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POLYNOMIAL INEQUALITIES IN L^p NORMS WITH GENERALIZED JACOBI WEIGHTS

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Abstract. We give concrete estimates of Schur- and Nikolskii-type inequalities with the best exponent of polynomial degree in L^p norms with generalized Jacobi weights. In particular, we obtain these inequalities with the Chebyshev weight, with the Gegenbauer weights and with the classical Jacobi ones.

1. Introduction

Let w be a generalized Jacobi weight

$$w(x) = g(x)\prod_{j=1}^{m}|x-x_j|^{r_j}, \qquad x\in(-1,1) \tag{1}$$

where $x_1=-1$, $x_m=1$, $r_1,r_m \geqslant -\frac{1}{2}$ and $x_j\in(-1,1)$, $r_j\geqslant 0$ for $j=2,\dots,m-1$ and $g=g(x)$ is a positive integrable function separated from 0 and from infinity for $x\in[-1,1]$. Consider q norms with the weights described above, i.e.

$$\|f\|_{q,w} := \left[\int_{-1}^{1}|f(x)|^q\, w(x)\, dx\right]^{1/q} \qquad \text{for } q\in[1,\infty)$$

$$\|f\|_{\infty,w} := \max_{x\in[-1,1]}\{|f(x)|\, w(x)\} \qquad \text{for } r_j\geqslant 0,\ j=1,\dots,m$$

for any function f continuous on $[-1,1]$. If $g\equiv 1$ and $r_1=r_2=\ldots=r_m=0$, then we have the usual norms $\|\cdot\|_q$ and $\|\cdot\|_\infty$ in L^q and L^∞. For $g\equiv 1$, $r_1=r_m=-\frac{1}{2}$, $r_2=\ldots=r_{m-1}=0$ we obtain the L^q norms for the equilibrium measure of $[-1,1]$ also called the q norms with the Chebyshev weight. When $r_2=\ldots=r_{m-1}=0$ we have the q norms with the classical Jacobi weight. If in addition $r_1=r_2=\alpha$, we get Gegenbauer norms

$$\|f\|_{q,\alpha} := \left(\int_{-1}^{1}|f(x)|^q\,(1-x^2)^\alpha dx\right)^{1/q}.$$

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We will also consider $\|\cdot\|_{q,w}$, $\|\cdot\|_q$ and $\|\cdot\|_{q,\alpha}$ when $q\in(0,1)$ although they are not norms in this case.

Let $\mathbb{N}=\{1,2,3,\ldots\}$, $\mathbb{N}_0=\{0\}\cup\mathbb{N}$ and $\mathscr{P}_n$ be the space of all algebraic polynomials of degree at most $n\in\mathbb{N}$. In this paper we adhere to the convention that $0^0=1$ and $\frac{1}{\infty}=0$.

Estimations of polynomials in q norms with the above weights have been studied by many authors, e.g., [15, Chap. VI], [16, Chap. 15], [17] and the references therein. This paper is inspired mainly by two kinds of estimates:

- *Schur inequality:*

$$\|p\|_\infty \leqslant (n+1)\max_{x\in[-1,1]}|\sqrt{1-x^2}\,p(x)| \quad \text{for} \quad p\in\mathscr{P}_n$$

  that is optimal because we have equality for the n-th Chebyshev polynomials of the second kind,

- *Nikolskii inequality:*

$$\|p\|_q \leqslant [(1+s)n^2]^{\frac{1}{s}-\frac{1}{q}}\|p\|_s \quad \text{for} \quad 0<s\leqslant q\leqslant\infty,\ p\in\mathscr{P}_n. \tag{2}$$

One of the key problems that we address in the paper is the estimation of $\|p\|_{q,\beta}$ by $\|p\|_{s,\alpha}$. We will study inequalities of the form

$$\|p\|_{q,\beta} \ \leqslant\ M_n(s,\alpha,q,\beta)\ \|p\|_{s,\alpha} \tag{3}$$

with $M_n(s,\alpha,q,\beta)$ independent of $p\in\mathscr{P}_n$. These estimates are worthy of interest because of their numerous applications. For references to the extensive literature on the subject one may refer to the book [15]. The dependence on n in $M_n(s,\alpha,q,\beta)$ was described almost 40 years ago, see [3], [11], [15, Sec. 6.1.8]. Namely, the constant

$$C(s,\alpha,q,\beta):=\sup\left\{\frac{\|p\|_{q,\beta}}{n^\gamma\|p\|_{s,\alpha}}: p\in\mathscr{P}_n,\ p\not\equiv 0,\ n\in\mathbb{N}\right\} \tag{4}$$

is finite for $q,s>0,\ \alpha,\beta\geqslant-\frac{1}{2}$ and

$$\gamma=2\left(\frac{1}{s}-\frac{1}{q}+\frac{\alpha}{s}-\frac{\beta}{q}\right) \quad \text{for} \quad q\geqslant s,\ \alpha\geqslant\beta \tag{5}$$

is the best possible exponent of n. However, the exact values of $C(s,\alpha,q,\beta)$ have been found only in some specific cases, see e.g. [7], [12] and the references in [17].

The aim of our paper is to give some concrete admissible bounds of the values $M_n(s,\alpha,q,\beta)$ and $C_0(s,\alpha,q,\beta)$ with the best possible exponent of n. Our estimates remain true for $q>0,\ s\geqslant1,\ \alpha,\beta\geqslant-\frac{1}{2}$ (see Theorem 1, Corollary 3, Theorem3). We also propose some bounds for corresponding constants with generalized Jacobi weights of type (1).

The paper is organized as follows. Section 2 is concerned with changing the Jacobi weights in integral norms. In particular, we estimate the constants $C_0(q,\alpha,q,\beta)$. We also give estimates of constants in Nikolskii-type inequalities with generalized Jacobi weights. The third Section deals with Schur-type inequalities.

2. Inequalities with Jacobi weights of type $(1-x)^a(1+x)^b$

2.1. Change of Jacobi weights

We start this Section with rather general estimates from which we will derive some inequalities that admit change of Jacobi weights with an appropriate control of the constants and with the best possible exponents of the polynomial degree n.

