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HEINZ MEANS AND TRIANGLES INSCRIBED IN A SEMICIRCLE IN BANACH SPACES

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Abstract. In this paper, we introduce two classes of new geometric constants for Banach spaces by using the Heinz means that interpolate between the geometric and arithmetic means. One of these constants is closely related to the modulus of convexity of the space and it seems to represent a useful tool to estimate the exact values of the James and von Neumann-Jordan constants of some Banach spaces, while the study of the other one seems to be more complicated. Moreover, we investigate some geometric properties related to these constants and calculate the precise values of these two constants for several Banach spaces. We also study the stability under norm perturbations of these constants.

1. Introduction

There are various ways for constructing the means between two positive numbers a and b. One of the most remarkable, which interpolates in a certain way between the arithmetic and geometric means is the so-called Heinz mean M_ν in the parameter $0 \leqslant \nu \leqslant 1$, defined by

$$M_\nu(a,b) = \frac{a^\nu b^{1-\nu} + a^{1-\nu} b^\nu}{2}.$$

One can easily show that the Heinz means are "inbetween" the geometric mean and the arithmetic mean, i.e.

$$\sqrt{ab} = M_{\frac{1}{2}}(a,b) \leqslant M_\nu(a,b) \leqslant M_1(a,b) = \frac{a+b}{2}, \qquad \nu \in [0,1].$$

It is easy to see that the Heinz mean is convex as a function of ν on the interval $[0,1]$, attains minimum at $\nu = \frac{1}{2}$, and attains maximum at $\nu = 0$ and $\nu = 1$. Moreover, $M_\nu(a,b)$ is symmetric with respect to $\nu = \frac{1}{2}$, i.e.

$$M_\nu(a,b) = M_{1-\nu}(a,b), \qquad \nu \in [0,1].$$

Throughout the paper, we shall assume that X is a Banach space with the dual space X^*. As usual, we will use $S_X = \{x \in X : \|x\| = 1\}$ and $B_X = \{x \in X : \|x\| \leqslant 1\}$ to denote the unit sphere and the closed unit ball of X, respectively.

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Recently many investigations have been devoted to geometric constants of a Banach space X, which enable us to make precise descriptions on various geometric properties of X. It should be noted that if y and $-y$ are antipodal points on the unit sphere S_X, then $\|x+y\|$, $\|x-y\|$ and 2 can be regarded as the lengths of the sides of the triangle T_{xy} with vertices x, y and $-y$ lying on S_X. Therefore, many geometric constants of Banach spaces can be regarded as the result of some kind of estimation of the lengths of the sides of these triangles when x and y move on S_X. For instance, if we consider the arithmetic mean of the lengths of the non-constant sides of T_{xy}, we get the constants

$$A_1(X) = \inf_{x\in S_X} \sup_{y\in S_X} M_1\big(\|x+y\|, \|x-y\|\big)$$

and

$$A_2(X) = \sup_{x,y\in S_X} M_1\big(\|x+y\|, \|x-y\|\big),$$

introduced by Baronti, Casini and Papini [4] in 2000.

Moreover, if we consider the geometric mean of the lengths of the non-constant sides of T_{xy}, we get the constants

$$t(X) = \inf_{x\in S_X} \sup_{y\in S_X} M_{\frac{1}{2}}\big(\|x+y\|, \|x-y\|\big)$$

and

$$T(X) = \sup_{x,y\in S_X} M_{\frac{1}{2}}\big(\|x+y\|, \|x-y\|\big),$$

introduced by Alonso and Llorens-Fuster [1] in 2008.

Following this line of research, we will introduce two classes of new geometric constants based on averaging the lengths of the sides of T_{xy} by considering the Heinz means, which are more general than the above constants. These constants are also proved to be connected with the well known modulus of convexity and other geometric constants which allow us to compute the precise values of these two constants for some interesting spaces. The results presented in this work are more general than the known results about the constants mentioned above.

2. Preliminaries

We start by reviewing some notions and definitions which will be needed in the sequel.

The Clarkson modulus of convexity of X [10] is the function $\delta_X : [0,2] \to [0,1]$ given by

$$\begin{aligned}\delta_X(\varepsilon) &= \inf\left\{1 - \frac{\|x+y\|}{2} : x,y \in B_X,\ \|x-y\| \geqslant \varepsilon\right\}\\ &= \inf\left\{1 - \frac{\|x+y\|}{2} : x,y \in S_X,\ \|x-y\| = \varepsilon\right\}.\end{aligned}$$

The function δ_X is continuous on $[0,2)$, increasing on $[0,2]$ and strictly increasing on $[\varepsilon_0(X),2]$, where $\varepsilon_0(X)=\sup\{\varepsilon\in[0,2]:\delta_X(\varepsilon)=0\}$ is the so-called the characteristic of convexity of X. A Banach space X is said to be uniformly convex if $\delta_X(\varepsilon)>0$ for $0<\varepsilon$
$\leqslant 2$, or equivalently, if $\varepsilon_0(X)=0$.

Recall that a Banach space X is called to be uniformly non-square (see [11]) if there exists $\delta\in(0,1)$ such that either $\|x+y\|\leqslant 2(1-\delta)$ or $\|x-y\|\leqslant 2(1-\delta)$ whenever $x,y\in S_X$, a property which in its turn implies that X is reflexive. It is easy to check that X is uniformly non-square if and only if $\varepsilon_0(X)<2$.

