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APPLICATIONS OF ONE INEQUALITY TO MEASURES OF NON–COMPACTNESS AND NARROW OPERATORS

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Abstract. We consider a generalization of an inequality from papers by Yu. A. Dubinskii, J.-L. Lions and E. Magenes. This inequality is of great importance for the proof of solvability of nonlinear elliptic and parabolic equations. In contrast to their works, we do not require the compactness of the embedding. We suggest a new approach to the problem of narrow operators. In particular, we find a new application of measures of non-compactness.

1. Introduction

The paper is devoted to several variants of an inequality from [5, 15, 16]:

- ([5, Lemma 1]) Let E, E_1 be linear normed spaces and E be embedded into E_1. Let $\Im$ by any subset of E such that $\lambda u \in \Im$ $\forall u \in \Im$ and $\forall \lambda \in \mathbb{R}$. Let $\Im$ be provided by function $M : \Im \to \mathbb{R}$ such that $M(u) \geqslant 0$, $M(u) = 0 \iff u = 0$, $M(\lambda u) = |\lambda| M(u)$ $\forall u \in \Im$ and $\forall \lambda \in \mathbb{R}$. Let $\Im$ be embedded into E, i.e. $\|u\|_E \leqslant KM(u)$ for some $K > 0$ and all $u \in \Im$. Let the embedding $\Im \subset E$ be compact, i.e. every sequence $\{u_n\}$ ($u_n \in \Im$, $M(u_n) \leqslant K_0$ for some $K_0 > 0$) contains a subsequence which converges in E. Then

$$\|u - v\|_E \leqslant \varepsilon(M(u) + M(v)) + c_\varepsilon \|u - v\|_{E_1} \tag{1.1}$$

  for all $u, v \in \Im$.

- ([15, Lemma 5.1] and [16, Theorem 16.4]) Let E_0, E, E_1 be Banach spaces, $E_0 \subset E \subset E_1$ and the embedding E_0 into E be compact. Then, for every $\varepsilon > 0$ there exists a constant c_ε such that

$$\|u\|_E \leqslant \varepsilon \|u\|_{E_0} + c_\varepsilon \|u\|_{E_1} \tag{1.2}$$

  for all $u \in E_0$.

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In [15, Lemma 5.1] E_0, E_1 are also reflexive.

As it was noted in [15, 12.2], Inequality (1.1) implies Inequality (1.2).

In [5, 15, 16] Inequalities (1.1) and (1.2) were used to obtain a priori estimates of solutions of nonlinear elliptic and parabolic equations (see for example [16, Remark 16.3]).

In this paper we investigate similar inequalities under assumptions that are less restrictive than in [5, 15, 16]. In particular we do not require the compactness of the embedding. We prove the inequalities and construct an example (Example 1) where the inequalities are still valid but the corresponding embedding is not compact.

The results obtained could have the same applications as the results from [5, 15, 16], e.g. to the proof of solvability of nonlinear elliptic and parabolic equations [5, 15, 16]. Moreover, in this paper we find a new application of this inequality and measures of non-compactness (MNCs for brevity), namely to narrow operators. For an account of the theory of measures of non-compactness, we forward the reader to [1, 3, 4] (see also [2, 6, 7, 8, 9, 10, 11, 12, 20, 21, 22]) and references therein.

Theorems 1-3 of this paper generalize previous results obtained in [6].

2. Main results

Let E and E_1 be normed spaces. Let $M\colon \mathfrak{I} \to \mathbb{R}_+$ be a function defined on some subset $\mathfrak{I} \subseteq E$ where $\mathbb{R}_+$ is the set of real non-negative numbers.

Consider maps $A\colon \mathfrak{I} \to E$, $T\colon \mathfrak{I} \to E_1$, and a subset $U \subseteq \mathfrak{I}$ satisfying the following condition:

$$\forall \varepsilon > 0\ \exists c_\varepsilon > 0 : \forall u \in U \qquad \|A(u)\|_E \leqslant \varepsilon M(u) + c_\varepsilon \|T(u)\|_{E_1}. \tag{2.1}$$

REMARK 1. If $\|A(u_0)\|_E = 0$ for some $u_0 \in \mathfrak{I}$ then (2.1) is trivial for $u = u_0$. Note that (2.1) also implies

1. $\|A(u_0)\|_E = 0$ for all $u_0 \in \mathfrak{I}$ with $\|T(u_0)\|_{E_1} = 0$;
2. $\lim\limits_{n\to\infty} \|A(u_n)\|_E = 0$ if $\lim\limits_{n\to\infty} \|T(u_n)\|_{E_1} = 0$ and $\lim\limits_{n\to\infty} M(u_n) = 0$ for some $\{u_n\}$, $u_n \in \mathfrak{I}$.

For a subset $U \subseteq \mathfrak{I}$ we define

$$\tau(U) = \sup_{v\in U} M(v), \quad \tilde{\tau}(U) = \inf_{v\in U} M(v).$$

A subset $U \subseteq \mathfrak{I}$ is called *M-bounded* if $\tau(U) < \infty$. We say that a set $U \subset \mathfrak{I}$ has *M-property*, if U is M-bounded and $\tilde{\tau}(U) > 0$. We say that an operator $A\colon \mathfrak{I} \to E$ is *M-bounded*, if A maps every M-bounded set $U \subset \mathfrak{I}$ into a bounded subset of E.