PROPOSITION 1. *For $q \in (0,\infty)$, $\alpha_i \geqslant \beta_i \geqslant -\frac{1}{2}$, $i=1,2$ and for all polynomials $p \in \mathscr{P}_n$ the following inequality holds*

$$\int_{-1}^{1} |p(x)|^q (1+x)^{\beta_1} (1-x)^{\beta_2} dx \leqslant A(\alpha_1,\alpha_2,\beta_1,\beta_2,n,q) \int_{-1}^{1} |p(x)|^q (1+x)^{\alpha_1} (1-x)^{\alpha_2} dx \tag{6}$$

where $A(\alpha_1,\alpha_2,\beta_1,\beta_2,n,q) > 0$ is such that for all $r \in (0,1)$

$$A(\alpha_1,\alpha_2,\beta_1,\beta_2,n,q) \leqslant \frac{2^{\alpha_1-\beta_1+\alpha_2-\beta_2}}{(1+r)^{2\delta_0}(1-r)^{2\delta}\, r^{2(nq+1+\beta_1+\beta_2)}} \tag{7}$$

where $\delta_0 := \min\{\alpha_1-\beta_1, \alpha_2-\beta_2\}$, $\delta := \max\{\alpha_1-\beta_1, \alpha_2-\beta_2\}$. In particular, if $\alpha_1-\beta_1 = \alpha_2-\beta_2$ then

$$A(\alpha_1,\alpha_2,\beta_1,\beta_2,n,q) \leqslant \frac{2^{2(\alpha_1-\beta_1)}}{(1-r^2)^{2(\alpha_1-\beta_1)}\, r^{2(1+\beta_1+\beta_2+nq)}} \quad \text{for all } r \in (0,1) \tag{8}$$

and

$$A(\alpha_1,\alpha_2,\beta_1,\beta_2,n,q) \leqslant \frac{1}{(\alpha_1-\beta_1)^{2(\alpha_1-\beta_1)}} \frac{(1+\alpha_1+\alpha_2+nq)^{1+\alpha_1+\alpha_2+nq}}{(1+\beta_1+\beta_2+nq)^{1+\beta_1+\beta_2+nq}}. \tag{9}$$

Proof. We can assume that $\deg p = n$. Set

$$J_{\gamma_1,\gamma_2}(\rho) := \int_0^{2\pi} u_{\gamma_1,\gamma_2}(\rho\, e^{it})\, dt \quad \text{for } \rho > 0,\ \gamma_1,\gamma_2 \geqslant -\frac{1}{2}$$

where
$$u_{\gamma_1,\gamma_2}(z) = \left| z^n p\left(\frac{1}{2}\left(z+\frac{1}{z}\right)\right)\right|^q \left|\frac{z+1}{\sqrt{2}}\right|^{1+2\gamma_1} \left|\frac{z-1}{\sqrt{2}}\right|^{1+2\gamma_2}.$$

Next, we have

$$2\int_{-1}^{1} |p(x)|^q (1+x)^{\beta_1}(1-x)^{\beta_2} dx = 2\int_{-\pi}^{0} |p(\cos t)|^q (1+\cos t)^{\beta_1}(1-\cos t)^{\beta_2} |\sin t|\, dt$$

$$= \int_0^{2\pi} \left| p\left(\frac{e^{it}+e^{-it}}{2}\right)\right|^q \left|\frac{2+e^{it}+e^{-it}}{2}\right|^{\beta_1} \left|\frac{2-e^{it}-e^{-it}}{2}\right|^{\beta_2} \left|\frac{e^{it}-e^{-it}}{2i}\right| dt = J_{\beta_1,\beta_2}(1).$$

Observe that $z \mapsto z^n p\left(\frac{1}{2}\left(z+\frac{1}{z}\right)\right)$ is a polynomial of degree $2n$, thus u_{γ_1,γ_2} is a subharmonic function in $\mathbb{C}$ and the function J_{γ_1,γ_2} is increasing in $(0,\infty)$ for each $\gamma_1,\gamma_2 \geqslant -\frac{1}{2}$, see [10, Th. 3.2.3]. Therefore, for any $r \in (0,1)$ we get

$$J_{\beta_1,\beta_2}(1) \leqslant J_{\beta_1,\beta_2}\left(\frac{1}{r}\right) = \int_0^{2\pi} \left|\frac{e^{int}}{r^n}\, p\left(\frac{1}{2}\left(\frac{e^{it}}{r}+\frac{r}{e^{it}}\right)\right)\right|^q \left|\frac{e^{it}+r}{\sqrt{2}r}\right|^{1+2\beta_1} \left|\frac{e^{it}-r}{\sqrt{2}r}\right|^{1+2\beta_2} dt$$

$$= \frac{1}{r^{2nq+2+2\beta_1+2\beta_2}} \int_0^{2\pi} \left|r^n\, p\left(\frac{1}{2}\left(\frac{e^{it}}{r}+\frac{r}{e^{it}}\right)\right)\right|^q \left|\frac{1+re^{-it}}{\sqrt{2}}\right|^{1+2\beta_1} \left|\frac{1-re^{-it}}{\sqrt{2}}\right|^{1+2\beta_2} dt.$$

From the inequality

$$\left|1+re^{-it}\right|^{1+2\beta_1} \left|1-re^{-it}\right|^{1+2\beta_2} = \frac{\left|1+re^{-it}\right|^{1+2\alpha_1} \left|1-re^{-it}\right|^{1+2\alpha_2}}{|1+re^{-it}|^{2(\alpha_1-\beta_1)}|1-re^{-it}|^{2(\alpha_2-\beta_2)}}$$

$$\leqslant \frac{\left|1+re^{-it}\right|^{1+2\alpha_1} \left|1-re^{-it}\right|^{1+2\alpha_2}}{(1-r^2)^{2\delta_0}(1-r)^{2\delta-2\delta_0}} = \frac{\left|1+re^{-it}\right|^{1+2\alpha_1} \left|1-re^{-it}\right|^{1+2\alpha_2}}{(1+r)^{2\delta_0}(1-r)^{2\delta}}$$