In order to measure the degree of uniform non-squareness of X, Gao [8] in 1982 defined the constant

$$J(X)=\sup_{x,y\in S_X}\min\{\|x+y\|,\|x-y\|\}$$

usually called the non-square or James constant. It is related to the Clarkson modulus by the equality

$$J(X)=\sup\left\{\varepsilon\in(0,2):\delta_X(\varepsilon)<1-\frac{\varepsilon}{2}\right\}.$$

Gao [8] also introduced the constant

$$j(X)=\inf_{x\in S_X}\sup_{y\in S_X}\max\{\|x+y\|,\|x-y\|\}.$$

In connection with the celebrated work of Jordan and von Neumann concerning inner products [12], the von Neumann-Jordan constant of X was introduced by Clarkson [5] as the smallest constant C for which

$$\frac{1}{C}\leqslant\frac{\|x+y\|^2+\|x-y\|^2}{2(\|x^2\|+\|y^2\|)}\leqslant C$$

holds for all $x,y\in X$ with $(x,y)\neq(0,0)$. If C is the best possible constant on the right-hand side of the above inequality, then so is $\frac{1}{C}$ on the left-hand one. An equivalent definition of the NJ constant is found in [13] as

$$C_{NJ}(X)=\sup\left\{\frac{\|x+y\|^2+\|x-y\|^2}{2(\|x^2\|+\|y^2\|)}:x\in S_X,y\in B_X\right\}.$$

As regards the above constants, we collect some basic properties of them (see [1, 4, 14]):

(i) $\sqrt{2}\leqslant J(X)\leqslant T(X)\leqslant A_2(X)\leqslant 2$ and $1\leqslant C_{NJ}(X)\leqslant 2$;

(ii) $\max\{J(X),\sqrt{2\varepsilon_0(X)}\}\leqslant T(X)\leqslant A_2(X)\leqslant\sqrt{2C_{NJ}(X)}$;

(iii) X is a Hilbert space if and only if $C_{NJ}(X)=1$;

(iv) If X is a Hilbert space, then $J(X)=\sqrt{2}$, $T(X)=\sqrt{2}$, $A_2(X)=\sqrt{2}$ and the converse is not true for each condition;

(v) $C_{NJ}(X) = C_{NJ}(X^*)$ and $A_2(X) = A_2(X^*)$, whereas $J(X) \neq J(X^*)$ and $T(X) \neq T(X^*)$ in general;

(vi) X is uniformly non-square if and only if one of the following is true:

$$(a)\, J(X) < 2, \quad (b)\, C_{NJ}(X) < 2, \quad (c)\, T(X) < 2, \quad (d)\, A_2(X) < 2.$$

Now, let us introduce the constants based on the Heinz means of the lengths of the sides of T_{xy}.

DEFINITION 1. For a given Banach space X, let

$$\begin{aligned} H_\nu(X) &= \sup_{x,y \in S_X} M_\nu\big(\|x+y\|, \|x-y\|\big) \\ &= \sup_{x,y \in S_X} \frac{\|x+y\|^\nu \|x-y\|^{1-\nu} + \|x+y\|^{1-\nu}\|x-y\|^\nu}{2} \end{aligned}$$

and

$$\begin{aligned} h_\nu(X) &= \inf_{x \in S_X} \sup_{y \in S_X} M_\nu\big(\|x+y\|, \|x-y\|\big) \\ &= \inf_{x \in S_X} \sup_{y \in S_X} \frac{\|x+y\|^\nu \|x-y\|^{1-\nu} + \|x+y\|^{1-\nu}\|x-y\|^\nu}{2}, \end{aligned}$$

where $0 \leqslant \nu \leqslant 1$.

REMARK 1. Obviously, $\sqrt{2} \leqslant H_\nu(X) \leqslant 2$ for $0 \leqslant \nu \leqslant 1$.

3. Properties of $H_\nu(X)$ and $h_\nu(X)$

From the definitions, it is clear that $h_\nu(X) \leqslant H_\nu(X)$ for any space X. Since

$$\begin{aligned} \min\big\{\|x+y\|, \|x-y\|\big\} &\leqslant \sqrt{\|x+y\|\,\|x-y\|} \\ &\leqslant \frac{\|x+y\|^\nu \|x-y\|^{1-\nu} + \|x+y\|^{1-\nu}\|x-y\|^\nu}{2} \\ &\leqslant \frac{\|x+y\| + \|x-y\|}{2}, \end{aligned}$$

it follows that

$$j(X) \leqslant t(X) \leqslant h_\nu(X) \leqslant A_1(X) \leqslant 2 \tag{3.1}$$

and

$$J(X) \leqslant T(X) \leqslant H_\nu(X) \leqslant A_2(X) \leqslant 2. \tag{3.2}$$

EXAMPLE 1. It is well known (see [1, 4]) that $t(\ell_p) = A_1(\ell_p) = 2^{\frac{1}{p}}$ for $1 \leqslant p \leqslant 2$. Hence, from (3.1) it follows that also $h_\nu(X) = 2^{\frac{1}{p}}$. Moreover, $T(\ell_p) = A_2(\ell_p) = \max\{2^{\frac{1}{p}}, 2^{1-\frac{1}{p}}\}$ for all $1 \leqslant p < \infty$. Consequently, (3.2) gives $H_\nu(\ell_p) = \max\{2^{\frac{1}{p}}, 2^{1-\frac{1}{p}}\}$.

EXAMPLE 2. For $X = (\mathbb{R}^2, \|\cdot\|_\infty)$, it is shown in [1] that $T(X) = 2$ and in [4, 9] that $A_2(X) = J(X) = T(X)$. Hence, from (3.2) it follows that $H_\nu(X) = 2$. Furthermore, it is well known (see [1, 4, 9]) that $t(X) = \sqrt{2}$, $A_1(X) = \frac{3}{2}$ and $j(X) = 1$. Therefore, thanks to (3.1), we have $\sqrt{2} \leqslant h_\nu(X) \leqslant \frac{3}{2}$. Note that this is a space for which $j(X) < h_\nu(X)$.