THEOREM 1. *Let E and E_1 be normed spaces, let A be an M-bounded operator from $\mathfrak{I}$ into E, and let T be an M-bounded operator from $\mathfrak{I}$ into E_1. Then the following conditions are equivalent:*

(i) For every subset $U \subset \mathfrak{I}$ possessing the M-property and for every $\varepsilon > 0$ there exists a constant $c_\varepsilon > 0$ such that (2.1) is true for all $u \in U$.

(ii) For any sequence $\{u_n\}$ of elements from $\mathfrak{I}$ such that $\{u_n\}$ possesses the M-property and

$$\lim_{n\to\infty} \|T(u_n)\|_{E_1} = 0, \tag{2.2}$$

we have

$$\lim_{n\to\infty} \|A(u_n)\|_E = 0. \tag{2.3}$$

(iii) For any sequence $\{u_n\}$ of elements from $\mathfrak{I}$ such that $\{u_n\}$ possesses the M-property and (2.2) holds we have

$$\varliminf_{n\to\infty} \|A(u_n)\|_E = 0. \tag{2.4}$$

Proof. Suppose that (i) is true. We claim that (ii) is true too.

Let a sequence $\{u_n\}$ have the M-property. Then there exist numbers $0 < r \leqslant R < \infty$ such that $r \leqslant M(u_n) \leqslant R$ for all n. Now we assume that (2.2) is true and claim that (2.3) is true too. Fix any $\varepsilon > 0$. Let c_ε be the constant from (2.1). Then there exists a number $n_\varepsilon > 0$ such that for all $n > n_\varepsilon$ we have $\|T(u_n)\|_{E_1} \leqslant \varepsilon / c_\varepsilon$. Now (2.1) implies that for all $n > n_\varepsilon$ we have also $\|A(u_n)\|_E \leqslant \varepsilon R + \varepsilon$.

Letting $\varepsilon \to 0$, we get (2.3), that is, Condition (ii) holds.

Condition (iii) obviously follows from (ii).

Suppose now that (iii) is true, but (i), that is, (2.1), is not true. Thus there exist a set U with the M-property, $\varepsilon > 0$, a sequence of elements $\{u_n\}$ ($u_n \in U$ for every n) and a sequence of numbers $\{c_n\}$ ($c_n \to \infty$) such that

$$\|A(u_n)\|_E > \varepsilon M(u_n) + c_n \|T(u_n)\|_{E_1} \tag{2.5}$$

holds for all n.

By the assumptions of Theorem 1, the operator A is M-bounded. Since $\{u_n\}$ has the M-property and hence M-bounded, the sequence $\{A(u_n)\}$ is bounded in the norm of E. Now (2.5) together with $c_n \to \infty$ implies $\|T(u_n)\|_{E_1} \to 0$. However by (2.5) we have $\|A(u_n)\|_E > \varepsilon r$ for all n, which contradicts (2.4). Therefore, (iii) indeed implies (i). $\square$

REMARK 2. In contrast to [5, 15, 16], we do not assume that set $\mathfrak{I}_1 = \{v \in \mathfrak{I} : M(v) \leqslant 1\}$ is relatively compact in E.

EXAMPLE 1. Let Ω be a subset of $\mathbb{R}^n$, let μ be a continuous measure in the sense of [14]. Suppose $\mu(\Omega) < \infty$. Denote by $L_p(\mu)$ the space of μ-measurable functions on Ω with the norm $\|u\|_{L_p(\mu)} = \left(\int_\Omega |u|^p d\mu \right)^{1/p}$ for $1 \leqslant p < \infty$ and

$$\|u\|_{L_\infty(\mu)} = \inf\{t \in \mathbb{R} : \mu(D(u,t)) = 0\},$$

where

$$D(u,t) = \{s \in \Omega : |u(s)| > t\}$$

for $t > 0$.

Let $\mathfrak{I} \subseteq L_\infty(\mu)$ be a set all functions u such that $|u(s)| = \gamma$ almost everywhere in Ω for any $\gamma \in \mathbb{R}_+$. Then for all $\varepsilon > 0$ there exists a constant $c_\varepsilon > 0$ such that

$$\|u\|_{L_p(\mu)} \leqslant \varepsilon \|u\|_{L_\infty(\mu)} + c_\varepsilon \|u\|_{L_q(\mu)}$$

for all $1 \leqslant p < \infty$, $1 \leqslant q < \infty$ and $u \in \mathfrak{I}$ belonging to an arbitrary spherical interlayer $r \leqslant \|u\|_{L_\infty(\mu)} \leqslant R$ $(0 < r \leqslant R < \infty)$.