we conclude that

$$\begin{aligned}
&J_{\beta_1,\beta_2}(1) r^{2(nq+1+\beta_1+\beta_2)} \\
&\leqslant \frac{2^{\alpha_1-\beta_1+\alpha_2-\beta_2}}{(1+r)^{2\delta_0}(1-r)^{2\delta}} \int_0^{2\pi} \left|r^n p\left(\frac{1}{2}\left(\frac{e^{it}}{r}+\frac{r}{e^{it}}\right)\right)\right|^q \left|\frac{1+re^{-it}}{\sqrt{2}}\right|^{1+2\alpha_1} \left|\frac{1-re^{-it}}{\sqrt{2}}\right|^{1+2\alpha_2} dt \\
&= \frac{2^{\alpha_1-\beta_1+\alpha_2-\beta_2}}{(1+r)^{2\delta_0}(1-r)^{2\delta}} \int_0^{2\pi} \left|r^n p\left(\frac{1}{2}\left(\frac{1}{re^{is}}+re^{is}\right)\right)\right|^q \left|\frac{1+re^{is}}{\sqrt{2}}\right|^{1+2\alpha_1} \left|\frac{1-re^{is}}{\sqrt{2}}\right|^{1+2\alpha_2} ds \\
&= \frac{2^{\alpha_1-\beta_1+\alpha_2-\beta_2}}{(1+r)^{2\delta_0}(1-r)^{2\delta}} \int_0^{2\pi} u_{\alpha_1,\alpha_2}(re^{is})\, ds = \frac{2^{\alpha_1-\beta_1+\alpha_2-\beta_2}}{(1+r)^{2\delta_0}(1-r)^{2\delta}} J_{\alpha_1,\alpha_2}(r).
\end{aligned}$$

By the monotonicity of the function J, we get

$$J_{\alpha_1,\alpha_2}(r) \leqslant J_{\alpha_1,\alpha_2}(1) = 2\int_{-1}^{1} |p(x)|^q (1+x)^{\alpha_1}(1-x)^{\alpha_2} dx$$

the last equality being a consequence of the same computation as in the first lines of the proof. Combining the last two estimates yields (7) for all $r \in (0,1)$. Now, set

$$f(r) := (1-r^2)^{2(\alpha_1-\beta_1)}\, r^{2(nq+1+\beta_1+\beta_2)}.$$

Since $f(0) = f(1) = 0$, there is $r_0 \in (0,1)$ such that $\sup_{(0,1)} f = f(r_0)$. One can calculate that $r_0 = \left(\frac{1+\beta_1+\beta_2+nq}{1+\alpha_1+\alpha_2+nq}\right)^{1/2}$ and

$$f(r_0) = \frac{[2(\alpha_1-\beta_1)]^{2(\alpha_1-\beta_1)}\,(1+\beta_1+\beta_2+nq)^{1+\beta_1+\beta_2+nq}}{(1+\alpha_1+\alpha_2+nq)^{1+\alpha_1+\alpha_2+nq}}.$$

From this and (8) it follows (9). □

THEOREM 1. *If* $\alpha \geqslant \beta \geqslant -\frac{1}{2}$, $q \in (0,\infty)$ *then for any polynomial* $p \in \mathscr{P}_n$

$$\|p\|_{q,\beta}^q \leqslant \frac{1}{(\alpha-\beta)^{2(\alpha-\beta)}} \frac{(1+2\alpha+nq)^{1+2\alpha+nq}}{(1+2\beta+nq)^{1+2\beta+nq}} \|p\|_{q,\alpha}^q \tag{10}$$

and

$$\|p\|_{q,\beta}^q \leqslant C_0(q,\alpha,q,\beta)^q\, n^{2(\alpha-\beta)}\, \|p\|_{q,\alpha}^q \tag{11}$$

for $n \geqslant 1$ *where*

$$C_0(q,\alpha,q,\beta)^q \leqslant \frac{1}{(\alpha-\beta)^{2(\alpha-\beta)}} \frac{(1+2\alpha+q)^{1+2\alpha+q}}{(1+2\beta+q)^{1+2\beta+q}} \leqslant e^{2/e} \frac{(1+2\alpha+q)^{1+2\alpha+q}}{(1+2\beta+q)^{1+2\beta+q}} \tag{12}$$

Moreover, the exponent $2(\alpha-\beta)$ *of* n *in (11) is optimal.*

Proof. Inequality (10) follows from (6) with $\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 = \beta_2 = \beta$. To show the estimates of $C_0(q,\alpha,q,\beta)$, we set

$$h(x) := \frac{(x+a)^{x+a}}{(x+b)^{x+b}}\, x^{b-a} \quad \text{for } x>0,\ a \geqslant b \geqslant 0.$$

Since $\frac{x+a}{x+b} = 1+\frac{a-b}{x+b} \leqslant 1+\frac{a-b}{x} \leqslant e^{\frac{a-b}{x}}$ and

$$\tfrac{d}{dx}[\log h(x)] = \log(x+a)+1-\log(x+b)-1+\tfrac{b-a}{x} = \log\tfrac{x+a}{x+b}+\tfrac{b-a}{x},$$

the function $\log h$ is decreasing in $(0,\infty)$ and

$$\sup\{h(x) : x \in [x_0,\infty)\} = h(x_0) \quad \text{for } x_0 > 0. \tag{13}$$

Therefore, by inequality (9) in Proposition 1, $nq \geqslant q$ implies

$$A(\alpha,\alpha,\beta,\beta,n,q)\,(nq)^{2\beta-2\alpha} \leqslant \frac{(1+2\alpha+q)^{1+2\alpha+q}}{(\alpha-\beta)^{2(\alpha-\beta)}\,(1+2\beta+q)^{1+2\beta+q}}\, q^{2\beta-2\alpha}$$

and so

$$A(\alpha,\alpha,\beta,\beta,n,q) \leqslant \frac{(1+2\alpha+q)^{1+2\alpha+q}}{(\alpha-\beta)^{2(\alpha-\beta)}(1+2\beta+q)^{1+2\beta+q}}\, n^{2(\alpha-\beta)}.$$

A standard verification shows that

$$\max\left\{\left(\frac{c}{x}\right)^x : x \in (0,\infty)\right\} = e^{c/e} \quad \text{for any } c>0. \tag{14}$$

Combining the above remark we obtain inequalities (12). The optimality of the exponent in (11) is a consequence of the results in [3] or [11], see (5). □

REMARK 1. Similarly to (3), let $M_{n,1}(2,0,2,0) := \sup\left\{\frac{\|p'\|_2}{\|p\|_2} : p \in \mathscr{P}_n\right\}$. By results from [3], [11], we have $\gamma = 2$. It is worth noticing that in 1990 Goetgheluck [8] proved that $M_{n,1}(2,0,2,0)/n^\gamma$ is a decreasing sequence in n and posed a question whether this is a general rule. This remains an open problem. Taking into account the proof of Theorem 1, especially property (13), we can see that $C_0(q,\alpha,q,\beta)^q \leqslant A(\alpha,\alpha,\beta,\beta,1,q)$ (see Proposition 1), i.e. it is sufficient to take $n=1$ in inequality (9) to get an estimate of the constant $C_0(q,\alpha,q,\beta)$ in (11) for all $n \in \mathbb{N}$. In this fashion our estimates agree with Goetgheluck's conjecture.