EXAMPLE 3. Let X_o be the space $\mathbb{R}^2$ endowed with a norm whose unit sphere is a regular octagon. In [1, 4] it has been shown that $t(X_o) = T(X_o) = A_1(X_o) = A_2(X_o) = \sqrt{2}$. Therefore, by applying the inequalities (3.1) and (3.2), we obtain that also $h_\nu(X_o) = H_\nu(X_o) = \sqrt{2}$.

We shall see that the constant $H_\nu(X)$ allows to characterize uniformly non-square spaces. It was shown that a Banach space X is uniformly non-square if and only if either $A_2(X) < 2$ or $T(X) < 2$. So bearing in mind inequalities (3.2), we get the following proposition.

PROPOSITION 1. *A Banach space X is uniformly non-square if and only if $H_\nu(X) < 2$ for $0 \leqslant \nu \leqslant 1$.*

PROPOSITION 2. *A Banach space X is not uniformly non-square if and only if any of the following properties hold:*

(i) $H_\nu(X) = 2$;

(ii) $J(X) = 2$.

Proof. It is known that both $A_2(X) = 2$ and $J(X) = 2$ characterize the spaces which are not uniformly non-square (see [4, 9]). From (3.2) it follows that the same is the case for $H_\nu(X) = 2$. □

With respect to the other bound of $H_\nu(X)$ in item (i) of Remark 1, the following proposition shows that it is attained in Hilbert spaces, but Example 3 tells us that $H_\nu(X) = \sqrt{2}$ is not characteristic of such spaces when $\dim(X) = 2$.

PROPOSITION 3. *If X is a Hilbert space, then $H_\nu(X) = \sqrt{2}$ for $0 \leqslant \nu \leqslant 1$ and the converse does not hold in general.*

It is natural to ask which necessary and sufficient conditions hold for Banach spaces X satisfying $H_\nu(X) = \sqrt{2}$ for $0 \leqslant \nu \leqslant 1$. It is shown in [14] that, for a Banach space of three or more dimensions, the James constant becomes $\sqrt{2}$ if and only if X is a Hilbert space. Using the same method in [14], we can get the following proposition, which answers a question posed by Alonso and Llorens-Fuster [1].

PROPOSITION 4. *If* $\dim(X) \geqslant 3$, *then* $H_\nu(X) = \sqrt{2}$ *for* $0 \leqslant \nu \leqslant 1$ *if and only if* X *is a Hilbert space.*

With respect to the bounds of $h_\nu(X)$, it is obvious that $h_\nu(X) \leqslant 2$. The following proposition provides some information about the extreme value 2.

PROPOSITION 5. *Let* $0 \leqslant \nu \leqslant 1$. *For a Banach space* X *the following properties are equivalent:*

(i) $j(X) = 2$;

(ii) $h_\nu(X) = 2$.

Moreover, any of the above properties implies that X *is a not-uniformly non-square infinite dimensional space. Consequently, in finite-dimensional spaces the two constants are less than* 2.

Proof. From the inequality

$$j(X) \leqslant h_\nu(X) \leqslant 2,$$

the above conditions are equivalent. In [1, Proposition 9] it is proved that X is a not-uniformly non-square infinite-dimensional space whenever $j(X) = 2$, which completes the proof. $\square$

In [4] it is shown that if X is any of the spaces c_0, c, or l_∞, then $A_1(X) = \frac{3}{2}$. This proves that for such spaces $h_\nu(X) \leqslant \frac{3}{2}$ and that the reciprocal result to the one in the second part of Proposition 5 is not true. Moreover, if X is any of the spaces $C[0,1]$, $C_0[0,1]$, or $L_1[0,1]$, then $A_1(X) = 2$ which implies that $h_\nu(X) = 2$. Also it is proved in [23] that if $\dim(X) = 2$, then $A_1(X) \leqslant \frac{3}{2}$, and then the same bound is valid for $h_\nu(X)$. Notice that this estimate is an improvement of a result given by Baronti, Casini and Papini [4].

PROPOSITION 6. *Let* $0 \leqslant \nu \leqslant 1$. *For any Banach space* X, $h_\nu(X) \geqslant M_\nu\left(1, \frac{3}{2}\right)$.

Proof. Suppose that $x \in S_X$. In any two-dimensional subspace of X that contains x we can find $z \in S_X$ such that $\|x - z\| = 1$. Let $w = x - z$. Thus, $3x = (x + w) + (x + z)$ which gives $3 \leqslant \|x + w\| + \|x + z\|$. Hence, either $\|x + w\| \geqslant \frac{3}{2}$ or $\|x + z\| \geqslant \frac{3}{2}$ with $\|x - w\| = \|x - z\| = 1$. Therefore, we have

$$\sup_{y \in S_X} M_\nu\left(\|x + y\|, \|x - y\|\right) \geqslant M_\nu\left(1, \frac{3}{2}\right).$$

Since x is arbitrary, so we obtain the desired inequality. $\square$

According to Proposition 19 of [1], we get the following result.

