on β and moreover

$$\begin{aligned}
\frac{1}{f_\mu} &\prec \frac{1}{f_{\mu+\beta}}, \quad && \mu \in (0, 1] \text{ where } \mu + \beta \leqslant 1. \\
\frac{1}{f_\mu} &\prec \frac{1}{f_{\mu-\beta}}, \quad && \mu \in [-1, 0].
\end{aligned}$$

So, Theorem 4 improves the result in [11, Theorem 2.2]. A similar comparison can be made for Theorem 5 with [11, Theorem 2.3].

Theorem 5 leads to a new definition of generalized Cesàro stable functions.

DEFINITION 1. (Generalized Cesàro Stable Function) A function $f \in \mathscr{A}_1$ is said to be n-generalized Cesàro stable with respect to $F \in \mathscr{A}_1$ if

$$\frac{\sigma_n^{(b-1,c)}(f,z)}{f(z)} \prec \frac{1}{F(z)} \tag{10}$$

holds for some $n \in \mathbb{N}$. We call f as n-generalized Cesàro stable if it is n-generalized Cesàro stable with respect to itself. If it is n-generalized Cesàro stable with respect to $F(z)$ for every n, then it is said to be generalized Cesàro stable with respect to $F(z)$.

REMARK 2. If we take $b = 1+\beta, c = 1$ then (10) reduces to

$$\frac{S_n^{\beta}(f,z)}{f} \prec \frac{1}{F(z)}$$

gives the (n,β) Cesàro-stability [11] of $f(z)$ about $F(z)$ which if $\beta = 0$ further reduces to stability [20] of $f(z)$ about $F(z)$.

LEMMA 2. [7, Proposition 5] *For* $\alpha,\beta > 0$. *If* $F \prec (1-z)^{\alpha}$ *and* $G \prec (1-z)^{\beta}$ *then* $FG \prec (1-z)^{\alpha+\beta}$, $z \in \mathbb{D}$.

Now for $0 < \mu \leqslant \rho \leqslant 1$, we have the following corollary of Theorem 4 following the same procedure as in [7, page 57].

COROLLARY 3. *For* $0 < \mu \leqslant \rho \leqslant 1$ *and* $b \geqslant \max\{c, 2c-1\} > 0$ *we have*

$$(1-z)^{\rho}\sigma_n^{(b-1,c)}(f_{\mu},z) \prec (1-z)^{\rho}, \quad z \in \mathbb{D}. \tag{11}$$

The relation (11) is sharp in the sense that it will not hold for $\mu > \rho$. It is clear when n becomes large then left hand side of (11) becomes unbounded and is subordinate to a bounded domain which is not possible.

If we change the right hand side of (11) by replacing the bounded function $(1-z)^{\rho}$, $0 \leqslant \rho < 1$ by the unbounded one $\left(\frac{1+z}{1-z}\right)^{\rho}$, $0 \leqslant \rho < 1$, then the subordination in (11) is still valid because $(1-z)^{\rho} \prec \left(\frac{1+z}{1-z}\right)^{\rho}$ in $\mathbb{D}$. Now this becomes a very interesting problem and leads to some new directions. This situation leads to the following definition.

DEFINITION 2. For $\rho \in (0,1]$, define $\mu(\rho, b-1, c)$ as the maximal number such that

$$(1-z)^{\rho}\sigma_n^{(b-1,c)}(f_{\mu},z) \prec \left(\frac{1+z}{1-z}\right)^{\rho}, \quad n \in \mathbb{N} \tag{12}$$

holds for all $0 < \mu \leqslant \mu(\rho, b-1, c)$.

Writing

$$(1-z)^{2\rho-1}\sigma_n^{b-1,c}(f_\mu,z) = (1-z)^{\rho}\sigma_n^{b-1,c}(f_\mu,z)\frac{1}{(1-z)^{1-\rho}}$$

Then (12) implies,

$$\mathrm{Re}\left[(1-z)^{2\rho-1}\sigma_n^{b-1,c}(f_\mu,z)\right] > 0, \quad z\in\mathbb{D} \quad \text{and} \quad n\in\mathbb{N}. \tag{13}$$

Motivated by Conjecture 1 given in [7], numerical evidences suggests the validity of the following conjecture given below.

CONJECTURE 1. For $\rho\in(0,1]$ we have $\mu(\rho,b-1,c)=\mu^*(\rho,b-1,c)$, where $\mu^*(\rho,b-1,c)$ is the unique solution in (0,1] of the equation

$$\int_0^{(\rho+1)\pi}\sin(t-\rho\pi)t^{\mu-1}\left(1-\frac{t}{(\rho+1)\pi}\right)^{b-c}dt=0. \tag{14}$$

Conjecture 1 for the case $\rho=1/2$ will be verify in Theorem 6, which justifies validity for the existence of conjecture 1. Note that the case $\rho=3/4$ and $1/4$ with $b=1, c=1$ are addressed in [7, 8]. The authors have provided affirmative answer for the conjecture for several ranges including the one given in [8] in a separate work. Conjecture 1 contains the following weaker one.