Without the assumption $\alpha_1 - \beta_1 = \alpha_2 - \beta_2$ in Proposition 1 we obtain

THEOREM 2. *For $q \in (0,\infty)$, $\alpha_i \geqslant \beta_i \geqslant -\frac{1}{2}$, $i=1,2$ and $p \in \mathscr{P}_n$ inequality (6) holds with*

$$A(\alpha_1,\alpha_2,\beta_1,\beta_2,n,q) \leqslant \frac{2^\delta}{\delta^{2\delta}} \frac{(1+\beta_1+\beta_2+q+2\delta)^{1+\beta_1+\beta_2+q+2\delta}}{(1+\beta_1+\beta_2+q)^{1+\beta_1+\beta_2+q}} n^{2\delta}$$

where δ is defined in Proposition 1. Moreover, the exponent 2δ of n is optimal.

Proof. From (7) in Proposition 1 we have

$$A(\alpha_1,\alpha_2,\beta_1,\beta_2,n,q) \leqslant \frac{2^{3\delta}}{(1-r^2)^{2\delta} r^{2(1+\beta_1+\beta_2+nq)}},$$

because $2^{\alpha_1-\beta_1+\alpha_2-\beta_2}(1+r)^{2\delta-2\delta_0} \leqslant 2^{\alpha_1-\beta_1+\alpha_2-\beta_2} 2^{2\delta-2\delta_0} \leqslant 2^{3\delta}$. The rest of the proof runs as in Proposition 1 and Theorem 1. The optimality of the exponent 2δ of n has been proved in [9]. □

COROLLARY 1. *If α,β and q are as in Theorem 1 then for $p \in \mathscr{P}_n$ inequality (11) holds with*

$$C_0(q,\alpha,q,\beta)^q \leqslant \left(\frac{2q}{\alpha-\beta}\right)^{2(\alpha-\beta)} \frac{(1+\alpha)^{2(1+\alpha)}}{(1+\beta)^{2(1+\beta)}} \leqslant e^{4q/e} \frac{(1+\alpha)^{2(1+\alpha)}}{(1+\beta)^{2(1+\beta)}} \qquad (15)$$

whenever $nq \geqslant 1$.

Proof. By inequality (9) and property (13) with $a = 1+2\alpha$, $b = 1+2\beta$, $x = nq \geqslant 1 = x_0$ we have

$$A(\alpha,\alpha,\beta,\beta,n,q)\ (nq)^{2\beta-2\alpha} \leqslant \frac{(nq)^{2\beta-2\alpha}}{(\alpha-\beta)^{2(\alpha-\beta)}} \frac{(1+2\alpha+nq)^{1+2\alpha+nq}}{(1+2\beta+nq)^{1+2\beta+nq}}$$

$$\leqslant \frac{1}{(\alpha-\beta)^{2(\alpha-\beta)}} \frac{(2+2\alpha)^{2+2\alpha}}{(2+2\beta)^{2+2\beta}} = \left(\frac{2}{\alpha-\beta}\right)^{2(\alpha-\beta)} \frac{(1+\alpha)^{2(1+\alpha)}}{(1+\beta)^{2(1+\beta)}}$$

and we get the first inequality in (15). The second one is a consequence of (14). □

COROLLARY 2. *For* α,β *as in Theorem* 1 *and for* $p\in\mathscr{P}_n$ *inequality* (11) *is fulfilled with*

$$C_0(q,\alpha,q,\beta)^q\leqslant 2^{1+2\alpha+q}e^q\quad for\ n\geqslant 2,\ q>0$$

and

$$C_0(q,\alpha,q,\beta)^q\leqslant 2^{2(\alpha-\beta)}e\,q^{2+2\alpha}(q-1)^{-2-2\beta}\quad for\ n\geqslant 1,\ q>1.$$

Proof. One can easily show that

$$C_0(q,\alpha,q,\beta)^q\ =\ \sup\{A(\alpha,\alpha,\beta,\beta,n,q)/n^{2(\alpha-\beta)}:n\in\mathbb{N}\}.$$

Now we apply inequality (8) with $\alpha_1=\alpha_2=\alpha$, $\beta_1=\beta_2=\beta$ and $r=\left(1-\frac{1}{n}\right)^{1/2}$ to show the first estimate of Corollary 2. We obtain the second one if we take $r=\sqrt{1-\frac{1}{nq}}$. □

2.2. Nikolskii-type inequalities with Gegenbauer weights

This subsection concerns estimates between different norms that are often called Nikolskii-type inequalities. Classical results admitting comparison between norms $\|\cdot\|_q$ and $\|\cdot\|_s$ can be found in [15, Chap. 5.3] and [2, Chap. A4]. The most convenient for us is estimate (2), see e.g. [4, Th. 2.6, Chap. 4].

PROPOSITION 2. *Assume that* $1\leqslant s\leqslant q$, $-\frac{1}{2}\leqslant\beta\leqslant\alpha$ *and* $\frac{\beta}{q}\leqslant\frac{k}{j}\leqslant\frac{\alpha}{s}$ *for some* $j\in\mathbb{N}$, $k\in\mathbb{N}_0$. *Then for any polynomial* $p\in\mathscr{P}_n$ *the Nikolskii-type inequality*

$$\|p\|_{q,\beta}\ \leqslant\ e^{\frac{2}{e}(\frac{1}{q}+\frac{1}{s})}\,\frac{(2q+2\alpha q)^{4+4\alpha}}{(2+2\beta)^{\frac{1}{q}(4+4\beta)}}\,\left[(j+s)j(1+\tfrac{2\alpha}{s})^2\right]^{\frac{1}{s}-\frac{1}{q}}\ \|p\|_{s,\alpha}\qquad(16)$$

and the exponent $d=2\left(\frac{1}{s}-\frac{1}{q}+\frac{\alpha}{s}-\frac{\beta}{q}\right)$ *of* n *is optimal.*