PROPOSITION 7. *For every Banach space X,*

$$2 \leqslant j(X)J(X) \leqslant h_v(X)J(X) \leqslant h_v(X)H_v(X).$$

REMARK 2. With regards the inequalities in Proposition 7 all the situations are possible. Gao [9] showed that $j(\ell_p) = J(\ell_p) = 2^{\frac{1}{p}}$ for $1 \leqslant p < 2$, whereas $j(\ell_p) = 2^{\frac{1}{p}}$ and $J(\ell_p) = 2^{1-\frac{1}{p}}$ for $2 \leqslant p < \infty$. Therefore, $j(\ell_p)J(\ell_p) > 2$ for $1 \leqslant p < 2$ and $j(\ell_p)J(\ell_p) = 2$ for $2 \leqslant p < \infty$. Example 1 shows that $j(\ell_p)J(\ell_p) = h_v(\ell_p)J(\ell_p) = h_v(\ell_p)H_v(\ell_p) = 2^{\frac{2}{p}} > 2$ for $1 \leqslant p < 2$. Example 2 shows that for $X = (\mathbb{R}^2, \|\cdot\|_\infty)$, we have $h_v(X) = \sqrt{2}$ and $H_v(X) = 2$ and from [9] we know that $j(X) = 1$ and $J(X) = 2$. Therefore for this space we deduce that $2 = j(X)J(X) < h_v(X)J(X) < H_v(X)J(X)$. Example 3 shows that all the inequalities can be identities.

Now, let us state an identity between the modulus of convexity and $H_v(X)$ which is motivated by [6].

THEOREM 1. *For any Banach space X,*

$$H_v(X) = \sup\Big\{M_v\big(\varepsilon, 2(1-\delta_X(\varepsilon))\big) \,:\, 0 \leqslant \varepsilon \leqslant 2\Big\},$$

where $0 \leqslant v \leqslant 1$.

Proof. Suppose that $0 \leqslant v \leqslant 1$ and

$$\sup\Big\{M_v\big(\varepsilon, 2(1-\delta_X(\varepsilon))\big) \,:\, 0 \leqslant \varepsilon \leqslant 2\Big\} = K.$$

From the definition of $\delta_X(\varepsilon)$, we have $\delta_X(\|x-y\|) \leqslant 1 - \frac{1}{2}\|x+y\|$ for $x, y \in S_X$. Then

$$\begin{aligned} &\frac{\|x+y\|^v\|x-y\|^{1-v} + \|x+y\|^{1-v}\|x-y\|^v}{2} \\ &\leqslant \frac{\varepsilon^v 2^{1-v}(1-\delta_X(\varepsilon))^{1-v} + \varepsilon^{1-v}2^v(1-\delta_X(\varepsilon))^v}{2} = K, \end{aligned}$$

from which it follows that $H_v(X) \leqslant K$. On the other hand, let $0 \leqslant \varepsilon \leqslant 2$. Then, for any $\eta > 0$ there exist $x, y \in S_X$ such that $\|x-y\| = \varepsilon$ and $1 - \frac{\|x+y\|}{2} \leqslant \delta_X(\varepsilon) + \eta$. Hence, we have

$$\begin{aligned} H_v(X) &\geqslant \frac{\|x+y\|^v\|x-y\|^{1-v} + \|x+y\|^{1-v}\|x-y\|^v}{2} \\ &\geqslant \frac{\varepsilon^v 2^{1-v}(1-\delta_X(\varepsilon))^{1-v} + \varepsilon^{1-v}2^v(1-\delta_X(\varepsilon))^v}{2}. \end{aligned}$$

Since η is arbitrary, it follows that $H_v(X) \geqslant K$ and hence the desired equality. □

COROLLARY 1. *For any Banach space X,*

$$\max\big(J(X), M_\nu(\varepsilon_0, 2)\big) \leqslant H_\nu(X),$$

where $0 \leqslant \nu \leqslant 1$.

The following example shows a space for which the modulus of convexity is known and for which we can compute $H_\nu(X)$ by applying Theorem 1.

EXAMPLE 4. Let W be the space $\mathbb{R}^2$ endowed with the norm

$$\big\|(x_1, x_2)\big\|_W = \max\Big\{2|x_1|, \sqrt{x_1^2 + x_2^2}\Big\}.$$

Hence, due to [3], we know that

$$\delta_W(\varepsilon) = \begin{cases} 0, & 0 \leqslant \varepsilon \leqslant \sqrt{3}, \\ 1 - 2\sqrt{1 - \frac{\varepsilon^2}{4}}, & \sqrt{3} \leqslant \varepsilon \leqslant \sqrt{\frac{16}{5}}, \\ 1 - \sqrt{1 - \frac{\varepsilon^2}{16}}, & \sqrt{\frac{16}{5}} \leqslant \varepsilon \leqslant 2. \end{cases}$$

By virtue of Theorem 1, we obtain

$$H_\nu(W) = M_\nu\big(\sqrt{3}, 2(1 - \delta_W(\sqrt{3}))\big) = 3^{\frac{\nu}{2}} \cdot 2^{-\nu} + 3^{\frac{1-\nu}{2}} \cdot 2^{\nu-1}.$$

In particular, it is elementary to check that $T(W) = \sqrt{2\sqrt{3}} \approx 1.8612$ and $A_2(W) = 1 + \frac{\sqrt{3}}{2} \approx 1.8660$. Therefore, $\sqrt{2\sqrt{3}} \leqslant H_\nu(W) \leqslant 1 + \frac{\sqrt{3}}{2}$. It is easy to check that $J(W) = \frac{4}{\sqrt{5}} \approx 1.7888$. Consequently, for this space W, we have $J(W) < H_\nu(W)$.

It is clear that for all Banach spaces X,

$$\max\Big\{J(X), \sqrt{2\varepsilon_0(X)}\Big\} \leqslant H_\nu(X) \leqslant \sqrt{2C_{NJ}(X)}, \tag{3.3}$$

where all the terms coincide if X is not uniformly non-square.