Indeed, consider the norm in $L_\infty(\mu)$ as a M-function on $\mathfrak{I}$. Let a sequence $\{u_n\}$ satisfy conditions: there exist constants $0 < r \leqslant R < \infty$ such that $r \leqslant \|u_n\|_{L_\infty(\mu)} \leqslant R$ for all n and

$$\lim_{n\to\infty} \|u_n\|_{L_q(\mu)} = 0.$$

It follows that

$$\lim_{n\to\infty} r(\mu(\mathrm{supp}\ u_n))^{1/q} \leqslant \lim_{n\to\infty} \|u_n\|_{L_q(\mu)} = 0$$

and

$$\lim_{n\to\infty} \mu(\mathrm{supp}\ u_n) = 0$$

and

$$\lim_{n\to\infty} \sqrt[p]{\int_\Omega |u_n|^p d\mu} = \lim_{n\to\infty} \|u_n\|_{L_p(\mu)} \leqslant R \lim_{n\to\infty} (\mu(\mathrm{supp}\ u_n))^{1/p} = 0,$$

that is, the condition (ii) of Theorem 1 is satisfied. By Theorem 1 it implies (i).

REMARK 3. It is well known that the sequence of Rademacher functions $r_n = \mathrm{sign}\sin(2^n \pi t)$ $(t \in [0,1]$, $n = 1,2,\ldots)$ is not compact in $L_p([0,1],\mathbb{R})$.

Analogously to Theorem 1, we can prove the assertion below.

THEOREM 2. *Let* (E,d) *and* (E_1,d_1) *be metric spaces,* A *be an* M*-bounded operator from* $\mathfrak{I}$ *into* E *and* T *be an* M*-bounded operator from* $\mathfrak{I}$ *into* E_1. *Then the following conditions are equivalent:*

(i) For every subset $U \subset \mathfrak{I}$ *possessing the* M*-property and every* $\varepsilon > 0$, *there exists a constant* $c_\varepsilon > 0$ *such that for all* $u, v \in U$

$$d(A(u),A(v)) \leqslant \varepsilon(M(u)+M(v)) + c_\varepsilon d_1(T(u),T(v)). \tag{2.6}$$

(ii) For any sequences $\{u_n\}$, $\{v_n\}$ *in* $\mathfrak{I}$ *possessing the* M*-property and satisfying*

$$\lim_{n\to\infty} d_1(T(u_n),T(v_n)) = 0,$$

we have

$$\lim_{n\to\infty} d(A(u_n),A(v_n)) = 0.$$

(iii) For any sequences $\{u_n\}$, $\{v_n\}$ *in* $\mathfrak{I}$ *possessing the* M*-property and satisfying*

$$\lim_{n\to\infty} d_1(T(u_n),T(v_n)) = 0,$$

we have

$$\varliminf_{n\to\infty} d(A(u_n),A(v_n)) = 0.$$

Proof. Suppose that (i) is true. We claim that (ii) is true too.

Let sequences $\{u_n\}$, $\{v_n\}$ in $\mathfrak{I}$ have the M-property. Then there exist numbers $0 < r \leqslant R < \infty$ such that $r \leqslant M(u_n), M(u_n) \leqslant R$ for all n. Now we assume that

$$\lim_{n\to\infty} d_1(T(u_n),T(v_n)) = 0$$

and we claim that

$$\lim_{n\to\infty} d(A(u_n),A(v_n)) = 0.$$

Fix any $\varepsilon > 0$. Let c_ε be the constant from (2.6). Then there exists a number $n_\varepsilon > 0$ such that for all $n > n_\varepsilon$ we have $d_1(T(u_n),T(v_n)) \leqslant \varepsilon/c_\varepsilon$. Now (2.6) implies that for all $n > n_\varepsilon$ we have also $d(A(u_n),A(v_n)) \leqslant \varepsilon 2R + \varepsilon$.

Letting $\varepsilon \to 0$, we get $\lim\limits_{n\to\infty} d(A(u_n),A(v_n)) = 0$, i.e. Condition (ii) holds.

Condition (iii) obviously follows from (ii).

Suppose now that (iii) is true, but (i), i.e. (2.6) is not true. Then there exist $\varepsilon > 0$, sequences of elements $\{u_n\}$, $\{v_n\}$ with the M-property and a sequence of numbers $\{c_n\}$ $(c_n \to \infty)$ such that

$$d(A(u_n),A(v_n)) > \varepsilon(M(u_n)+M(v_n)) + c_n d_1(T(u_n),T(v_n)) \tag{2.7}$$

holds for all n.

By the assumptions of Theorem 2, the operator A is M-bounded. Since $\{u_n\}$ and $\{v_n\}$ have the M-property and hence M-bounded, the sequence $\{d(A(u_n),A(v_n))\}$ is bounded in E. Now (2.7) together with $c_n \to \infty$ implies $d_1(T(u_n),T(v_n)) \to 0$. However by (2.7) we have $d(A(u_n),A(v_n)) > \varepsilon 2r$ for all n, which contradicts

$$\varliminf_{n\to\infty} d(A(u_n),A(v_n)) = 0.$$

Therefore, (iii) indeed implies (i). □

3. Measures of non-compactness in the inequality

Let (E,d) be any metric space. Let us recall that a set U is totally bounded if for each $\delta > 0$ the set may be covered by a finite number of balls of radius $r < \delta$. In a complete metric space a totally bounded set is precompact (relatively compact), that is, its closure is compact.