Proof. Proposition 1 leads us to

$$\int_{-1}^{1}|p(x)|^q(1-x^2)^\beta dx\ \leqslant\ A(\tfrac{kq}{j},\tfrac{kq}{j},\beta,\beta,n,q)\int_{-1}^{1}\left[|p(x)|^j(1-x^2)^k\right]^{q/j}dx.$$

Since $Q(x):=p(x)^j(1-x^2)^k$ is a polynomial of degree $nj+2k$, from inequality (2) we have

$$\|Q\|_{q/j}^{q/j}\leqslant\left[\left(1+\tfrac{s}{j}\right)(jn+2k)^2\right]^{(\frac{j}{s}-\frac{j}{q})\frac{q}{j}}\|Q\|_{s/j}^{q/j}=\left[(j+s)\,j(n+\tfrac{2k}{j})^2\right]^{(\frac{1}{s}-\frac{1}{q})q}\|Q\|_{s/j}^{q/j}.$$

Again by Proposition 1 we get

$$\|Q\|_{s/j}^{s/j}=\int_{-1}^{1}|p(x)|^s(1-x^2)^{ks/j}dx\ \leqslant\ A(\alpha,\alpha,\tfrac{ks}{j},\tfrac{ks}{j},n,s)\int_{-1}^{1}|p(x)|^s(1-x^2)^\alpha dx.$$

By the above, for all $0 < s \leqslant q$ we obtain

$$\|p\|_{q,\beta} \leqslant \Big(A(\tfrac{kq}{j},\tfrac{kq}{j},\beta,\beta,n,q)\Big)^{\frac{1}{q}} \Big[(j+s)\,j(n+\tfrac{2k}{j})^2\Big]^{\frac{1}{s}-\frac{1}{q}} \Big(A(\alpha,\alpha,\tfrac{ks}{j},\tfrac{ks}{j},n,s)\Big)^{\frac{1}{s}} \|p\|_{s,\alpha}$$

To prove (16), we use the same argument as in the proof of Theorem 1 and we get

$$E(s,\alpha,q,\beta,n,j,k) \leqslant \Big(A(\tfrac{kq}{j},\tfrac{kq}{j},\beta,\beta,1,q)\Big)^{\frac{1}{q}} \Big[(j+s)\,j(1+\tfrac{2\alpha}{s})^2\Big]^{\frac{1}{s}-\frac{1}{q}} \Big(A(\alpha,\alpha,\tfrac{ks}{j},\tfrac{ks}{j},1,s)\Big)^{\frac{1}{s}} n^d$$

with the exponent of n equal to

$$2\left(\frac{kq}{jq}-\frac{\beta}{q}\right)+2\left(\frac{1}{s}-\frac{1}{q}\right)+2\left(\frac{\alpha}{s}-\frac{ks}{js}\right)=2\left(\frac{\alpha}{s}-\frac{\beta}{q}+\frac{1}{s}-\frac{1}{q}\right)$$

because $\left(n+\frac{2k}{j}\right)^2 \leqslant \left(1+\frac{2\alpha}{s}\right)^2 n^2$. From inequality (9) we have

$$\Big(A(\tfrac{kq}{j},\tfrac{kq}{j},\beta,\beta,1,q)\Big)^{\frac{1}{q}} \Big(A(\alpha,\alpha,\tfrac{ks}{j},\tfrac{ks}{j},1,s)\Big)^{\frac{1}{s}}$$

$$\leqslant \frac{1}{(\frac{kq}{j}-\beta)^{\frac{2}{q}(\frac{kq}{j}-\beta)}} \frac{1}{(\alpha-\frac{ks}{j})^{\frac{2}{s}(\alpha-\frac{ks}{j})}} \left[\frac{(1+2\frac{kq}{j}+q)^{1+2\frac{kq}{j}+q}}{(1+2\beta+q)^{1+2\beta+q}}\right]^{\frac{1}{q}} \left[\frac{(1+2\alpha+s)^{1+2\alpha+s}}{(1+2\frac{ks}{j}+s)^{1+2\frac{ks}{j}+s}}\right]^{\frac{1}{s}}$$

$$\leqslant e^{\frac{2}{e}(\frac{1}{q}+\frac{1}{s})} \left[\frac{(1+2\alpha\frac{q}{s}+q)^{1+2\alpha\frac{q}{s}+q}}{(1+2\beta+q)^{1+2\beta+q}}\right]^{\frac{1}{q}} \left[\frac{(1+2\alpha+s)^{1+2\alpha+s}}{(1+2\beta\frac{s}{q}+s)^{1+2\beta\frac{s}{q}+s}}\right]^{\frac{1}{s}}$$

the last inequality being a consequence of (14) and the inequality $\frac{\beta}{q} \leqslant \frac{k}{j} \leqslant \frac{\alpha}{s}$. Since $1 \geqslant \frac{s}{q} \geqslant \frac{1}{q}$, we obtain $1+2\alpha\frac{q}{s}+q \geqslant 1$, $1+2\beta\frac{s}{q}+s \geqslant 1$ and

$$\left[\frac{(1+2\alpha\frac{q}{s}+q)^{1+2\alpha\frac{q}{s}+q}}{(1+2\beta+q)^{1+2\beta+q}}\right]^{\frac{1}{q}} \left[\frac{(1+2\alpha+s)^{1+2\alpha+s}}{(1+2\beta\frac{s}{q}+s)^{1+2\beta\frac{s}{q}+s}}\right]^{\frac{1}{s}}$$

$$\leqslant \left[\frac{(q+2\alpha q+q)^{q+2\alpha q+q}}{(1+2\beta+1)^{1+2\beta+1}}\right]^{\frac{1}{q}} \left[\frac{(s+2\alpha s+s)^{s+2\alpha s+s}}{(\frac{s}{q}+2\beta\frac{s}{q}+\frac{s}{q})^{\frac{s}{q}+2\beta\frac{s}{q}+\frac{s}{q}}}\right]^{\frac{1}{s}} = \frac{(qs)^{2+2\alpha}(2+2\alpha)^{4+4\alpha}}{(s/q)^{\frac{1}{q}(2+2\beta)}(2+2\beta)^{\frac{1}{q}(4+4\beta)}}.$$

Hence and by the inequality $2+2\alpha \geqslant \frac{1}{q}(2+2\alpha) \geqslant \frac{1}{q}(2+2\beta)$, estimate (16) follows. The optymality of exponent d is a consequence of the results in [3] or [11]. □

REMARK 2. In the case of $0 \leqslant \alpha$ the assumption $\frac{\beta}{q} \leqslant \frac{k}{j} \leqslant \frac{\alpha}{s}$ for some $j \in \mathbb{N}$, $k \in \mathbb{N}_0$ is always true, because $\frac{\beta}{q} < \frac{\alpha}{s}$ whenever $\alpha \neq \beta$ or $q \neq s$.