The inequalities (3.3) allows us to obtain $H_\nu(X)$ for some spaces for which $J(X)$ and $C_{NJ}(X)$ are well known.

EXAMPLE 5. Let V be the space $\mathbb{R}^2$ endowed with the norm

$$\big\|(x_1, x_2)\big\|_V = \max\Big\{\sqrt{x_1^2 + 2x_2^2}, \sqrt{2x_1^2 + x_2^2}\Big\}.$$

The closed unit ball of V is just the set

$$B_V = \max\big\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + 2x_2^2 \leqslant 1, 2x_1^2 + x_2^2 \leqslant 1\big\}.$$

Notice that $\varepsilon_0(V) = 0$. In [16] it was shown that

$$C_{NJ}(V) = \frac{4}{3} \qquad \text{and} \qquad J(V) = \sqrt{\frac{8}{3}}.$$

Therefore, $J(V) = \sqrt{2C_{NJ}(V)}$ and from (3.3), we get

$$H_\nu(V) = \sqrt{\frac{8}{3}}.$$

EXAMPLE 6. For $\beta \geqslant 1$, let X_β be the space ℓ_2 endowed with the norm

$$|x|_\beta = \max\left\{\|x\|_2, \beta\|x\|_\infty\right\}.$$

The spaces X_β have been extensively studied because they play a major role in Metric Fixed Point Theory. It is shown in [13] that

$$C_{NJ}(X_\beta) = \min\{2, \beta^2\} \qquad \text{and} \qquad J(X_\beta) = \min\{2, \beta\sqrt{2}\}.$$

In particular, if $1 < \beta < \sqrt{2}$, $C_{NJ}(X_\beta) = \beta^2$ and $J(X_\beta) = \beta\sqrt{2} > \sqrt{2\varepsilon_0(X_\beta)} = \sqrt{2\sqrt{\beta^2 - 1}}$. Therefore, thanks to (3.3), we obtain

$$H_\nu(X_\beta) = \beta\sqrt{2}.$$

EXAMPLE 7. For $\lambda > 0$, let Z_λ be the space $\mathbb{R}^2$ endowed with the norm

$$|x|_\lambda = \left(\|x\|_p^2 + \lambda\|x\|_q^2\right)^{\frac{1}{2}}.$$

It is shown in [24] that

(i) if $2 \leqslant p \leqslant q \leqslant \infty$, then $J(Z_\lambda) = 2\sqrt{\frac{\lambda+1}{2^{\frac{2}{p}} + \lambda 2^{\frac{2}{q}}}}$ and $C_{NJ}(Z_\lambda) = \frac{2(\lambda+1)}{2^{\frac{2}{p}} + \lambda 2^{\frac{2}{q}}}$.

Thus, $J(Z_\lambda) = \sqrt{2C_{NJ}(Z_\lambda)}$ and from (3.3), we get

$$H_\nu(Z_\lambda) = 2\sqrt{\frac{\lambda+1}{2^{\frac{2}{p}} + \lambda 2^{\frac{2}{q}}}}.$$

(ii) if $1 \leqslant p \leqslant q \leqslant 2$, then $J(Z_\lambda) = \sqrt{\frac{2^{\frac{2}{p}} + \lambda 2^{\frac{2}{q}}}{\lambda+1}}$ and $C_{NJ}(Z_\lambda) = \frac{2^{\frac{2}{p}} + \lambda 2^{\frac{2}{q}}}{2(\lambda+1)}$.

Hence, $J(Z_\lambda) = \sqrt{2C_{NJ}(Z_\lambda)}$ and (3.3) gives

$$H_\nu(Z_\lambda) = \sqrt{\frac{2^{\frac{2}{p}} + \lambda 2^{\frac{2}{q}}}{\lambda+1}}.$$

Moreover, it is proved in [25] that

(iii) if $1 \leqslant p \leqslant 2 \leqslant q \leqslant \infty$, then $C_{NJ}(Z_\lambda) \leqslant \frac{2^{\frac{2}{p}}+2\lambda}{2^{\frac{2}{q}}\lambda+2}$.

Thus, by applying (3.3), we have

$$H_v(Z_\lambda) \leqslant \sqrt{2C_{NJ}(Z_\lambda)} \leqslant \sqrt{\frac{2(2^{\frac{2}{p}}+2\lambda)}{2^{\frac{2}{q}}\lambda+2}}.$$

On the one hand, let $x_1 = (a,a)$ and $y_1 = (a,-a)$, where $a = \frac{1}{\sqrt{2^{\frac{2}{p}}+\lambda 2^{\frac{2}{q}}}}$. Then $\|x_1\|_\lambda = \|y_1\|_\lambda = 1$, $x_1+y_1 = (2a,0)$ and $x_1-y_1 = (0,2a)$. Hence, we obtain

$$H_v(Z_\lambda) \geqslant 2\sqrt{\frac{\lambda+1}{2^{\frac{2}{p}}+\lambda 2^{\frac{2}{q}}}}.$$

On the other hand, let $x_2 = (b,0)$ and $y_2 = (0,-b)$, where $b = \frac{1}{\sqrt{\lambda+1}}$. Then $\|x_2\|_\lambda = \|y_2\|_\lambda = 1$, $x_2+y_2 = (b,-b)$ and $x_2-y_2 = (b,b)$. Hence, we get

$$H_v(Z_\lambda) \geqslant \sqrt{\frac{2^{\frac{2}{p}}+\lambda 2^{\frac{2}{q}}}{\lambda+1}}.$$

Therefore, we have

$$\max\left\{2\sqrt{\frac{\lambda+1}{2^{\frac{2}{p}}+\lambda 2^{\frac{2}{q}}}}, \sqrt{\frac{2^{\frac{2}{p}}+\lambda 2^{\frac{2}{q}}}{\lambda+1}}\right\} \leqslant H_v(Z_\lambda) \leqslant \sqrt{\frac{2(2^{\frac{2}{p}}+2\lambda)}{2^{\frac{2}{q}}\lambda+2}}.$$

In the next example, we calculate the constant $H_v(X)$ for two-dimensional Lorentz sequence space $X = d^{(2)}(w,q)$ where $2 \leqslant q < \infty$. For more detailed discussion and some results concerning Lorentz sequence spaces, we refer the reader to [7, 18, 22].