CONJECTURE 2. Let $\rho\in(0,1]$ and $\mu^*(\rho,b-1,c)$ be as in Conjecture 1, then

$$\mathrm{Re}\left[(1-z)^{2\rho-1}\sigma_n^{(b-1,c)}(f_\mu,z)\right] > 0, \quad z\in\mathbb{D}, \quad n\in\mathbb{N} \tag{15}$$

holds for $0<\mu\leqslant\mu^*(\rho,b-1,c)$ and $\mu^*(\rho,b-1,c)$ is the largest number with this property.

If we take $b=1+\beta$ and $c=1$ then $\sigma_n^{(b-1,c)}(z)$ reduces to Cesàro mean of order β. Figure 2 shows the graphs of $\mu^*(\rho,\beta,1)$ for $\beta=0,1,2,3$.

For $\beta=0$, Figure 1 (a) is same as graph of μ^* given in [7]. For $\rho=1$, both conjectures are equivalent and reduces to

$$(1-z)\sigma_n^{(b-1,c)}(f_\mu,z)\prec\left(\frac{1+z}{1-z}\right)$$

which holds for $0<\mu\leqslant 1$.

For $\mu(\rho,b-1,c)$ and $\mu^*(\rho,b-1,c)$, we have the following proposition.

PROPOSITION 1. *For* $0<\rho<1$, *we have* $\mu^*(\rho,b-1,c)\geqslant\mu(\rho,b-1,c)$.

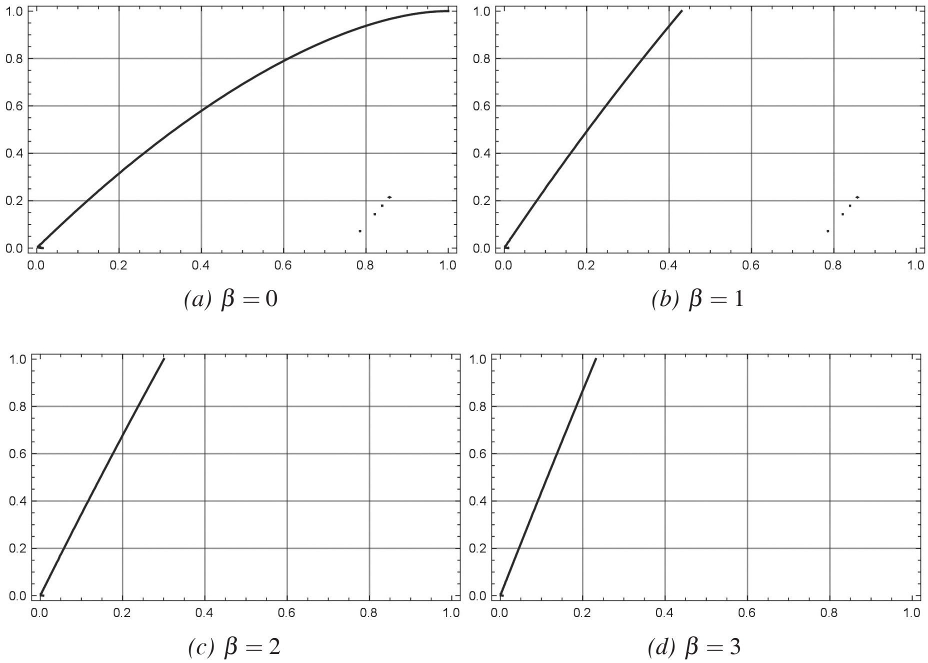


Figure 1: *Graph of* $\mu^*(\rho,\beta,1)$ *for (a)* $\beta=0$*, (b)* $\beta=1$*, (c)* $\beta=2$ *and (d)* $\beta=3$ *with respect to* ρ*.*

Proof. For $z=e^{i\phi}$, (13) is equivalent to

$$\sum_{k=0}^{n}\frac{B_{n-k}}{B_n}\frac{(\mu)_k}{k!}\sin\left[(k+\rho-1/2)\phi-\rho\pi\right]<0,\qquad \text{for } 0<\phi<2\pi. \tag{16}$$

Now limiting case of this inequality can be obtained using the asymptotic formula,

$$\lim_{n\to\infty}\left(\frac{\phi}{n}\right)^{\mu}\sum_{k=0}^{n}\frac{B_{n-k}}{B_n}\frac{(\mu)_k}{k!}\sin\left[(k+\rho-1/2)\frac{\phi}{n}-\rho\pi\right]$$
$$=\frac{1}{\Gamma(\mu)}\int_0^{\phi}t^{\mu-1}\left(1-\frac{t}{\phi}\right)^{b-c}\sin(t-\rho\pi)dt \tag{17}$$

Hence a necessary condition for the validity of (16) is the non positivity of the integral (17). In particular, $\phi=(\rho+1)\pi$ gives

$$I^{(b-1,c)}(\mu)=\int_0^{(\rho+1)\pi}\sin(t-\rho\pi)t^{\mu-1}\left(1-\frac{t}{(\rho+1)\pi}\right)^{b-c}dt.$$