3. Schur-type inequalities with weights

We begin this Section with inequalities concerning specific cases and then we will present an estimate for general Jacobi weights. Two propositions below give some bounds of the constants in inequalities regarding change of weights of type $|x-x_0|^\beta$ for $x_0 \in [-1,1]$. Similar estimates are often called division inequalities or Schur inequalities and have been investigated, e.g., in [6], [9], [5], [13], [14], [1].

PROPOSITION 3. *For $\alpha \geqslant \beta \geqslant 0$, $q \in (0,\infty)$ and $p \in \mathscr{P}_n$ the inequality*

$$\int_{-1}^{1} |p(x)|^q |x|^\beta dx \leqslant B_0(\alpha,\beta,n,q) \int_{-1}^{1} |p(x)|^q |x|^\alpha dx \tag{17}$$

holds with $B_0(\alpha,\beta,n,q)$ satisfying

$$B_0(\alpha,\beta,n,q) \leqslant \left(\frac{2}{\alpha-\beta}\right)^{\alpha-\beta} \frac{(1+\alpha+nq)^{1+\alpha+nq}}{(1+\beta+nq)^{1+\beta+nq}} \leqslant \left(\frac{2}{\alpha-\beta}\right)^{\alpha-\beta} \frac{(1+\alpha+q)^{1+\alpha+q}}{(1+\beta+q)^{1+\beta+q}}\, n^{\alpha-\beta}.$$

Moreover, the exponent $\alpha-\beta$ of n is optimal.

Proof. The proof of the first estimate of $B_0(\alpha,\beta,n,q)$ is similar to that of Proposition 1. Again we have

$$2\int_{-1}^{1} |p(x)|^q |x|^\beta dx = J_\beta(1) \leqslant J_\beta\left(\frac{1}{r}\right) \quad \text{for} \quad r \in (0,1),$$

$$J_\gamma(\rho) := \int_0^{2\pi} u_\gamma(\rho\, e^{it})\, dt \quad \text{for } \rho > 0,\ \gamma \geqslant 0$$

and

$$u_\gamma(z) := \left| z^n p\left(\frac{1}{2}\left(z+\frac{1}{z}\right)\right)\right|^q \left|\frac{z^2+1}{2}\right|^\gamma \left|\frac{z^2-1}{2}\right|.$$

Since $|1+r^2e^{2it}| \geqslant 1-r^2$, it follows

$$J_\beta\left(\frac{1}{r}\right) = \frac{J_\beta(r)}{r^{2+2\beta+2nq}} \leqslant \frac{2^{\alpha-\beta} J_\alpha(r)}{r^{2+2\beta+2nq}(1-r^2)^{\alpha-\beta}}.$$

Consequently, by the monotonicity of the mean value J_α of subharmonic functions (e.g. [10, Th. 3.2.3]), we have

$$J_\beta\left(\frac{1}{r}\right) \leqslant \frac{2^{\alpha-\beta} J_\alpha(1)}{r^{2+2\beta+2nq}(1-r^2)^{\alpha-\beta}} = \frac{2^{\alpha-\beta}}{r^{2+2\beta+2nq}(1-r^2)^{\alpha-\beta}}\, 2\int_{-1}^{1} |p(x)|^q |x|^\alpha dx$$

We now need to find the minimum value of $\frac{2^{\alpha-\beta}}{r^{2+2\beta+2nq}(1-r^2)^{\alpha-\beta}}$ over all $r \in (0,1)$. It is attained at $r_0 = \sqrt{\frac{1+\beta+nq}{1+\alpha+nq}}$ and a short computation gives the desired formula.

The proof of the second estimate of $B_0(\alpha,\beta,n,q)$ can be derived from the first one and (13) as in the proof of Theorem 1. The optimality of the exponent $\alpha-\beta$ is a consequence of [9, Th. 2]. □

In order to find an estimate for weights of the type $|x-x_0|^\beta$ for $x_0\in(0,1)$ (or $x_0\in(-1,0)$), $\alpha\geqslant\beta$, we can apply Proposition 3 and the following standard reasoning:

$$\begin{aligned}
&\int_{-1}^{1}|p(x)|^q|x-x_0|^\beta dx=\int_{-1}^{2x_0-1}+\int_{2x_0-1}^{1}\\
&=\int_{-1}^{2x_0-1}|p(x)|^q\frac{|x-x_0|^\alpha}{|x-x_0|^{\alpha-\beta}}dx+(1-x_0)^{1+\beta}\int_{-1}^{1}|p((1-x_0)t+x_0)|^q|t|^\beta dt\\
&\leqslant\frac{1}{(1-x_0)^{\alpha-\beta}}\int_{-1}^{2x_0-1}|p(x)|^q|x-x_0|^\alpha dx+\frac{B_0(\alpha,\beta,n,q)}{(1-x_0)^{\alpha-\beta}}\int_{2x_0-1}^{1}|p(x)|^q|x-x_0|^\alpha dx.
\end{aligned}$$

By inequality (17) for the polynomial $p\equiv1$, we get

$$1\leqslant\frac{1+\alpha}{1+\beta}\leqslant B_0(\alpha,\beta,n,q).$$

We have thus proved

COROLLARY 3. *For α,β,q as in Proposition 3 and for $x_0\in(0,1)$, $p\in\mathscr{P}_n$ the inequality*

$$\int_{-1}^{1}|p(x)|^q|x-x_0|^\beta dx\ \leqslant\ \frac{B_0(\alpha,\beta,n,q)}{(1-x_0)^{\alpha-\beta}}\int_{-1}^{1}|p(x)|^q|x-x_0|^\alpha dx$$

holds with the constant $B_0(\alpha,\beta,n,q)$ being described in Proposition 3.

An inequality for weights of the type $(1-x)^\beta$ or $(1+x)^\beta$ can be derived from Theorem 2. However, a better estimate of this kind is given below.