The MNC $\beta_E(U) = \beta(U)$ *of* $U \subset E$ or *the separation MNC* is defined as the supremum of all numbers $r > 0$ such that there exists an infinite sequence in U with $d(u_n,u_m) \geqslant r$ for every $n \neq m$ (see, for example, [1, 3, 4]).

The MNC β satisfies the regularity property: $\beta_E(U) = 0$ if and only if U is a totally bounded set in E.

THEOREM 3. *Let (E,d) and (E_1,d_1) be metric spaces and let $A\colon \mathfrak{I} \to E$ and $T\colon \mathfrak{I} \to E_1$ be M-bounded operators. Let $U \subset \mathfrak{I}$ be a subset possessing the M-property. Suppose that for every $\varepsilon > 0$ there exists a constant $c_\varepsilon > 0$ such that the inequality (2.6) holds for all $u,v \in U$. Then for every $\varepsilon > 0$ and every subset $V \subseteq U$ we have*

$$\beta_E(A(V)) \leqslant 2\varepsilon\tau(V) + c_\varepsilon \beta_{E_1}(T(V)). \tag{3.1}$$

Proof. Let (2.6) be true for all u,v from U. Let V be an arbitrary subset of U. If $\beta_E(A(V)) = 0$, then the assertion is trivial. Let $\beta_E(A(V)) > 0$. By the definition of the MNC β_E, for any $0 < \delta < \beta_E(A(V))$ there exists a sequence $\{A(v_n)\} \subset A(V)$ such that

$$\beta_E(A(V)) - \delta \leqslant d(A(v_n), A(v_m))$$

for all $n \neq m$.

By the definition of β_{E_1} we can choose in $\{v_n\}$ elements $\tilde{v}_n$, $\tilde{v}_m$, satisfying the inequality

$$d_1(T(\tilde{v}_n), T(\tilde{v}_m)) \leqslant \beta_{E_1}(T(V)) + \delta.$$

Applying (2.6) for any $\varepsilon > 0$, we obtain

$$\begin{aligned}\beta_E(A(V)) - \delta &\leqslant d(A(\tilde{v}_n), A(\tilde{v}_m)) \\ &\leqslant \varepsilon(M(\tilde{v}_n) + M(\tilde{v}_m)) + c_\varepsilon d_1(T(\tilde{v}_n), T(\tilde{v}_m)) \\ &\leqslant 2\varepsilon\tau(V) + c_\varepsilon(\beta_{E_1}(T(V)) + \delta),\end{aligned}$$

which implies (3.1) since δ is arbitrary. □

COROLLARY 1. *If (3.1) be true then $\beta(T(U)) = 0$ implies $\beta(A(U)) = 0$ for all subsets $U \subset \mathfrak{I}$ possessing the M-property. It is obviously that the last remains valid for every MNC equivalent to the MNC β.*

Let us recall ϕ is an MNC equivalent to the MNC β, that is, there exist constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1\phi(V) \leqslant \beta(V) \leqslant c_2\phi(V).$$

Below we consider any non-negative function ψ_E (not necessarily equivalent to β) defined on bounded subsets of a normed space E with $\psi_E(U) < \infty$ for all bounded subsets of a normed space E. For example, ψ_E may be the measure of nonequiabsolute continuity ν_E in a regular space E ([2, 7, 8, 9, 10, 20, 21, 22]). It is well known that the equality $\nu_E(U) = 0$ is possible on non-compact sets.

THEOREM 4. *Let E and E_1 be linear normed spaces, A be an M-bounded operator from $\mathfrak{I}$ into E, and let T be an M-bounded operator from $\mathfrak{I}$ into E_1. Then the following conditions are equivalent:*

(i) For every subset $U \subset \mathfrak{I}$ possessing the M-property and for every $\varepsilon > 0$ there exists a constant $c_\varepsilon > 0$ such that

$$\psi_E(A(V)) \leqslant 2\varepsilon\tau(V) + c_\varepsilon \psi_{E_1}(T(V)) \tag{3.2}$$

is true for every subset $V \subseteq U$.