We prove that $I^{(b-1,c)}(\mu)$ is strictly increasing function in $(0,1)$. Now differentiation under integral sign gives

$$\begin{aligned}
I^{(b-1,c)}(\mu)' &= \int_0^{(\rho+1)\pi} \sin(t-\rho\pi)\left(1-\frac{t}{(\rho+1)\pi}\right)^{b-c} t^{\mu-1}\log(1/t)dt \\
&= \left(1-\frac{t}{(\rho+1)\pi}\right)^{b-c} \int_0^{(\rho+1)\pi} \frac{\sin(t-\rho\pi)}{t^{1-\mu}}\log(1/t)dt \\
&\quad + (b-c)\int_0^{(\rho+1)\pi}\left(1-\frac{t}{(\rho+1)\pi}\right)^{b-c-1} \int_0^{(\rho+1)\pi} \frac{\sin(t-\rho\pi)}{t^{1-\mu}}\log(1/t)dt
\end{aligned}$$

The positivity of $I^{(b-1,c)}(\mu)'$ follows from the increasing property of the integral $I(\mu)$ in [7, Lemma 1] using the method given in [24, V. 2.29]. So $I^{(b-1,c)}(\mu)$ is strictly increasing in $(0,1)$ and if we choose $b=c$ then $I(0)=-\infty$ and $I(1)>0$, so $I(\mu)=0$ has only one solution in $(0,1]$ which is $\mu^*(\rho,b-1,c)$ given by (14). Hence the best possible bound for μ in Conjecture 2 cannot be greater than $\mu^*(\rho,b-1,c)$. This proves the assertion. $\square$

Since the conditions in Conjecture 1 and Conjecture 2 turns out to be the positivity of trigonometric polynomials. So it follows from summation by parts that both conjectures need to established only for $\mu=\mu^*(\rho,b-1,c)$. We discuss some particular cases of these conjectures.

THEOREM 6. *Conjecture 1 holds for* $\rho=1/2$.

Proof. If $\rho=1/2$ then (12) is equivalent to

$$\mathrm{Re}[(1-z)\sigma_n^{(b-1,c)}(f_\mu,z)^2]>0 \tag{18}$$

Using minimum principle for harmonic functions it is sufficient to establish (18) for $z=e^{2i\phi}, 0<\phi<\pi$. Let

$$P_n(\phi) := (1-e^{2i\phi})\left(\sum_{k=0}^{n}\frac{B_{n-k}}{B_n}\frac{(\mu)_k}{k!}e^{2ik\phi}\right)^2 \tag{19}$$

and we want to prove $\mathrm{Re}P_n(\phi)>0$ for all $n\in\mathbb{N}$, $0<\phi<\pi$.

For arbitrary number $d_k=c_{2k}=c_{2k+1}$, $k=0,1,2,\ldots,n$, we have

$$\begin{aligned}
(1+z)\sum_{k=0}^{n} d_k z^{2k} &= \sum_{k=0}^{2n+1} c_k z^k, \\
\text{and } (1-z)\sum_{k=0}^{n} d_k z^{2k} &= \sum_{k=0}^{2n+1} (-1)^k c_k z^k, \\
\text{so that } (1-z^2)\left[\sum_{k=0}^{n} d_k z^{2k}\right]^2 &= \left(\sum_{k=0}^{2n+1} c_k z^k\right)\left(\sum_{k=0}^{2n+1}(-1)^k c_k z^k\right).
\end{aligned}$$

Choosing $z = e^{i\phi}, -z = e^{-i(\pi-\phi)}$ we have

$$(1-e^{2i\phi})\left(\sum_{k=0}^{n} d_k e^{2ik\phi}\right)^2 = \left(\sum_{k=0}^{2n+1} c_k e^{ik\phi}\right)\left(\sum_{k=0}^{2n+1} (-1)^k c_k e^{ik\phi}\right),$$

which implies

$$\begin{aligned}&\mathrm{Re}(P_n(\phi))\\ &= \left(\sum_{k=0}^{2n+1} c_k \cos k\phi\right)\left(\sum_{k=0}^{2n+1} c_k \cos k(\pi-\phi)\right) + \left(\sum_{k=1}^{2n+1} c_k \sin k\phi\right)\left(\sum_{k=1}^{2n+1} c_k \sin k(\pi-\phi)\right).\end{aligned}$$

Since $c_{2k} = c_{2k+1}$, we have

$$\sin\frac{\phi}{2}\sum_{k=0}^{2n+1} c_k \cos k\phi = \cos\frac{\phi}{2}\sum_{k=1}^{2n+1} c_k \sin k(\pi-\phi).$$

This leads to the fact that

$$\sum_{k=0}^{2n+1} c_k \cos k\phi > 0, \quad 0<\phi<\pi \tag{20}$$

and

$$\sum_{k=1}^{2n+1} c_k \sin k\phi > 0, \quad 0<\phi<\pi \tag{21}$$

are equivalent. When $d_k = \frac{B_{n-k}}{B_n}\frac{(\mu)_k}{k!}$ then positivity of (20) and (21) hold respectively from Theorem 1 and Theorem 2 for $0<\mu\leqslant\mu_0'$ and $0<\phi<\pi$. So $\mathrm{Re}(P_n(\phi))>0$ which means Conjecture 1 is true for $\rho = 1/2$. $\square$

We observe that Theorem 6 becomes equivalent to the extension of Vietori's theorem [21]. We interpret extension of Vietoris' theorem in terms of generalized Cesàro stable functions.

