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EMBEDDING GENERALIZED WIENER CLASSES INTO LIPSCHITZ SPACES

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Abstract. In this note, we give a necessary and sufficient condition for embedding the classes $\Lambda\mathrm{BV}^{(p_n\uparrow p)}$ into the generalized Lipschitz spaces H_q^ω $(1\leqslant q<p\leqslant\infty)$.

In 1972, Waterman [8] introduced the class of functions of Λ-bounded variation. The major motivation behind this generalization of the concept of bounded variation stemmed from the study of convergence of Fourier series for any change of variable. On the other hand, aiming to study the convergence of Fourier series for classes larger than Wiener's class, Kita and Yoneda [4] defined a new function space $BV^{(p_n\uparrow p)}$, which was a direct generalization of Wiener's class BV_p. Combining these two notions leads to the following definition.

DEFINITION 1. Let $\Lambda=\{\lambda_j\}$ be a nondecreasing sequence of positive numbers such that $\sum_{j=1}^\infty \frac{1}{\lambda_j}=\infty$. Such a sequence is called a Waterman sequence. Let $\{p_n\}_{n=1}^\infty$ and $\{\delta_n\}_{n=1}^\infty$ be sequences of positive real numbers such that $1\leqslant p_n\uparrow p\leqslant\infty$ and $2\leqslant\delta_n\uparrow\infty$. A real-valued function f on $[a,b]$ is said to be of p_n-Λ-bounded variation if

$$V_\Lambda(f)=V_\Lambda(f;p_n\uparrow p,\delta):=\sup_{n\geqslant 1}\sup_{\{I_j\}}\left(\sum_{j=1}^{s}\frac{|f(I_j)|^{p_n}}{\lambda_j}\right)^{\frac{1}{p_n}}<\infty,$$

where the $\{I_j\}_{j=1}^s$ are collections of nonoverlapping subintervals of $[a,b]$ such that $\inf_j|I_j|\geqslant\frac{b-a}{\delta_n}$. The class of functions of p_n-Λ-bounded variation is denoted by $\Lambda\mathrm{BV}^{(p_n\uparrow p)}$ and was introduced in [6]. In the sequel we will assume that $[a,b]=[0,1]$ and that functions are 1-periodic.

When $p_n=p$ for all n, we get the Waterman–Shiba classes $\Lambda\mathrm{BV}^{(p)}$ (see [5]). More specifically, when $p=1$, the well-known Waterman class $\Lambda\mathrm{BV}$ is obtained (see [8]). In the case $\lambda_j=1$ for all j and $\delta_n=2^n$ for all n, we get the class $\mathrm{BV}^{(p_n\uparrow p)}$ – introduced by Kita and Yoneda (see [4]) – which in turn recedes to the Wiener class BV_p, when $p_n=p$ for all n.

DEFINITION 2. A function $\omega:[0,1]\to\mathbb{R}$ is said to be a modulus of continuity if it is nondecreasing, continuous and $\omega(0)=0$. We denote by H_q^ω the class of all

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1-periodic functions $f:[0,1]\to\mathbb{R}$ with the property that $\omega_q(f,\delta)=\mathrm{O}(\omega(\delta))$ as $\delta\to 0^+$, where

$$\omega_q(f,\delta):=\begin{cases}\sup_{0\leqslant\gamma\leqslant\delta}\left(\int_0^1|f(t+\gamma)-f(t)|^q dt\right)^{\frac{1}{q}}, & 1\leqslant q<\infty,\\ \sup_{0\leqslant\gamma\leqslant\delta}\sup_{t\in[0,1]}|f(t+\gamma)-f(t)|, & q=\infty.\end{cases}$$

The problem of determining when various classes of functions of generalized bounded variation embed into generalized Lipschitz classes H_q^ω has been investigated by several authors. In particular, Goginava [1] characterized the embedding $\Lambda BV\subseteq H_q^\omega$. Hormozi [3] extended this result to give a necessary and sufficient condition for the embedding $\Lambda BV^{(p)}\subseteq H_q^\omega$, and Wang [7] characterized the embedding $\Lambda_\varphi BV\subseteq H_q^\omega$. Recently, Wu characterized the embedding $\Phi BV\subseteq H_q^\omega$ in [9].