(ii) For any sequence $\{V_n\}$ of subsets from $\mathfrak{I}$ such that $\cup V_n$ possesses the M-property and

$$\lim_{n\to\infty} \psi_{E_1}(T(V_n)) = 0, \tag{3.3}$$

we have

$$\lim_{n\to\infty} \psi_E(A(V_n)) = 0. \tag{3.4}$$

(iii) For any sequence $\{V_n\}$ of subsets from $\mathfrak{I}$ such that $\cup V_n$ possesses the M-property and (3.3) holds we have

$$\underline{\lim}_{n\to\infty} \psi_E(A(V_n)) = 0. \tag{3.5}$$

Proof. Suppose that (i) is true. We claim that (ii) is true too. Let a consequence $\{V_n\}$ such that $\cup V_n$ have the M-property. Then there exist numbers $0 < r \leqslant R$ such that $r \leqslant M(u) \leqslant R$ for all $u \in \cup V_n$. Now we assume that (3.3) is true and claim that (3.4) is true too. Fix any $\varepsilon > 0$. Let c_ε be the constant from (3.2). Then there exists a number $n_\varepsilon > 0$ such that for all $n > n_\varepsilon$ we have $\psi_{E_1}(T(V_n)) \leqslant \varepsilon / c_\varepsilon$. Now (3.2) implies that for all $n > n_\varepsilon$ we have also $\psi_E(A(V_n)) \leqslant \varepsilon R + \varepsilon$. Letting $\varepsilon \to 0$, we get (3.4), that is, Condition (ii) holds.

Condition (iii) obviously follows from (ii).

Suppose now that (iii) is true, but (i), that is, (3.2), is not true. Then there exist a set U with the M-property, $\varepsilon > 0$, a sequence of subsets $\{V_n\}$ ($V_n \subseteq U$ for every n) and a sequence of numbers $\{c_n\}$ ($c_n \to \infty$) such that

$$\psi_E(A(V_n)) > 2\varepsilon\tau(V_n) + c_n \psi_{E_1}(T(V_n)) \tag{3.6}$$

holds for all n. By the assumptions of Theorem 4, the operator A is M-bounded. Since U has the M-property and hence M-bounded, $V_n \subseteq U$ for every n, the sequence $\{A(V_n)\}$ is bounded in the norm of E. Now (3.6) together with $c_n \to \infty$ implies $\psi_{E_1}(T(V_n)) \to 0$. However by (3.6) we have $\psi_E(A(V_n)) > 2\varepsilon r$ for all n, which contradicts (3.5). Therefore, (iii) indeed implies (i). $\square$

4. Application to narrow operators

We first recall basic definitions and notation in a form convenient for us.

Let $(\Omega, \sum, \mu)$ be a measure space where Ω is a subset of $\mathbb{R}^n$ and μ is a continuous measure in the sense of [14]; that is, each subset $D \subseteq \Omega$, $\mu(D) > 0$, can be split into two subsets of the same measure. Assume also that $\mu(\Omega) < \infty$.

Let $\sum^+$ be the set of all $D \in \sum$ with $\mu(D) > 0$ and let $S(\mu)$ be the linear space of all equivalence classes of μ-measurable functions $u\colon \Omega \to K$, where $K \in \{\mathbb{R}, C\}$.

Let κ_D be the characteristic function of a set $D \in \sum$. A function $u \in S(\mu)$ is called a mean zero sign on D if $u^2 = \kappa_D$ and $\mu(\{t \in D : u(t) = 1\}) = \mu(\{t \in D : u(t) = -1\})$.

Recall that a normed space X which is a linear subspace of $S(\mu)$ is called an *ideal space* if $\kappa_\Omega \in X$ and for each $u \in S(\mu)$ and $v \in X$ the condition $|u| \leqslant |v|$ implies $u \in X$ and $\|u\|_X \leqslant \|v\|_X$.

Let $D \in \Sigma^+$ be any subset and let X be an ideal space on (Ω, Σ, μ). Denote by $\mathfrak{I}_D \subset X$ be set of all mean zero signs on D. Since $|u| \leqslant \kappa_\Omega$ for all $u \in \mathfrak{I}_D$ and $D \in \Sigma^+$, the set $\mathfrak{I}_D$ is bounded in X for all $D \in \Sigma^+$.

In addition, the embedding $I\colon L_\infty(\mu) \to X$ is bounded for all ideal spaces X.

The spaces $L_p(\mu)$ (see Example 1) are examples of ideal Banach spaces.

The elements of the sequence of Rademacher functions $r_n = \operatorname{sign}\sin(2^n \pi t)$ ($t \in [0,1]$, $n = 1, 2, \dots$) in $L_p([0,1], \mathbb{R})$ are examples of mean zero signs on $[0,1]$.

The definition of a mean zero sign implies the following assertion:

PROPOSITION 1. *Let $D_1, D_2 \in \Sigma^+$ be any sets with $D_1 \cap D_2 = \varnothing$. Then for any $u_1 \in \mathfrak{I}_{D_1}$, $u_2 \in \mathfrak{I}_{D_2}$ the function $u = u_1 \pm u_2$ belongs to $\mathfrak{I}_{D_1 \cup D_2}$.*

For normed spaces E and E_1 denote by $\mathscr{L}(E, E_1)$ the space of all linear continuous operators $A\colon E \to E_1$.

Let X be an ideal space on (Ω, Σ, μ) and let Y be a Banach space.

An operator $A \in \mathscr{L}(X,Y)$ is called *narrow* if for each $\delta > 0$ and each $D \in \Sigma^+$ there exists a mean zero sign u on D with $\|Au\|_Y < \delta$.

The notion of a narrow operator was introduced by A. Plichko and M. Popov [18]. Narrow operators were studied by many mathematicians (for details and bibliography see, for example, [13, 17, 19]).