For further generalization of Theorem 5, we define for $\mu>0$,

$$\mathscr{F}_\mu := \left\{ f\in\mathscr{A}_0 : \mathrm{Re}\left(\frac{zf'}{f}\right) > \frac{-\mu}{2}, z\in\mathbb{D}\right\},$$

and $f_\mu = \frac{1}{(1-z)^\mu}$ taken as an extremal function for $\mathscr{F}_\mu$. For all $f\in\mathscr{F}_\mu$ we get $f\prec f_\mu$. It is obvious that $f\in\mathscr{F}_\mu \Leftrightarrow zf\in\mathscr{S}^*(1-\mu/2)$. We define

$$\mathscr{P}\mathscr{F}_\mu = \{f\in\mathscr{A}_0 : f*f_\mu\in\mathscr{F}_\mu\}.$$

Clearly $\mathscr{P}\mathscr{F}_1 = \mathscr{F}_1$. The functions of $\mathscr{F}$ and $\mathscr{P}\mathscr{F}$ behaves same as the functions of starlike and prestarlike classes respectively. Before going to proceed further we recall some results on starlike and prestarlike class.

LEMMA 3. [13] *For* $0 < \mu \leqslant \rho$, *we have*

1. $\mathscr{F}_\mu \subset \mathscr{F}_\rho$
2. $\mathscr{P}\mathscr{F}_\mu \supset \mathscr{P}\mathscr{F}_\rho$
3. *If* $h \in \mathscr{P}\mathscr{F}_\mu$ *and* $f \in \mathscr{F}_\mu$ *then* $h * f \in \mathscr{F}_\mu$.

Lemma 1 also holds good in context with the class $\mathscr{F}_\mu$ and $\mathscr{P}\mathscr{F}_\mu$. We need the following lemma.

We define $\tilde{f}_\mu \in \mathscr{A}_0$ be the unique solution of $f_\mu * \tilde{f}_\mu = \frac{1}{1-z}$. It is clear that $f \in \mathscr{F}_\mu \Longleftrightarrow f * \tilde{f}_\mu \in \mathscr{P}\mathscr{F}_\mu$.

THEOREM 7. *Let* $\rho \in (0,1]$ *and* $zf \in \mathscr{S}^*(1-\mu/2)$ *with* $0 < \mu \leqslant \rho$, *then for* $b \geqslant \max\{c, 2c-1\} > 0$,

$$\frac{\sigma_n^{(b-1,c)}(f,z)}{\phi_{\rho,\mu} * f} \prec (1-z)^\rho, \quad n \in \mathbb{N}, \tag{22}$$

where $\phi_{\rho,\mu}(z) = F(1,\rho;\mu;z)$, *where* F *is the Gaussian hypergeometric function can also be defined by the equation,*

$$\frac{z}{(1-z)^\mu} * z\phi_{\rho,\mu} = \frac{z}{(1-z)^\rho}.$$

Proof. Let $\phi_{\rho,\mu}(z) = \sum_{k=0}^{\infty} \frac{(\rho)_k}{(\mu)k} z^k = f_\rho * \tilde{f}_\mu$ where $\tilde{f}_\mu$ is defined as $f_\mu * \tilde{f}_\mu = \frac{1}{1-z}$. For $0 < \mu < \rho \leqslant 1$, $f_{\rho-\mu} = \frac{1}{(1-z)^{\rho-\mu}}$ maps $\mathbb{D}$ univalently into a convex domain. $f \in \mathscr{F}_\mu \Rightarrow f * \tilde{f}_\mu \in \mathscr{P}\mathscr{F}_\mu$ and $f_\mu \in \mathscr{F}_\mu$. Clearly,

$$\frac{\phi_{\rho,\mu} * f}{f} = \frac{f_\rho * \tilde{f}_\mu * f}{f_\mu * \tilde{f}_\mu * f} = \frac{f * \tilde{f}_\mu * f_\mu f_{\rho-\mu}}{f * \tilde{f}_\mu * f_\mu} \in \overline{co}\left(f_{\rho-\mu}(\mathbb{D})\right),$$

i.e. $\frac{\phi_{\rho,\mu} * f}{f} \prec \frac{1}{(1-z)^{\rho-\mu}}$. Since $f \in \mathscr{F}_\mu \Rightarrow \frac{\sigma_n^{(b-1,c)}(f,z)}{f} \prec (1-z)^\mu$. So using Lemma 2,

$$\frac{\sigma_n^{(b-1,c)}(f,z)}{\phi_{\rho,\mu} * f} \prec (1-z)^\rho.$$

If we take $zf \in \mathscr{S}^*(1-\mu/2)$ we get that,

$$\frac{\sigma_n^{(b-1,c)}(f,z)}{\phi_{\rho,\mu} * f} \prec (1-z)^\rho. \quad \square$$

REMARK 3. If we take $\rho = \mu = 2-2\lambda$, then (22) becomes (9). This means Theorem 7 can be regarded as a generalization of Theorem 5.