PROPOSITION 4. *For $\alpha\geqslant\beta\geqslant-\frac{1}{2}$, $q\in(0,\infty)$ and $p\in\mathscr{P}_n$ the inequality*

$$\int_{-1}^{1}|p(x)|^q(1-x)^\beta dx\ \leqslant\ B_1(\alpha,\beta,n,q)\int_{-1}^{1}|p(x)|^q(1-x)^\alpha dx$$

holds with $B_1(\alpha,\beta,n,q)>0$ such that

$$\begin{aligned}
B_1(\alpha,\beta,n,q)&\leqslant\left(\frac{2}{(\alpha-\beta)^2}\right)^{\alpha-\beta}\frac{(1+\alpha+nq)^{2(1+\alpha+nq)}}{(1+\beta+nq)^{2(1+\beta+nq)}}\\
&\leqslant\left(\frac{2}{(\alpha-\beta)^2}\right)^{\alpha-\beta}\frac{(1+\alpha+q)^{2(1+\alpha+q)}}{(1+\beta+q)^{2(1+\beta+q)}}\ n^{2(\alpha-\beta)}.
\end{aligned}$$

Moreover, the exponent $2(\alpha-\beta)$ of n is optimal. An inequality with the same $B_1(\alpha,\beta,n,q)$ holds for the q norms (or metrics) with the weight $(1+x)^\beta$ and $(1+x)^\alpha$ respectively.

Proof. We use inequality (7) from Proposition 1 for $\alpha_1 = 0$, $\alpha_2 = \alpha$, $\beta_1 = 0$, $\beta_2 = \beta$ and we get

$$A(0,\alpha,0,\beta,n,q) \leqslant \frac{2^{\alpha-\beta}}{(1-r)^{2(\alpha-\beta)}\, r^{2(1+\beta+nq)}}.$$

To minimize $\frac{2^{\alpha-\beta}}{(1-r)^{2(\alpha-\beta)}\, r^{2(1+\beta+nq)}}$ over all $r \in (0,1)$, we take $r_0 = \frac{1+\beta+nq}{1+\alpha+nq}$. The optimality of the exponent is a consequence of [9, Th. 2]. □

Taking into account Corollary 3 and Proposition 4, there are many different ways to obtain a polynomial estimate for generalized Jacobi weights of type (1). We propose below an inequality for the most general case. However, it is possible to find better estimates for specific cases with precise values of the zeros of the weight.

THEOREM 3. *Let* $-1 = x_1 < x_2 < \ldots < x_{m-1} < x_m = 1$, $\alpha_i \geqslant \beta_i \geqslant -\frac{1}{2}$ *for* $i = 1$, $i = m$ *and* $\alpha_i \geqslant \beta_i \geqslant 0$ *for* $i = 2, \ldots, m-1$. *The weights* u *and* w *are defined by*

$$u(x) = \prod_{i=1}^{m} |x - x_i|^{\alpha_i}, \qquad w(x) = \prod_{i=1}^{m} |x - x_i|^{\beta_i}. \tag{18}$$

Then for $q \in (0,\infty)$ *and* $p \in \mathscr{P}_n$ *the inequality*

$$\|p\|_{q,w}^{q} \leqslant D(u,w,n,q)\ \|p\|_{q,u}^{q}$$

holds with $D(u,w,n,q)$ *satisfying*

$$D(u,w,n,q) \leqslant \max\left\{K_1\, B_1(\alpha_1,\beta_1,n,q), K_m\, B_1(\alpha_m,\beta_m,n,q), \max_{j=2,\ldots,m-1}\left\{\frac{I_j}{J_j}\, B_0(\alpha_j,\beta_j,n,q)\right\}\right\}$$

where

$$K_1 := \frac{\prod_{i=2}^{m}(1+x_i)^{\beta_i}}{\prod_{i=2}^{m}(1+x_i)^{\alpha_i}}\, \frac{2^{\alpha_2+\ldots+\alpha_m} 4^{\alpha_1-\beta_1}}{(1+x_2)^{\alpha_1-\beta_1}}, \quad K_m := \left[\prod_{i=1}^{m-1} \frac{(1-x_i)^{\beta_i}}{(1-x_i)^{\alpha_i}}\right] \frac{2^{\alpha_1+\ldots+\alpha_{m-1}} 4^{\alpha_m-\beta_m}}{(1-x_{m-1})^{\alpha_m-\beta_m}},$$

$$I_j := 2^{\alpha_{j-1}+\alpha_{j+1}} \prod_{i=1}^{j-1}(x_{j+1}-x_i)^{\beta_i} \prod_{i=j}^{m}(x_i - x_{j-1})^{\beta_i},$$

$$J_j := \prod_{i=1}^{j-2}(x_{j-1}-x_i)^{\alpha_i}(x_j - x_{j-1})^{\alpha_{j-1}+\alpha_j}(x_{j+1}-x_j)^{\alpha_{j+1}} \prod_{i=j+2}^{m}(x_i - x_{j+1})^{\alpha_i}$$

(as usual, any product over the empty set is equal to 1) and B_0, B_1 *are estimated in Propositions* 3 *and* 4. *Moreover,*

$$D(u,w,n,q) \leqslant \max\left\{K_1\, B_1(\alpha_1,\beta_1,1,q), K_m\, B_1(\alpha_m,\beta_m,1,q), \max_{j=2,\ldots,m-1}\left\{\frac{I_j}{J_j}\, B_0(\alpha_j,\beta_j,1,q)\right\}\right\} n^d$$

and the exponent $d := \max\{2(\alpha_1-\beta_1), \alpha_2-\beta_2, \ldots, \alpha_{m-1}-\beta_{m-1}, 2(\alpha_m-\beta_m)\}$ *of* n *is optimal.*

Proof. We divide the interval $[-1,1]$ into m subintervals and integrate separately on each of them:

$$\int_{-1}^{1}|p(x)|^q\, w(x)dx = \int_{-1}^{\frac{x_2-1}{2}} + \int_{\frac{x_2-1}{2}}^{\frac{x_2+x_3}{2}} + \ldots + \int_{\frac{x_{m-2}+x_{m-1}}{2}}^{\frac{x_{m-1}+1}{2}} + \int_{\frac{x_{m-1}+1}{2}}^{1}.$$