EXAMPLE 8. (Lorentz sequence space). Let $w = (w_1, w_2)$ with $w_1 \geqslant w_2 > 0$. For $2 \leqslant q < \infty$, the two-dimensional Lorentz sequence space $d^{(2)}(w,q)$ is defined as the space $\mathbb{R}^2$ endowed with the norm

$$\|(x,y)\|_{w,q} = \left\{w_1|x|^{*q} + w_2|y|^{*q}\right\}^{\frac{1}{q}},$$

where $(|x|^*, |y|^*)$ is the rearrangement of $(|x|, |y|)$ satisfying $|x|^* \geqslant |y|^*$. One has that

$$J(d^{(2)}(w,q)) = 2\left(\frac{w_1}{w_1+w_2}\right)^{\frac{1}{q}} \quad \text{and} \quad C_{NJ}(d^{(2)}(w,q)) = 2\left(\frac{w_1}{w_1+w_2}\right)^{\frac{2}{q}}.$$

Therefore, $J(d^{(2)}(w,q)) = \sqrt{2C_{NJ}(d^{(2)}(w,q))}$ and due to (3.3), we get

$$H_v(d^{(2)}(w,q)) = 2\left(\frac{w_1}{w_1+w_2}\right)^{\frac{1}{q}}.$$

We now compute the constant $H_\nu(X)$ in the case when X is a two-dimensional Cesàro sequence space $\mathrm{ces}_2^{(2)}$. The Cesàro sequence space was defined by Shue [21] in 1970. It is very useful in the theory of matrix operators and others.

Let ℓ be the space of real sequences. For $1 < p < \infty$, the Cesàro sequence space ces_p is defined by

$$\mathrm{ces}_p = \left\{ x \in \ell : \|x\| = \big\|\big(x(i)\big)\big\| = \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{n} |x(i)| \right)^p \right)^{\frac{1}{p}} < \infty \right\}.$$

The geometry of Cesàro sequence spaces and their generalizations have been extensively studied in [7, 17, 19, 20, 21]. Let us restrict ourselves to the two-dimensional Cesàro sequence space $\mathrm{ces}_p^{(2)}$ which is just $\mathbb{R}^2$ equipped with the norm defined by

$$\big\|(x,y)\big\| = \left(|x|^p + \left(\frac{|x|+|y|}{2} \right)^p \right)^{\frac{1}{p}}$$

for all $(x,y) \in \mathbb{R}^2$.

EXAMPLE 9. (Cesàro sequence space). The two-dimensional Cesàro sequence space $\mathrm{ces}_2^{(2)}$ is defined as the space $\mathbb{R}^2$ endowed with the norm

$$\big\|(x,y)\big\| = \left(|x|^2 + \left(\frac{|x|+|y|}{2} \right)^2 \right)^{\frac{1}{2}}.$$

It is shown in [19] that

$$C_{NJ}\big(\mathrm{ces}_2^{(2)}\big) = 1 + \frac{1}{\sqrt{5}} \qquad \text{and} \qquad J\big(\mathrm{ces}_2^{(2)}\big) = \sqrt{2 + \frac{2}{\sqrt{5}}}.$$

Therefore, $J\big(\mathrm{ces}_p^{(2)}\big) = \sqrt{2C_{NJ}\big(\mathrm{ces}_p^{(2)}\big)}$ and thanks to (3.3), we obtain

$$H_\nu\big(\mathrm{ces}_2^{(2)}\big) = \sqrt{2 + \frac{2}{\sqrt{5}}}.$$

The equality $J(X) = \sqrt{2C_{NJ}(X)}$ holds in the above examples. Although it seems to suggest some kind of symmetry, it is untrue in general, as the following example shows.

EXAMPLE 10. (Day-James $\ell_\infty - \ell_1$ space). Let $\ell_\infty - \ell_1$ be $\mathbb{R}^2$ endowed with the norm defined by

$$\|(x_1,x_2)\| = \begin{cases} \|(x_1,x_2)\|_\infty, & x_1x_2 \geqslant 0, \\ \|(x_1,x_2)\|_1, & x_1x_2 \leqslant 0. \end{cases}$$

Thus, $\delta_{\ell_\infty-\ell_1}(\varepsilon)=\max\left\{0,\frac{\varepsilon-1}{2}\right\}$. In [13] it was shown that $J(\ell_\infty-\ell_1)=\frac{3}{2}$. From Theorem 1 it immediately follows that $H_\nu(\ell_\infty-\ell_1)=\frac{3}{2}$. The calculation of the von Neumann-Jordan constant of this space was a part of Problem 2 in [13]. It was solved independently in two recent papers, namely [26] and [2], and the value is $C_{NJ}(\ell_\infty-\ell_1)=\frac{3+\sqrt{5}}{4}$. Therefore, we have

$$J(\ell_\infty-\ell_1)=H_\nu(\ell_\infty-\ell_1)=\frac{3}{2}<\sqrt{2C_{NJ}(\ell_\infty-\ell_1)}\approx 1.618.$$