3. Matrix representation

Cesàro mean of type $(b-1,c)$ can be written in terms of lower triangular matrix (g_{ij}) defined as,

$$g_{i0}=1, \qquad g_{ik}=\begin{cases}\frac{B_{i-k}}{B_i}, & 1\leqslant k\leqslant i;\\ 0, & k\geqslant i+1.\end{cases}$$

Then the entries in $(n+1)th$ row of the matrix induces Cesàro mean of type $(b-1,c)$ of order n is given by,

$$\sigma_n^{(b-1,c)}(z)=\sum_{k=0}^{n}\frac{B_{n-k}}{B_n}z^k, \quad z\in\mathbb{D}.$$

Consider,

$$G=\begin{pmatrix}1 & 0 & 0 & 0 & \cdots & 0\\ 1 & \frac{B_0}{B_1} & 0 & 0 & \cdots & 0\\ 1 & \frac{B_1}{B_2} & \frac{B_0}{B_2} & 0 & \cdots & 0\\ 1 & \frac{B_2}{B_3} & \frac{B_1}{B_3} & \frac{B_0}{B_3} & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots\end{pmatrix}\times\begin{pmatrix}a_0\\ a_1z\\ a_2z^2\\ a_3z^3\\ \vdots\end{pmatrix}$$

Then $(n+1)th$ row of G generates the Cesàro mean of type $(b-1,c)$ of $f(z)=\sum_{k=0}^{\infty}a_kz^k$ of order n for $n\geqslant 0$. Then the concept of stable function can be generalized in terms of lower triangular matrix as well.

For $n\in\mathbb{N}$, $\mathscr{H}_n$ be the set of lower triangular matrix (h_{ij}) of order $(n+1)$ satisfying $h_{ij}\geqslant 0, i,j=0,1,2\ldots,n$, and satisfy the following conditions:

1. $h_{i0}=1$ for every $i=0,1,\ldots,n$,
2. for each fixed $i\geqslant 1$, $h_{ij}=h_{i1}h_{i-1,j-1}$, $j=1,\ldots,n$,
3. for each fixed $i\geqslant 1$, $\{h_{ij}\}$ is a decreasing sequence.

Then $(n+1)th$ row of (h_{ij}) induces a polynomial H_n of degree n is

$$H_n(z):=\sum_{k=0}^{n}h_{nk}z^k,$$

and for $f(z)=\sum_{k=0}^{\infty}a_kz^k\in\mathscr{A}_1$ the polynomial

$$H_n(f,z)=\sum_{k=0}^{n}h_{nk}a_kz^k=H_n(z)*f(z). \tag{23}$$

Following the same procedure as in Theorem 4 we can obtain the following theorem for H_n defined by lower triangular matrix. We state the result without proof.

THEOREM 8. *Let H_n be given by (23), and $f_\mu = 1/(1-z)^\mu$. Suppose $h_{n1} \leqslant 1$, then for $\mu \in [-1,1]$,*

$$(1-z)^\mu H_n(f_\mu, z) \prec (1-z)^\mu. \tag{24}$$

4. Application in geometric properties of Cesàro mean of type $(b-1,c)$

For finding the geometric properties of Cesàro mean of type $(b-1,c)$, instead of $\sigma_n^{(b-1,c)}(z)$ we will use normalized Cesàro mean of type $(b-1,c)$ denoted by $s_n^{(b-1,c)}(z)$ because the geometric properties like convexity, starlikeness and close-to-convexity remains intact under such normalization. For $b+1 > c > 0$, let

$$s_n^{(b-1,c)}(z) := z + \sum_{k=2}^{n} \frac{B_{n-k}}{B_{n-1}} z^k, \quad z \in \mathbb{D}.$$

For $f \in \mathscr{A}$, it is easy to obtain that

$$s_n^{(b-1,c)}(f,z)' = \sigma_{n-1}^{(b-1,c)}(f', z) = \sigma_{n-1}^{(b-1,c)}(z) * f'(z).$$

Note that $s_n^{(\beta,1)} = s_n^{\beta}(z)$ was studied in [16]. Among the results available in the literature regarding $s_n^{\beta}(z)$, the interesting result is given by Lewis [9] is that for $\beta \geqslant 1$ and $n \in \mathbb{N}$, $s_n^{\beta}(z) \in \mathscr{K}$. Using the convolution between convex and close-to-convex functions, it is clear that for $f \in \mathscr{C}$, $(n+\beta)s_n^{\beta}(f,z)/n \in \mathscr{K}, \beta \geqslant 1$. Ruscheweyh and Salinas [17] also discussed the geometric property of $(n+\beta)s_n^{\beta}(f,z)/n$ when $0 < \beta < 1$. It is interesting to discuss the geometric property of Cesàro mean of type $(b-1,c)$ of $f(z)$, where $f(z)$ belongs to some class of functions. Note that certain geometric properties of $s_n^{(b-1,c)}(z)$ are given in [22], mainly using the positivity results that are consequences of [22]. In this section, we provide some more geometric properties as consequences of Theorem 4 and Theorem 5 which are fundamental in the formulation of concept of Cesàro stable functions.