In this note, we characterize the embedding $\Lambda BV^{(p_n\uparrow p)}\subseteq H_q^\omega$ for $1\leqslant q<p\leqslant\infty$. Before proceeding to the main result we state the following lemma which will be needed in the proof.

LEMMA. *Let n be a positive integer and $1\leqslant q<\infty$. Then*

$$\left(\sum_{j=1}^n x_j^q z_j\right)^{\frac{1}{q}}\leqslant\sum_{j=1}^n x_j y_j\max_{1\leqslant k\leqslant n}\left(\sum_{j=1}^k z_j\right)^{\frac{1}{q}}\left(\sum_{j=1}^k y_j\right)^{-1},\tag{1}$$

where $\{x_j\}$, $\{y_j\}$ and $\{z_j\}$ are positive nonincreasing sequences.

Proof. See [2, Proposition 2.1]. □

Our main result can be formulated as follows.

THEOREM. *Let $1\leqslant q<p\leqslant\infty$. Then the embedding $\Lambda\mathrm{BV}^{(p_n\uparrow p)}\subseteq H_q^\omega$ holds if and only if*

$$\limsup_{n\to\infty}\left\{\omega(1/\delta_n)^{-1}\min_{1\leqslant s\leqslant n}\left\{\delta_s^{-\frac{1}{q}}\max_{1\leqslant k\leqslant\delta_s}k^{\frac{1}{q}}\Lambda(k)^{-\frac{1}{p_s}}\right\}\right\}<\infty,\tag{2}$$

where $\Lambda(r):=\sum_{j=1}^{\lfloor r\rfloor}\frac{1}{\lambda_j}$ for $r\geqslant 1$.

Proof. Sufficiency. Let $n\in\mathbb{N}$, $1\leqslant s\leqslant n$ and $h\in(0,\frac{1}{\delta_n})$. Set

$$L:=\lceil\tfrac{1}{h\delta_s}\rceil+1,\quad K:=\lfloor L^{-1}\lfloor\tfrac{1}{h}\rfloor\rfloor.$$

If we put

$$x_{j,k}:=x+((j-1)L+k)h,$$

for positive integers j and k, then

$$\int_0^1 |f(x+h)-f(x)|^q dx \leqslant \sum_{k=1}^{\lceil \frac{1}{h} \rceil} \int_0^h |f(x_{1,k})-f(x_{1,k-1})|^q dx$$
$$= \sum_{k=1}^{L} \sum_{j=1}^{K} \int_0^h |f(x_{j,k})-f(x_{j,k-1})|^q dx$$
$$+ \sum_{k=KL+1}^{\lceil \frac{1}{h} \rceil} \int_0^h |f(x_{1,k})-f(x_{1,k-1})|^q dx.$$

We will now estimate the latter two summands. Since $\frac{1}{\delta_s} \leqslant (L-1)h < Lh \leqslant \frac{3}{\delta_s}$, one can observe that

$$\begin{aligned}
&\sum_{k=1}^{L} \sum_{j=1}^{K} \int_0^h |f(x_{j,k})-f(x_{j,k-1})|^q dx \\
&\leqslant 2^q \sum_{k=1}^{L} \sum_{j=1}^{K} \int_0^h |f(x_{j,k})-f(x_{j+1,k-1})|^q dx + 2^q \sum_{k=1}^{L} \sum_{j=1}^{K} \int_0^h |f(x_{j+1,k-1})-f(x_{j,k-1})|^q dx \\
&\leqslant 2^{q+1} \sum_{k=1}^{L} \int_0^h V_\Lambda(f) \max_{1\leqslant l\leqslant \delta_s} l\Lambda(l)^{\frac{-q}{p_s}} dx \\
&\leqslant 2^{q+1} Lh V_\Lambda(f) \max_{1\leqslant l\leqslant \delta_s} l\Lambda(l)^{-\frac{q}{p_s}} \\
&\leqslant \frac{2^{q+3}}{\delta_s} V_\Lambda(f) \max_{1\leqslant l\leqslant \delta_s} l\Lambda(l)^{\frac{-q}{p_s}},
\end{aligned}$$

where the second inequality is obtained by an application of the above lemma with q/p_s in place of q,

$$z_j := 1, \quad y_j := 1/\lambda_j \quad \text{and} \quad x_j := |f(x_{j,k})-f(x_{j+1,k-1})|^{p_s}, |f(x_{j+1,k-1})-f(x_{j,k-1})|^{p_s}.$$

(Here we have assumed, without loss of generality, that the x_j are arranged in descending order.)