The following assertion is a consequence of the definition.

PROPOSITION 2. *Let X be an ideal space on (Ω, Σ, μ) and let $D \in \Sigma^+$ be any subset. Then the following conditions are equivalent:*

(i) $A \in \mathscr{L}(X,Y)$ is a narrow operator;

(ii) $\forall D \in \Sigma^+ \quad \inf\limits_{u \in \mathfrak{I}_D} \|Au\|_Y = 0$;

(iii) $\forall D \in \Sigma^+ \quad \exists \{u_n\} : u_n \in \mathfrak{I}_D,\ n = 1, 2, \dots,\ \lim\limits_{n\to\infty} \|Au_n\|_Y = 0.$

We suggest a new approach to the problem of narrow operators via inequality (2.1). (Recall that by $\mathfrak{I}_D \subset X$ we denote the set of all mean zero signs on D.)

THEOREM 5. *Let X be an ideal space on (Ω, Σ, μ) and let Y be a Banach space. Suppose that for every $D \in \Sigma^+$ there exists a bounded function $M_D : \mathfrak{I}_D \to \mathbb{R}_+$ and an M_D-bounded operator (not necessarily linear) $T_D\colon \mathfrak{I}_D \to E_1$ (where E_1 is some normed space) such that for any $\delta > 0$ there exists $u \in \mathfrak{I}_D$ with $\|T_D(u)\|_{E_1} < \delta$. Then every operator $A \in \mathscr{L}(X,Y)$, satisfying for all $D \in \Sigma^+$ the condition:*

$$\forall \varepsilon > 0\ \exists c_\varepsilon > 0 : \forall u \in \mathfrak{I}_D \qquad \|Au\|_Y \leqslant \varepsilon M_D(u) + c_\varepsilon \|T_D(u)\|_{E_1} \tag{4.1}$$

is a narrow operator.

Proof. Let $D \in \Sigma^+$ be any subset and suppose $A \in \mathscr{L}(X,Y)$ satisfies (4.1). Since T_D is M_D-bounded, (4.1) implies that A is M_D-bounded too. Putting $\delta \to 0$ in the assumptions of Theorem 5, we get a sequence $\{u_n\}$ of elements of $\mathfrak{I}_D$ such that $\lim\limits_{n\to\infty} \|T_D(u_n)\|_{E_1} = 0$.

If $\varliminf_{n\to\infty} M_D(u_n) > 0$, then $U = \{u_n\} \subseteq \mathfrak{I}_D$ possesses M_D-property. Applying Theorem 1 to $U = \{u_n\}$, we obtain $\lim_{n\to\infty} \|Au_n\|_Y = 0$.

If $\varliminf_{n\to\infty} M_D(u_n) = 0$, then $\varliminf_{n\to\infty} \|Au_n\|_Y = 0$ by (4.1), i.e. A is narrow by Proposition 2. $\square$

REMARK 4. Let $L_p(\mu)$, $1 \leqslant p < \infty$, be as in Example 1. Since $\|u\|_{L_p(\mu)} = (\mu(D))^{1/p}$ for all $u \in \mathfrak{I}_D$ and $D \in \Sigma^+$, the identity operator $I\colon L_p(\mu) \to L_p(\mu)$ and the embedding $I\colon L_p(\mu) \to L_q(\mu)$ $(q < p)$ are not narrow.

Recall that an operator $A\colon E \to E_1$, where E, E_1 are normed spaces, is *compact* if it maps bounded subsets of E into totally bounded subsets of E_1. Below we show that the restriction on $\mathfrak{I}_D$ of the embedding $I\colon L_\infty(\mu) \to L_q(\mu)$ $(1 \leqslant q < \infty)$ is not compact for all $D \in \Sigma^+$.

EXAMPLE 2. Fix any $D \in \Sigma^+$. We construct a sequence $\{u_n\}, u_n \in \mathfrak{I}_D, n = 1,2,\ldots$, defined in a manner analogous to the sequence of Rademacher functions.

Since μ is a continuous measure we can split D into subsets $D = D_1 \cup D_2$, $D_1 \cap D_2 = \emptyset$, $\mu(D_1) = \mu(D_2)$. Putting

$$u_1(s) = \begin{cases} 1 & \text{if } s \in D_1, \\ -1 & \text{if } s \in D_2, \\ 0 & \text{if } s \notin D, \end{cases} \tag{4.2}$$

we obtain some function from $\mathfrak{I}_D$. By analogy with (4.2) we can construct $u_{11} \in \mathfrak{I}_{D_1}$ and $u_{12} \in \mathfrak{I}_{D_2}$. Denote $u_2 := u_{11} + u_{12} \in \mathfrak{I}_D$. Iterating the above process, we obtain a sequence $\{u_n\}$, $u_n \in \mathfrak{I}_D, n = 1,2,\ldots$, defined in a manner analogous to the sequence of Rademacher functions.