THEOREM 9. *Let $F_\lambda(z) = z + \sum_{k=2}^{\infty} (2-2\lambda)_{k-1} \frac{z^k}{k!}$, $\lambda \in [1/2, 1)$. Then for $b \geqslant \max\{c, 2c-1\} > 0$,*

$$\left| 1 - (1-z) \cdot \left((s_n^{(b-1,c)}(F_\lambda, z))' \right)^{\frac{1}{2-2\lambda}} \right| \leqslant 1.$$

In particular, $s_n^{(b-1,c)}(F_\lambda, z) \in \mathscr{K}(\lambda)$.

Proof. It is given that,

$$F_\lambda(z) = z + \sum_{k=2}^{\infty} (2-2\lambda)_{k-1} \frac{z^k}{k!}, \quad \lambda \in [1/2, 1).$$

By Alexander transform it is obvious that,

$$F_\lambda(z) \in \mathscr{C}(\lambda) \Longleftrightarrow zF_\lambda' = \frac{z}{(1-z)^{2-2\lambda}} \in \mathscr{S}^*(\lambda). \tag{25}$$

Substituting $2-2\lambda = \mu$, we obtain

$$(1-z)^{2-2\lambda}\sigma_{n-1}^{(b-1,c)}\left(\frac{1}{(1-z)^{2-2\lambda}}, z\right) \prec (1-z)^{2-2\lambda}.$$

Since

$$\begin{aligned}(1-z)^{2-2\lambda}\sigma_{n-1}^{(b-1,c)}\left(\frac{1}{(1-z)^{2-2\lambda}}, z\right) &= (1-z)^{2-2\lambda}\sigma_{n-1}^{(b-1,c)}(F_\lambda', z)\\ &= (1-z)^{2-2\lambda}s_n^{(b-1,c)}(F_\lambda, z)',\end{aligned}$$

we get, using Theorem 4,

$$\left|1-\left((1-z)^{2-2\lambda}\cdot s_n^{(b-1,c)}(F_\lambda, z)'\right)^{\frac{1}{2-2\lambda}}\right| \leqslant 1,$$

which is equivalent to,

$$\mathrm{Re}\left((1-z)^{2-2\lambda}\cdot s_n^{(b-1,c)}(F_\lambda, z)'\right) > 0.$$

This expressions together with (25) and the analytic characterization of $\mathscr{K}(\lambda)$ guarantees that $s_n^{(b-1,c)}(F_\lambda, z) \in \mathscr{K}(\lambda)$ with respect to the starlike function given in (25). $\square$

In particular if $\lambda = 1/2$, $F_{1/2}(z) = -\log(1-z)$, then

$$s_n^{(b-1,c)}(-\log(1-z), z) \in \mathscr{K}(1/2).$$

THEOREM 10. *If $f \in \mathscr{C}(\lambda)$, $\lambda \in [1/2, 1)$ and $b \geqslant \max\{c, 2c-1\}$, then for $n \geqslant 1$,*

$$\frac{s_n^{(b-1,c)}(f,z)'}{f'(z)} \prec (1-z)^{2-2\lambda}.$$

In particular, $s_n^{(b-1,c)}(z,f) \in \mathscr{K}(\lambda)$.

Proof. If $f \in \mathscr{C}(\lambda)$, then by Alexander transform, $g(z) = zf'(z) \in \mathscr{S}^*(\lambda)$, then

$$\frac{s_n^{(b-1,c)}(f,z)'}{f'(z)} = \frac{z\sigma_{n-1}^{(b-1,c)}(f', z)}{g(z)} = \frac{z\sigma_{n-1}^{(b-1,c)}(g/z, z)}{g(z)}.$$

If $g(z) \in \mathscr{S}^*(\lambda)$, $\lambda \in [1/2, 1)$, then from Theorem 5,

$$\frac{s_n^{(b-1,c)}(f,z)'}{f'(z)} \prec (1-z)^{2-2\lambda} \Rightarrow \mathrm{Re}\left(\frac{zs_n^{(b-1,c)}(f,z)'}{g(z)}\right) > 0$$

This means $s_n^{(b-1,c)}(f,z) \in \mathscr{K}(\lambda)$. $\square$

If we substitute $b = 1+\beta$ and $c = 1$ in Theorem 10 then we obtain the following corollary.

COROLLARY 4. *If* $f \in \mathscr{C}(\lambda)$, $\lambda \in [1/2,1)$ *and* $\beta \geqslant 0$ *then for* $n \geqslant 1$, $s_n^{\beta}(f,z) \in \mathscr{K}(\lambda)$.