On the first interval we have

$$\frac{w(x)}{(1+x)^{\beta_1}} \leqslant \prod_{i=2}^{m}(1+x_i)^{\beta_i} \quad \text{and} \quad \frac{u(x)}{(1+x)^{\alpha_1}} \geqslant \prod_{i=2}^{m}\left(\frac{1+x_i}{2}\right)^{\alpha_i}$$

and by Proposition 4,

$$\begin{aligned}
&\int_{-1}^{\frac{x_2-1}{2}}|p(x)|^q\, w(x)dx \leqslant \prod_{i=2}^{m}(1+x_i)^{\beta_i}\int_{-1}^{\frac{x_2-1}{2}}|p(x)|^q\,(1+x)^{\beta_1}dx\\
&=\prod_{i=2}^{m}(1+x_i)^{\beta_i}\left(\frac{1+x_2}{4}\right)^{1+\beta_1}\int_{-1}^{1}\left|p\left(\frac{1+x_2}{4}t+\frac{x_2-3}{4}\right)\right|^q(1+t)^{\beta_1}dt\\
&\leqslant\prod_{i=2}^{m}(1+x_i)^{\beta_i}\left(\frac{1+x_2}{4}\right)^{1+\beta_1}B_1(\alpha_1,\beta_1,n,q)\int_{-1}^{1}\left|p\left(\frac{1+x_2}{4}t+\frac{x_2-3}{4}\right)\right|^q(1+t)^{\alpha_1}dt.
\end{aligned}$$

Therefore,

$$\int_{-1}^{\frac{x_2-1}{2}}|p(x)|^q\, w(x)dx \leqslant K_1\, B_1(\alpha_1,\beta_1,n,q)\int_{-1}^{\frac{x_2-1}{2}}|p(x)|^q\, u(x)dx.$$

Analogously, for the integral over the last segment we get an inequality as above with the constant K_m instead of K_1. If $x\in\left(\frac{x_{j-1}+x_j}{2},\frac{x_j+x_{j+1}}{2}\right)$ for $j\in\{2,\ldots,m-1\}$ then

$$\begin{aligned}
\frac{w(x)}{|x-x_j|^{\beta_j}} &\leqslant \prod_{i=1}^{j-1}(x_{j+1}-x_i)^{\beta_i}\prod_{i=j+1}^{m}(x_i-x_{j-1})^{\beta_i}\\
&=\frac{I_j}{2^{\alpha_{j-1}+\alpha_{j+1}}(x_j-x_{j-1})^{\beta_j}},\\
\frac{u(x)}{|x-x_j|^{\alpha_j}} &\geqslant \prod_{i=1}^{j-2}(x_{j-1}-x_i)^{\alpha_i}\left(\frac{x_j-x_{j-1}}{2}\right)^{\alpha_{j-1}}\left(\frac{x_{j+1}-x_j}{2}\right)^{\alpha_{j+1}}\prod_{i=j+2}^{m}(x_i-x_{j+1})^{\alpha_i}\\
&=\frac{J_j}{2^{\alpha_{j-1}+\alpha_{j+1}}(x_j-x_{j-1})^{\alpha_j}}
\end{aligned}$$

and by Corollary 3,

$$\begin{aligned}
&\int_{\frac{x_{j-1}+x_j}{2}}^{\frac{x_j+x_{j+1}}{2}} |p(x)|^q w(x)dx \\
&\leqslant \frac{I_j}{2^{\alpha_{j-1}+\alpha_{j+1}}(x_j-x_{j-1})^{\beta_j}} \int_{\frac{x_{j-1}+x_j}{2}}^{\frac{x_j+x_{j+1}}{2}} |p(x)|^q |x-x_j|^{\beta_j} dx \\
&= \frac{(x_{j+1}-x_{j-1})^{1+\beta_j} I_j}{2^{\alpha_{j-1}+\alpha_{j+1}+1+\beta_j}(x_j-x_{j-1})^{\beta_j}} \\
&\quad \times \int_{-1}^{1} \left| p\left(\frac{x_{j+1}-x_{j-1}}{4}t+\frac{2x_j+x_{j+1}+x_{j-1}}{4}\right)\right|^q \left|t+\frac{x_{j-1}-2x_j+x_{j+1}}{x_{j+1}-x_{j-1}}\right|^{\beta_j} dt \\
&\leqslant \frac{(x_{j+1}-x_{j-1})^{1+\beta_j} I_j}{2^{\alpha_{j-1}+\alpha_{j+1}+1+\beta_j}(x_j-x_{j-1})^{\beta_j}} \frac{(x_{j+1}-x_{j-1})^{\alpha_j-\beta_j} B_0(\alpha_j,\beta_j,n,q)}{[2(x_j-x_{j-1})]^{\alpha_j-\beta_j}} \int_{-1}^{1} |p(\ldots)|^q |\ldots|^{\alpha_j} dt \\
&= \frac{2^{\alpha_j-\beta_j} I_j}{2^{\alpha_{j-1}+\alpha_{j+1}}(x_j-x_{j-1})^{\beta_j}} \frac{B_0(\alpha_j,\beta_j,n,q)}{[2(x_j-x_{j-1})]^{\alpha_j-\beta_j}} \int_{\frac{x_{j-1}+x_j}{2}}^{\frac{x_j+x_{j+1}}{2}} |p(x)|^q \left|x-x_j\right|^{\alpha_j} dx.
\end{aligned}$$

Taking into account the estimate for $\frac{u(x)}{|x-x_j|^{\alpha_j}}$ we obtain the inequality

$$\int_{\frac{x_{j-1}+x_j}{2}}^{\frac{x_j+x_{j+1}}{2}} |p(x)|^q w(x)dx \leqslant \frac{I_j}{J_j} B_0(\alpha_j,\beta_j,n,q) \int_{\frac{x_{j-1}+x_j}{2}}^{\frac{x_j+x_{j+1}}{2}} |p(x)|^q u(x)dx.$$

The optimality of the exponent d has been proved in [9, Th. 2]. □

An analogue of Theorem 3 can be easily proved also for weights of form (1).

We can also obtain similar estimates to these in Proposition 2 for generalized Jacobi weights (18). It is sufficient to take $j_i \in \mathbb{N}$, $k_i \in \mathbb{N}_0$ such that $\frac{\beta_i}{q} \leqslant \frac{k_i}{j_i} \leqslant \frac{\alpha_i}{s}$ for $i = 1,\ldots,m$. Then we can apply Theorem 3) and use analogous arguments as in the proof of Proposition 2.

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