EXAMPLE 11. (Day-James $\ell_2-\ell_1$ space). Let $\ell_2-\ell_1$ be $\mathbb{R}^2$ endowed with the norm defined by

$$\|(x_1,x_2)\|=\begin{cases}\|(x_1,x_2)\|_2, & x_1x_2\geqslant 0,\\ \|(x_1,x_2)\|_1, & x_1x_2<0.\end{cases}$$

In [13] it was shown that $J(\ell_2-\ell_1)=\sqrt{\frac{8}{3}}\approx 1.633$. Moreover, from [2] and [26] we know that $C_{NJ}(\ell_2-\ell_1)=\frac{3}{2}$. It is well known [10] that $\varepsilon_0(\ell_2-\ell_1)=\sqrt{2}$ and that the modulus of convexity $\delta_{\ell_2-\ell_1}$ is the function

$$\delta_{\ell_2-\ell_1}(\varepsilon)=\begin{cases}0, & 0\leqslant\varepsilon\leqslant\sqrt{2},\\ \min\left\{1-\sqrt{2-\frac{\varepsilon^2}{2}},1-\sqrt{1-\frac{\varepsilon^2}{8}}\right\}, & \sqrt{2}\leqslant\varepsilon\leqslant 2.\end{cases}$$

Owing to Theorem 1 it is easy to derive that

$$H_\nu(\ell_2-\ell_1)=M_\nu\left(\sqrt{2},2(1-\delta_{\ell_2-\ell_1}(\sqrt{2}))\right)=2^{-\frac{\nu}{2}}+2^{\frac{\nu-1}{2}}.$$

In particular, we have $T(\ell_2-\ell_1)=\sqrt{2\sqrt{2}}$ (see [1]) and $A_2(\ell_2-\ell_1)=1+\frac{\sqrt{2}}{2}$ (see [1]). Thus, $\sqrt{2\sqrt{2}}\leqslant H_\nu(\ell_2-\ell_1)\leqslant 1+\frac{\sqrt{2}}{2}$. Note that $T(\ell_2-\ell_1)<A_2(\ell_2-\ell_1)$. With the help of inequalities (3.3), for this space one has

$$J(\ell_2-\ell_1)<\sqrt{2\varepsilon_0(\ell_2-\ell_1)}\leqslant H_\nu(\ell_2-\ell_1)<\sqrt{2C_{NJ}(\ell_2-\ell_1)}.$$

On the other hand, it is easy to check that, in fact, the dual space $(\ell_2-\ell_1)^*$ is just $\mathbb{R}^2$ endowed with the $\ell_2-\ell_\infty$ norm defined by

$$\|(x_1,x_2)\|=\begin{cases}\|(x_1,x_2)\|_2, & x_1x_2\geqslant 0,\\ \|(x_1,x_2)\|_\infty, & x_1x_2<0.\end{cases}$$

Again from [13] we know that $J((\ell_2-\ell_1)^*)\geqslant 1+\frac{1}{\sqrt{2}}\approx 1.7071>J(\ell_2-\ell_1)$. It is easy to see that $C_{NJ}(\ell_2-\ell_1)=C_{NJ}((\ell_2-\ell_1)^*)=\frac{3}{2}$ and $\varepsilon_0((\ell_2-\ell_1)^*)=1$. Bearing in mind inequalities (3.3), we deduce that

$$\begin{aligned}1+\frac{1}{\sqrt{2}}&\leqslant J((\ell_2-\ell_1)^*)\leqslant T((\ell_2-\ell_1)^*)\leqslant H_\nu((\ell_2-\ell_1)^*)\\ &\leqslant A_2((\ell_2-\ell_1)^*)=A_2(\ell_2-\ell_1)=1+\frac{1}{\sqrt{2}}.\end{aligned}$$

Hence, $J((\ell_2-\ell_1)^*) = H_\nu((\ell_2-\ell_1)^*) = 1+\frac{1}{\sqrt{2}} \geqslant H_\nu(\ell_2-\ell_1)$. In particular, we have $T((\ell_2-\ell_1)^*) = 1+\frac{1}{\sqrt{2}} > \sqrt{2\sqrt{2}} = T(\ell_2-\ell_1)$ and $A_2((\ell_2-\ell_1)^*) = A_2(\ell_2-\ell_1) = 1+\frac{1}{\sqrt{2}}$. Note that the space $\ell_2-\ell_1$ is an example for which $H_\nu(\ell_2-\ell_1) \neq H_\nu((\ell_2-\ell_1)^*)$ in general.

4. Stability under norm perturbations

THEOREM 2. *For any Banach space X,*

$$H_\nu(X) = \sup_{x,y\in B_X} M_\nu\big(\|x+y\|,\|x-y\|\big)$$

and

$$h_\nu(X) = \inf_{x\in S_X}\sup_{y\in B_X} M_\nu\big(\|x+y\|,\|x-y\|\big),$$

where $0 \leqslant \nu \leqslant 1$.