Theorem 10 guarantees that for $g(z) = z$, for $f \in \mathscr{C}(\lambda)$ where $\lambda \in [1/2,1)$ then,

$$\mathrm{Re}(s_n^{(b-1,c)}(f,z))' > 0 \Longrightarrow s_n^{(b-1,c)}(f,z)' \neq 0.$$

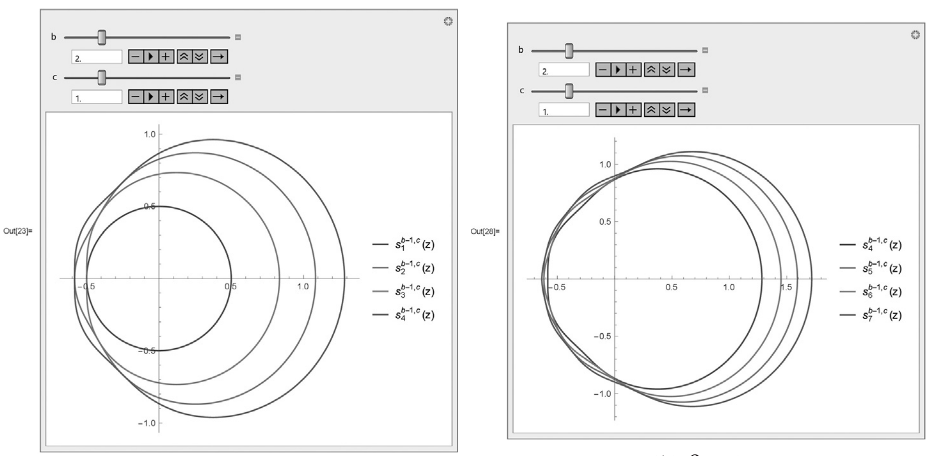
Since every close-to-convex function is univalent [2, p.47], the generalized Cesàro mean $s_n^{(b-1,c)}(f,z)$ for the convex function f is also univalent. In this situation for $b = 1, c = 1$, a subordination chain was provided by Ruscheweyh and Salinas [17] which is given in the following result.

THEOREM 11. [17] *If* $f \in \mathscr{C}(1/2)$, *then*

$$s_1^{(\alpha+k)}(f,z) \prec s_2^{(\alpha+k)}(f,z) \prec \cdots s_n^{(\alpha+k)}(f,z) \prec \cdots f(z), \quad k \in \mathbb{N}.$$

holds for $\alpha \geqslant 0$ *and* $z \in \mathbb{D}$.

An extension of Theorem 11 to $\sigma_n^{(b-1,c)}(f,z)$ can provide more information on the geometric nature of $\sigma_n^{(b-1,c)}(z)$ and we state this as a problem.



(a) $\beta = 0$ *(b)* $\beta = 1$

Figure 2: *Univalent subordination chain for* $-\log(1-z)$, *(a)* $n = 1,2,3,4$ *and (b)* $n = 5,6,7,8$.

Open Problem. For $b \geqslant \max\{c, 2c-1\} > 0$ and $f \in \mathscr{C}(\lambda)$ where $\lambda \in [1/2, 1)$ we have the following subordination chain.

$$s_1^{(b-1+k,c)}(f,z) \prec s_2^{(b-1+k,c)}(f,z) \prec \cdots s_n^{(b-1+k,c)}(f,z) \prec \cdots f(z), \quad k \in \mathbb{N}. \tag{26}$$

We do not have the proof of this problem but the graphical justification of the problem is provided here. If we take $f(z) = -\log(1-z) = \sum_{k=1}^{\infty} \frac{z^k}{k} \in \mathscr{C}(1/2)$. Figure 2 shows the univalent subordination chain for $f(z)$ when $k = 1$ and $b = 2$, $c = 1$. Figure 2 (a) shows the subordination chain for $n = 1, 2, 3, 4$ and Figure 2 (b) for $n = 4, 5, 6, 7$.

5. Concluding remarks

In this section, we define a set Ω be the set of nonnegative real numbers having the following property.

$$\Omega := \{\mu_k \in [0,1] : \text{ such that } \sum_{k=1}^{n} \mu_k = 1\}.$$

In the context of generalization of Kakeya-Eneström theorem given in [14], we have the following corollary of Theorem 5.

LEMMA 4. [14] *Let $n \in \mathbb{N}$ and $f(z) = z \sum_{k=0}^{\infty} b_k z^k \in \mathscr{S}^*(1/2)$. Then $\exists$ a number $\rho = \rho(n, f) \geqslant 1$ such that for every sequence $a_k \in \mathbb{R}, k = 0, 1, 2 \cdots, n$, with*

$$1 = a_0 \geqslant a_1 \geqslant \cdots \geqslant a_n \geqslant 0,$$

we have

$$P(z) = \sum_{k=0}^{n} a_k b_k z^k \neq 0, \quad |z| < \rho.$$

We get the following consequences of Theorem 5 using Lemma 4.