Note that $u_1 - u_2 = 2\tilde{u}$ where $\tilde{u} \in \mathfrak{I}_{\tilde{D}_{12}}$ and $2\mu(\tilde{D}_{12}) = \mu(D)$, $2\mu(\tilde{D}_{12} \cap D_i) = \mu(D_i)$, $i = 1,2$. For all $n \neq m$

$$u_n - u_m = 2\tilde{v},\ \tilde{v} \in \mathfrak{I}_{\tilde{D}_{nm}},\ 2\mu(\tilde{D}_{nm}) = \mu(D) \tag{4.3}$$

where $2\mu(\tilde{D}_{nm} \cap D_i) = \mu(D_i)$, $i = 1,2$.

Note that $\|u_1 - u_2\|_{L_q(\mu)} = (2^{q-1}\mu(D))^{1/q}$ for all $1 \leqslant q < \infty$. For the sequence $\{u_n\}$, $u_n \in \mathfrak{I}_D$, $n = 1,2,\ldots$, constructed in Example 2 we have $\beta_{L_q(\mu)}\{u_n\} = (2^{q-1}\mu(D))^{1/q}$ since $\|u_k - u_m\|_{L_q(\mu)} = (2^{q-1}\mu(D))^{1/q}$ for all $k \neq m$ and $1 \leqslant q < \infty$. Hence the restriction on $\mathfrak{I}_D$ of the embedding $I\colon L_\infty(\mu) \to L_q(\mu)$ $(1 \leqslant q < \infty)$ is not compact for all $D \in \Sigma^+$.

The idea of the proof of the following sufficient condition for an operator to be narrow is based on the definition of the MNC β as well as on the proof of Theorem 3.

We assume below that μ is *σ-additive*, i.e. for all countable collections $\{D_i\}_{i=1}^\infty$ of pairwise disjoint sets in Σ we have

$$\mu\left(\bigcup_{k=1}^\infty D_k\right) = \sum_{k=1}^\infty \mu(D_k). \tag{4.4}$$

THEOREM 6. *Let X be an ideal space on (Ω,Σ,μ) where the measure μ is σ-additive, and let Y be a Banach space. Denote by Z the set of all mean zero signs u in X. Suppose $A\in\mathscr{L}(X,Y)$ is compact on Z and*

$$\forall\varepsilon>0\ \exists c_\varepsilon>0\ \ \|Au\|_Y\leqslant\varepsilon+c_\varepsilon\mu(\operatorname{supp}u) \tag{4.5}$$

for all $u\in Z$. Then A is a narrow operator.

Proof. Fix any $D\in\Sigma^+$. Let $\{u_n\}$, $u_n\in\mathfrak{I}_D$, $n=1,2,\ldots$, be a sequence constructed in Example 2. Then the sequence $\{Au_n\}$ is relatively compact, that is, $\beta_Y\{Au_n\}=0$. Hence for any $\delta>0$ there exist $u_{\tilde n}$ and $u_{\tilde m}$ ($\tilde n\neq\tilde m$) such that $\|Au_{\tilde n}-Au_{\tilde m}\|_Y\leqslant\delta$. Hence by (4.3) and linearity of A

$$\delta\geqslant\|Au_{\tilde n}-Au_{\tilde m}\|_Y=\|A(u_{\tilde n}-u_{\tilde m})\|_Y=\|A(2\tilde v)\|_Y=2\|A\tilde v\|_Y$$

where $\tilde v\in\mathfrak{I}_{D_{\tilde n\tilde m}}$ and $2\mu(D_{\tilde n\tilde m})=\mu(D)$.

For simplicity of notation put $\Delta_1=D_{\tilde n\tilde m}$, $v_1=\tilde v$, $\Delta_1^+=\{\omega\in D: v_1=1\}$ and $\Delta_1^-=\{\omega\in D: v_1=-1\}$.

Thus we find the sets Δ_1, Δ_1^+ and Δ_1^- with $\Delta_1^+\cup\Delta_1^-=\Delta_1$, $\Delta_1^+\cap\Delta_1^-=\varnothing$, $2^2\mu(\Delta_1^+)=2^2\mu(\Delta_1^-)=\mu(D)$ and the function $v_1=\kappa_{\Delta_1^+}-\kappa_{\Delta_1^-}$, $v_1\in\mathfrak{I}_{\Delta_1}$ such that $\|Av_1\|_Y\leqslant\delta/2$. We repeat our argument for $D\setminus\Delta_1$ and we find the sets $\Delta_2\subset D\setminus\Delta_1$, Δ_2^+, Δ_2^- with $\Delta_2^+\cup\Delta_2^-=\Delta_2$, $\Delta_2^+\cap\Delta_2^-=\varnothing$, $2^3\mu(\Delta_2^+)=2^3\mu(\Delta_2^-)=\mu(D)$ and the function $v_2=\kappa_{\Delta_2^+}-\kappa_{\Delta_2^-}$, $v_2\in\mathfrak{I}_{\Delta_2}$ such that $\|Av_2\|_Y\leqslant\delta/2^2$.