Proof. Suppose that $u \in S_X$ and $v \in B_X$. It follows from [15, p. 60] that there exist $x,y \in S_X$ such that

$$\|u-v\| = \|x-y\| \qquad \text{and} \qquad \|u+v\| \leqslant \|x+y\|.$$

Hence, we have

$$\begin{aligned} M_\nu\big(\|u-v\|,\|u+v\|\big) &\leqslant M_\nu\big(\|x+y\|,\|x-y\|\big) \\ &\leqslant \sup_{x,y\in S_X} M_\nu\big(\|x+y\|,\|x-y\|\big) \\ &= H_\nu(X), \end{aligned}$$

from which it follows that

$$H_\nu(X) \geqslant \sup_{x\in S_X,y\in B_X} M_\nu\big(\|x+y\|,\|x-y\|\big).$$

On the other hand, let $u,v \in B_X$ assume without loss of generality that $\|u\| \geqslant \|v\| > 0$. Thus, we have

$$\begin{aligned} M_\nu\big(\|u-v\|,\|u+v\|\big) &= \|u\| M_\nu\left(\left\|\frac{u}{\|u\|}-\frac{v}{\|u\|}\right\|,\left\|\frac{u}{\|u\|}+\frac{v}{\|u\|}\right\|\right) \\ &\leqslant \sup_{x\in S_X,y\in B_X} M_\nu\big(\|x+y\|,\|x-y\|\big). \end{aligned}$$

This implies that

$$\sup_{x\in S_X,y\in B_X} M_\nu\big(\|x+y\|,\|x-y\|\big) \geqslant \sup_{x,y\in B_X} M_\nu\big(\|x+y\|,\|x-y\|\big)$$

and so the first identity follows. Similarly, we get the second identity. $\square$

Recall the Banach-Mazur distance between isomorphic Banach spaces X and Y is defined as

$$d(X,Y) = \inf\left\{\|T\|\|T^{-1}\|\right\},$$

where the infimum is taken over all bicontinuous linear operators T from X onto Y.

THEOREM 3. *Let X and Y be isomorphic Banach spaces. Then for $0 \leqslant \nu \leqslant 1$,*

$$\frac{H_\nu(X)}{d(X,Y)} \leqslant H_\nu(Y) \leqslant H_\nu(X)d(X,Y).$$

In particular, $H_\nu(X) = H_\nu(Y)$ if X and Y are isometric.

Proof. Suppose that $x,y \in S_X$. By the definition of Banach-Mazur distance, for each $\varepsilon > 0$, there exists an operator T from X onto Y such that

$$\|T\|\|T^{-1}\| \leqslant d(X,Y)(1+\varepsilon).$$

Consider

$$y_1 = \frac{Tx_1}{\|T\|} \in B_Y \qquad \text{and} \qquad y_2 = \frac{Tx_2}{\|T\|} \in B_Y.$$

According to Theorem 2, we obtain

$$\begin{aligned} M_\nu\left(\|x_1+x_2\|,\|x_1-x_2\|\right) &= \|T\|M_\nu\left(\|T^{-1}(y_1+y_2)\|,\|T^{-1}(y_1-y_2)\|\right) \\ &\leqslant d(X,Y)(1+\varepsilon)M_\nu\left(\|y_1+y_2\|,\|y_1-y_2\|\right) \\ &\leqslant d(X,Y)(1+\varepsilon)H_\nu(Y), \end{aligned}$$

which implies that

$$H_\nu(X) \leqslant d(X,Y)(1+\varepsilon)H_\nu(Y).$$

The last inequality is true for every $\varepsilon > 0$, so we obtain the left-hand side of our assertion. The right-hand side of our assertion follows by simply interchanging X and Y. $\square$

COROLLARY 2. *Let $X_1 = (X,\|\cdot\|_1)$ and $X_2 = (X,\|\cdot\|_2)$, where $\|\cdot\|_1$ and $\|\cdot\|_2$ are two equivalent norms in X such that*

$$\alpha\|\cdot\|_1 \leqslant \|\cdot\|_2 \leqslant \beta\|\cdot\|_1 \qquad (0 < \alpha \leqslant \beta).$$

Then

$$\frac{\alpha}{\beta}H_\nu(X_1) \leqslant H_\nu(X_2) \leqslant \frac{\beta}{\alpha}H_\nu(X_1),$$

where $0 \leqslant \nu \leqslant 1$.

Proof. This follows from Theorem 3 and the fact that $d(X_1,X_2) \leqslant \frac{\beta}{\alpha}$. $\square$

A Banach space X is finitely representable in a Banach space Y if for every $\varepsilon > 0$ and for every finite-dimensional subspace X_0 of X, there exists a finite-dimensional subspace Y_0 of Y with $\dim(X_0) = \dim(Y_0)$ such that $d(X_0,Y_0) \leqslant 1+\varepsilon$.

COROLLARY 3. *Let X be a Banach space which is finitely representable in Y. Then for $0 \leqslant \nu \leqslant 1$,*

(i) $H_\nu(X) \leqslant H_\nu(Y)$.

(ii) $H_\nu(X^{**}) = H_\nu(X)$.

Proof. (i) For any $x,y \in S_X$, let X_0 be a two-dimensional subspace that contains x and y. For every $\varepsilon > 0$, since X is finitely representable in Y, there exists a two-dimensional subspace Y_0 of Y such that $d(X_0,Y_0) \leqslant 1+\varepsilon$. By virtue of Theorem 3 to the pair of X_0 and Y_0, we obtain

$$H_\nu(X) \leqslant (1+\varepsilon)H_\nu(Y).$$

The last inequality is true for every $\varepsilon > 0$, so we obtain the desired inequality.

(ii) For any Banach space X, by using the principle of local reflexivity, X^{**} is always finitely representable in X. Hence, due to (i), we have $H_\nu(X^{**}) \leqslant H_\nu(X)$. On the other hand, X is isometric to a subspace of X^{**} and so $H_\nu(X) \leqslant H_\nu(X^{**})$. $\square$

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