In a similar way, noting that $\lceil \frac{1}{h} \rceil - KL \leqslant L$ we obtain

$$\begin{aligned}
&\sum_{k=KL+1}^{\lceil \frac{1}{h} \rceil} \int_0^h |f(x_{1,k})-f(x_{1,k-1})|^q dx \\
&\leqslant 2^q \sum_{k=KL+1}^{\lceil \frac{1}{h} \rceil} \int_0^h |f(x_{1,k})-f(x_{1,k-L})|^q dx + 2^q \sum_{k=KL+1}^{\lceil \frac{1}{h} \rceil} \int_0^h |f(x_{1,k-L})-f(x_{1,k-1})|^q dx \\
&\leqslant 2^{q+1} Lh V_\Lambda(f) \max_{1\leqslant l\leqslant \delta_s} l\Lambda(l)^{-\frac{q}{p_s}} \\
&\leqslant \frac{2^{q+3}}{\delta_s} V_\Lambda(f) \max_{1\leqslant l\leqslant \delta_s} l\Lambda(l)^{\frac{-q}{p_s}}.
\end{aligned}$$

Consequently,

$$\omega_q(\tfrac{1}{\delta_n}, f)^q \leqslant \frac{2^{q+5}}{\delta_s} V_\Lambda(f) \max_{1 \leqslant l \leqslant \delta_s} l\Lambda(l)^{\frac{-q}{ps}},$$

from which it follows that condition (2) is sufficient for $\Lambda\mathrm{BV}^{(p_n \uparrow p)} \subseteq H_q^\omega$.

Necessity. To proceed by contraposition, it is enough to show that if (2) does not hold, then we can construct a function $g \in \Lambda\mathrm{BV}^{(p_n \uparrow p)}$ such that $g \notin H_q^\omega$. So, let us suppose that condition (2) is not satisfied. Then there exist sequences $(n_k)_k$ and $(m_s)_s$ such that for all k and s

$$\delta_{n_k} \geqslant 2^{k+2} \tag{3}$$

and

$$\omega\left(\frac{1}{\delta_{n_k}}\right)^{-1} \Phi_k > 2^{8k}, \tag{4}$$

where

$$1 \leqslant m_s \leqslant \delta_s,$$

$$\Phi_k := \min_{1 \leqslant s \leqslant n_k} \delta_s^{-\frac{1}{q}} m_s^{\frac{1}{q}} \Lambda(m_s)^{\frac{-1}{ps}},$$

and

$$\max_{1 \leqslant \rho \leqslant \delta_s} \rho^{\frac{1}{q}} \Lambda(\rho)^{\frac{-1}{ps}} = m_s^{\frac{1}{q}} \Lambda(m_s)^{\frac{-1}{ps}}.$$

Let N_k be the largest integer such that $2N_k - 1 \leqslant \frac{\delta_{n_k}}{2^k}$ and define

$$g_k(y) := \begin{cases} 2^{-k-2}\Phi_k, & y \in [\frac{1}{2^k} + \frac{2j-2}{\delta_{n_k}}, \frac{1}{2^k} + \frac{2j-1}{\delta_{n_k}}); \quad 1 \leqslant j \leqslant N_k, \\ 0, & \text{otherwise.} \end{cases}$$

Since the g_k's have disjoint supports, $g(x) := \sum_{k=1}^\infty g_k(x)$ is a well-defined function on $[0,1]$. Moreover, $V_\Lambda(g) \leqslant \sum_{k=1}^\infty V_\Lambda(g_k) < \infty$, as $m_{n_k} = \lfloor \delta_{n_k} \rfloor$ for sufficiently large k and