COROLLARY 5. *Let $zf \in \mathscr{S}^*(\lambda), \lambda \in [1/2, 1)$ and $b \geqslant \max\{c, 2c-1\}$. Then for any $\{\mu_k\}_{k=1}^{n} \in \Omega$, we have*

$$\sum_{k=1}^{n} \mu_k \sigma_k^{(b-1,c)}(f,z) \neq 0, \quad z \in \overline{\mathbb{D}}.$$

Proof. Clearly $\{\mu_k\}_{k=1}^{n} \in \Omega$, implies $\sum_{k=1}^{n} \mu_k = 1$. We consider

$$\sum_{k=1}^{n} \mu_k \sigma_k^{(b-1,c)}(f,z) = \sum_{k=0}^{n} \delta_k a_k z^k, \quad z \in \mathbb{D}.$$

By simple calculation we can get that,

$$1 = \delta_0 \geqslant \delta_1 \geqslant \delta_2 \geqslant \cdots \geqslant \delta_n > 0.$$

Therefore δ_k satisfies the conditions of Lemma 4, hence we proved that

$$\sum_{k=1}^{n} \mu_k \sigma_k^{(b-1,c)}(f,z) \neq 0, \quad z \in \overline{\mathbb{D}}. \quad \square \tag{27}$$

Among several other consequences possible we would like to provide an application involving Gegenbauer polynomials. Note that, for $0 < \lambda < 1/2$ and $-1 \leqslant x \leqslant 1$,

$$G(z) = \frac{z}{(1-2xz+z^2)^{\lambda}} = z \sum_{k=0}^{\infty} C_k^{\lambda}(x) z^k \in \mathscr{S}^*(1-\lambda),$$

where C_k^{λ} are the Gegenbauer polynomial of degree k and order λ. Therefore (choosing $\mu_n = 1$ and rest μ_k are all zero) we obtain,

$$\sum_{k=0}^{n} \frac{B_{n-k}}{B_n} C_k^{\lambda}(x) z^k \neq 0, \quad z \in \mathbb{D}. \tag{28}$$

The inequality (5.1) contains the result by Koumandos [4] that the partial sum of $G(z)/z$ i.e. $\sum_{k=0}^{n} C_k^{\lambda}(x) z^k$ are non-vanishing in the closed unit disc for $0 < \lambda < 1/2$. This result enables us to show that certain polynomials in z having Gegenbauer polynomials as a coefficients are zero free in the unit disc. This result will also be helpful in proving positivity of Jacobi polynomial sums [9]. The inequality (5.1) further can be sharpened in Corollary 6.

COROLLARY 6. *Let $zf \in \mathscr{S}^*(\lambda), \lambda \in [1/2,1)$ and $b \geqslant \max\{c, 2c-1\}$. Then for any $\{\mu_k\}_{k=1}^{n} \in \Omega$, we have*

$$\left| \arg \sum_{k=1}^{n} \mu_k \sigma_k^{(b-1,c)}(f,z) \right| \leqslant 2\pi(1-\lambda), \quad z \in \overline{\mathbb{D}}.$$

Proof. From Theorem 5 we have for $zf \in \mathscr{S}^*(\lambda), \lambda \in [1/2,1)$,

$$\sigma_n^{(b-1,c)}(f,z) = \left(\frac{1-\omega(z)}{1-z} \right)^{2-2\lambda}, \quad \text{where } |\omega(z)| \leqslant |z|.$$

Choose $\mu_k, k = 1,2,\ldots,n \in \Omega$ and taking the convex combination, we get

$$\sum_{k=1}^{n} \mu_k \sigma_k^{(b-1,c)}(f,z) = \left(\frac{1-\omega(z)}{1-z} \right)^{2-2\lambda}.$$

This implies

$$\left|\arg \sum_{k=1}^{n} \mu_k \sigma_k^{(b-1,c)}(f,z)\right| = (2-2\lambda)\left|\arg\left(\frac{1-\omega(z)}{1-z}\right)\right|$$
$$\Longrightarrow \left|\arg \sum_{k=1}^{n} \mu_k \sigma_k^{(b-1,c)}(f,z)\right| \leqslant 2\pi(1-\lambda). \quad \square$$

Note that if $\lambda \in [3/4, 1)$ and $zf \in \mathscr{S}^*(\lambda)$ then,

$$\left|\arg \sum_{k=1}^{n} \mu_k \sigma_k^{(b-1,c)}(f,z)\right| \leqslant \pi/2 \Longrightarrow \operatorname{Re} \sum_{k=1}^{n} \mu_k \sigma_k^{(b-1,c)}(f,z) > 0.$$

Choose $\mu_n = 1$ and rest of μ_k are zero.

$$\operatorname{Re}(\sigma_n^{(b-1,c)}(f,z)) > 0, \quad z \in \mathbb{D} \text{ and } n \in \mathbb{N}.$$

Further in context of Gegenbauer polynomials this would imply for $\lambda \in (0, 1/4]$, $n \in \mathbb{N}$,

$$\sum_{k=0}^{n} \frac{B_{n-k}}{B_n} C_k^{\lambda}(x) \cos k\theta > 0, \quad \theta \in (0,\pi), n \in \mathbb{N}. \tag{29}$$

This estimate of the upper bound on λ in (29) is not sharp. The theory of starlike functions ensure that the upper bound will be evaluated at $x = 1$ for the large values of n. However, for the case $b = c = 1$, this problem was solved by Koumandos and Ruscheweyh [6]. For that case, the upper bound for λ is $\lambda = 0.345778\ldots$. In general to find the upper bound for λ, for values of b and c, will lead to new problem which will have further implications.

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(Received June 30, 2017)

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