Iterating this process we get sequences of sets $\{\Delta_n\}$, $\{\Delta_n^+\}$, $\{\Delta_n^-\}$ and of functions $\{v_n\}$ such that $\Delta_n\subset D\setminus(\Delta_1\cup\ldots\cup\Delta_{n-1})$, $\Delta_n^+\cup\Delta_n^-=\Delta_n$, $\Delta_n^+\cap\Delta_n^-=\varnothing$, $2^{n+1}\mu(\Delta_n^+)=2^{n+1}\mu(\Delta_n^-)=\mu(D)$, $v_n=\kappa_{\Delta_n^+}-\kappa_{\Delta_n^-}$, $v_n\in\mathfrak{I}_{\Delta_n}$ and $\|Av_n\|_Y\leqslant\delta/2^n$ for all n.

Note that this process does not terminate. Define

$$P_m^+=\bigcup_{n=m}^{\infty}\Delta_n^+;\ P_m^-=\bigcup_{n=m}^{\infty}\Delta_n^-;\ w_m=\kappa_{P_m^+}-\kappa_{P_m^-} \tag{4.6}$$

for all m. Note that $P_m^+\cap P_m^-=\varnothing$ and (4.4) implies $\mu(P_m^+)=\mu(P_m^-)=\mu(D)/2^m$.

Denote $y_m=Aw_m$. By the construction, the sequence $\{y_m\}_{m=1}^\infty$ is fundamental, that is, it satisfies the Cauchy condition:

$$\begin{aligned}\|y_k-y_l\|_Y&=||Aw_k-Aw_l||_Y=||A(w_k-w_l)||_Y\\&=\left\|A\left(\sum_{n=l}^{k}v_n\right)\right\|_Y\leqslant\sum_{n=l}^{k}\|Av_n\|_Y\leqslant\sum_{n=l}^{k}\frac{\delta}{2^n}\end{aligned}$$

for any $k>l$. Hence $\lim\limits_{m\to\infty}Aw_m$ and as consequence $\lim\limits_{m\to\infty}||Aw_m||_Y$ exist.

Moreover, by (4.4)

$$\lim_{m\to\infty}\mu(\operatorname{supp}w_m)=0. \tag{4.7}$$

By (4.7) and (4.5) we have

$$\lim_{m\to\infty}||Aw_m||_Y=0. \tag{4.8}$$

By the construction, for all m

$$w_1 = \kappa_{P_1^+} - \kappa_{P_1^-} = \sum_{n=1}^{m-1} v_n + w_m$$

and $w_1 \in \mathfrak{I}_D$.

For all m we have

$$\|Aw_1\|_y = \left\| A\left(\sum_{n=1}^{m-1} v_n\right) + Aw_m \right\|_Y \leqslant \sum_{n=1}^{m-1} \|Av_n\|_Y + \|Aw_m\|_Y .$$

Letting $m \to \infty$, we get by (4.8)

$$\|Aw_1\|_y \leqslant \sum_{n=1}^{\infty} \|Av_n\|_Y \leqslant \sum_{n=1}^{\infty} \frac{\delta}{2^n} = \delta.$$

Thus for each $\delta > 0$ and each $D \in \Sigma^+$ there exists a mean zero sign w_1 on D with $\|Aw_1\|_Y < \delta$. Then A is narrow. □

REMARK 5. In contrast to [19, Proposition 2.1], in Theorem 6 we suppose that Y is a Banach space and μ is σ-additive measure. However, we do not require from the norm in X to be absolutely continuous on the unit.

Since we do not use the property $\|\lambda u\|_X = |\lambda|\|u\|_X$ where $\lambda \in \mathbb{R}$ or C, then X could be a Köthe F-space.

We recall briefly that an F-space is a complete metric linear space X with an invariant metric ρ (i.e. $\rho(u,v) = \rho(u+z, v+z)$ for each $u, v, z \in X$).

An F-space X which is a linear subspace of $S(\mu)$ is called a Köthe F-space on (Ω, Σ, μ) if $\kappa_\Omega \in X$ and for each $u \in S(\mu)$ and $v \in X$ the condition $|u| \leqslant |v|$ implies $u \in X$ and $\|u\|_X \leqslant \|v\|_X$ (see for example [13] and [19]).

Here $\|u\|_X = \rho(u,0)$ and the property $\|\lambda u\|_X = |\lambda|\|u\|_X$ could be not satisfied in general.

The condition (4.5) in Theorem 6 is essential, since the non-narrow continuous linear functional, constructed in [17], obviously does not satisfy (4.5).

Also note that if

$$\lim_{\mu(D)\to 0} \|A\kappa_D\|_Y = 0, \tag{4.9}$$

then the condition (4.5) will be clearly satisfied.

In particular, if the norm in X is absolutely continuous on the unit as in [19, Proposition 2.1], then $\lim\limits_{\mu(D)\to 0} \|\kappa_D\|_X = 0$ and we have (4.9) for all $A \in \mathscr{L}(X,Y)$ both compact and not compact.

As a consequence we obtain the following result: every operator $A \in \mathscr{L}(L_p(\mu), Y)$, $1 \leqslant p < \infty$, with a compact restriction on $L_\infty(\mu)$ is narrow by Theorem 6.

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