$$\begin{aligned}
V_\Lambda(g_k) &\leqslant \sup_r \Big(\sum_{j=1}^{\lfloor \delta_r \rfloor} \frac{(2^{-k-2}\Phi_k)^{p_r}}{\lambda_j} \Big)^{\frac{1}{p_r}} \\
&\leqslant 2^{-k-2} \left\{ \sup_{r \leqslant n_k} \Phi_k \, (2\Lambda(\delta_r))^{\frac{1}{p_r}} + \Phi_k \cdot \sup_{r > n_k} (2\Lambda(N_k))^{\frac{1}{p_r}} \right\} \\
&\leqslant 2^{-k-1} \left\{ \sup_{r \leqslant n_k} \inf_{s \leqslant n_k} \left(\frac{m_s}{\delta_s}\right)^{\frac{1}{q}} \Lambda(m_s)^{\frac{-1}{ps}} \Lambda(\delta_r)^{\frac{1}{pr}} + \left(\frac{m_{n_k}}{\delta_{n_k}}\right)^{\frac{1}{q}} \Lambda(m_{n_k})^{\frac{-1}{p_{n_k}}} \sup_{r > n_k} \Lambda(N_k)^{\frac{1}{pr}} \right\} \\
&\leqslant 2^{-k-1} \left\{ \sup_{r \leqslant n_k} \left(\frac{m_r}{\delta_r}\right)^{\frac{1}{q}} \Lambda(m_r)^{\frac{-1}{pr}} \Lambda(\delta_r)^{\frac{1}{pr}} + \Lambda(\delta_{n_k})^{\frac{-1}{p_{n_k}}} \sup_{r > n_k} \Lambda(N_k)^{\frac{1}{pr}} \right\} \\
&\leqslant 2^{-k-1} \left\{ \sup_{r \leqslant n_k} \left(\frac{m_r}{\delta_r}\right)^{\frac{1}{q}} \Lambda(m_r)^{\frac{-1}{pr}} \left(\frac{\delta_r}{m_r}\right)^{\frac{1}{pr}} \Lambda(m_r)^{\frac{1}{pr}} + \Lambda(\delta_{n_k})^{\frac{-1}{p_{n_k}}} \Lambda(\delta_{n_k})^{1/p_{n_k}} \right\} \\
&= O(2^{-k}),
\end{aligned}$$

where $\left(\frac{\delta_r}{m_r}\right)^{\frac{1}{p_r}-\frac{1}{q}}$ is bounded since $p_r < q$ for finitely many integers r.

In conclusion, we show that $g \notin H_q^\omega$. Note that from the definition of N_k and (3), one can easily observe that $\frac{2N_k-1}{\delta_{n_k}} \geqslant 2^{-k-1}$ for all k. Thus

$$\begin{aligned}
\omega_q(\tfrac{1}{\delta_{n_k}}, g)^q &= \sup_{0<\gamma\leqslant\frac{1}{\delta_{n_k}}} \int_0^1 |g(x+\gamma)-g(x)|^q\,dx \geqslant \int_0^1 \left|g\left(x+\frac{1}{\delta_{n_k}}\right)-g(x)\right|^q dx \\
&\geqslant \int_{\frac{1}{2^k}}^{\frac{1}{2^k}+\frac{2N_k-1}{\delta_{n_k}}} \left|g\left(x+\frac{1}{\delta_{n_k}}\right)-g(x)\right|^q dx \\
&= \frac{2N_k-1}{\delta_{n_k}} \cdot 2^{-kq-2q}|\Phi_k|^q \geqslant \frac{1}{2^{kq+2q+k+1}}|\Phi_k|^q.
\end{aligned}$$

This along with (4) implies

$$\frac{\omega_q(\frac{1}{\delta_{n_k}}, g)}{\omega(\frac{1}{\delta_{n_k}})} \geqslant \frac{1}{2^{3k+3}} \cdot \frac{1}{\omega(\frac{1}{\delta_{n_k}})} \cdot |\Phi_k| \geqslant 2^k \quad \text{for all } k,$$

which means that $g \notin H_q^\omega$. $\square$

The following result was first obtained in [3].

COROLLARY. *Let $1 \leqslant q \leqslant p < \infty$. Then the embedding $\Lambda\mathrm{BV}^{(p)} \subseteq H_q^\omega$ holds if and only if*

$$\limsup_{n\to\infty} \frac{1}{\omega(1/n)\left(\sum_{j=1}^n \frac{1}{\lambda_j}\right)^{\frac{1}{p}}} < \infty.$$

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