Mathematical
Inequalities
& Applications
Volume 22, Number 1 (2019), 353–360 doi:10.7153/mia-2019-22-26

INEQUALITIES INVOLVING GEGENBAUER POLYNOMIALS AND THEIR TANGENT LINES

TOMASZ HRYCAK AND SEBASTIAN SCHMUTZHARD

(*Communicated by T. Erdélyi*)

Abstract. On the interval $[-1,1]$, the Gegenbauer polynomial C_n^λ $(\lambda > 0)$ is greater than or equal to its tangent line at the point $z_0 = 1$. We derive a lower bound for the difference of C_n^λ and this tangent line.

1. Introduction

Our main objective is to derive estimates from below on the difference of the Gegenbauer polynomial C_n^λ and its tangent line at the point $z_0 = 1$. For Chebyshev polynomials, such an inequality is proved in [2], and used to obtain error estimates for a three-term recurrence. In this paper, we generalize that result to Gegenbauer polynomials C_n^λ with an arbitrary $\lambda > 0$.

In Theorem 3.1, we demonstrate that if $\lambda > 0$, $n \geqslant 2$ and $-1 \leqslant z \leqslant 1$, then

$$C_n^\lambda(z) - \left[C_n^\lambda(1) + \left(C_n^\lambda\right)'(1)\,(z-1)\right] \tag{1}$$

$$\geqslant \frac{2(\lambda+1)}{2\lambda+1}\, C_n^\lambda(1)\, \min\left(\frac{n^2(n+2\lambda)^2}{16(\lambda+1)^2}\,(1-z)^2, \left(1-\cos\frac{\pi}{\lambda+2}\right)^2\right). \tag{2}$$

For a proof of (1)–(2), we study a solution of a certain nonlinear equation involving the Gegenbauer functions C_ν^λ with a real degree $\nu > 1$. In Theorem 2.1, we show that the solution is a decreasing function of ν, and thus the general case is reduced to the case $n = 2$.

Setting $\lambda = \frac{1}{2}$ in (1)–(2), we obtain the following inequality for Legendre polynomials P_n, $n \geqslant 2$,

$$P_n(z) - \left[1 + \frac{1}{2}n(n+1)(z-1)\right] \geqslant \min\left(\frac{1}{24}n^2(n+1)^2(1-z)^2, \frac{15}{16}\left(3-\sqrt{5}\right)\right). \tag{3}$$

We note that the expression on the left-hand side of (3) is $\mathscr{O}(n^2)$ as $n \to \infty$, while the first argument of the minimum asymptotes to a multiple of n^4, so the inequality is generally invalid without taking the minimum. In the limiting case $\lambda \to 0^+$, we have the following bound for Chebyshev polynomials T_n, $n \geqslant 2$ [2, Theorem 2],

$$T_n(z) - \left[1 + n^2(z-1)\right] \geqslant \min\left(\frac{1}{8}n^4(1-z)^2, 2\right). \tag{4}$$

Mathematics subject classification (2010): 33C45, 26D05.
Keywords and phrases: Gegenbauer polynomial, Gegenbauer function, tangent line, inequalities.

2. Properties of Gegenbauer functions

The main result of this section is Theorem 2.1, which establishes monotonicity of solutions of an equation involving Gegenbauer functions of the first kind C_ν^λ. We use this theorem in the proof of Theorem 3.1.

The Gegenbauer function C_ν^λ can be defined in terms of the hypergeometric function [3, 15.9.15]

$$C_\nu^\lambda(z) = \frac{\Gamma(\nu+2\lambda)}{\Gamma(2\lambda)\,\Gamma(\nu+1)}\,{}_2F_1\left(\nu+2\lambda,-\nu;\lambda+\frac{1}{2};\frac{1}{2}(1-z)\right). \tag{5}$$

From this representation, we deduce that

$$C_\nu^\lambda(1) = \frac{\Gamma(\nu+2\lambda)}{\Gamma(2\lambda)\,\Gamma(\nu+1)}, \tag{6}$$

$$\left(C_\nu^\lambda\right)'(1) = \frac{2\lambda\,\Gamma(\nu+2\lambda+1)}{\Gamma(2\lambda+2)\,\Gamma(\nu)}, \tag{7}$$

where $\left(C_\nu^\lambda\right)'(z) = \frac{d}{dz}C_\nu^\lambda(z)$. The expressions in equations (5)–(7) are well-defined for any $\lambda,\nu \in \mathbb{C}$ as long as $\nu+2\lambda \neq 0,-1,-2,\ldots$.

In the next two lemmas, we derive integral representations for C_ν^λ and $\left(C_\nu^\lambda\right)'$ of the Dirichlet-Mehler type. The assumption $\lambda > 0$ guarantees convergence of the integrals.

LEMMA 2.1. *If* $\lambda > 0$, $\nu+2\lambda \neq 0,-1,\ldots,$ *and* $0<\theta<\pi$, *then*

$$C_\nu^\lambda(\cos\theta) = \frac{2^\lambda\,\Gamma(\lambda+\frac{1}{2})\,(\sin\theta)^{1-2\lambda}}{\sqrt{\pi}\,\Gamma(\lambda)} \tag{8}$$

$$\times\; C_\nu^\lambda(1)\int_0^\theta \cos(\nu+\lambda)t\ (\cos t-\cos\theta)^{\lambda-1}\,dt. \tag{9}$$

Proof. The claim follows by substituting (6) into the following formula [1, 3.15.2 (23)]

$$C_\nu^\lambda(\cos\theta) = \frac{2^\lambda\,\Gamma(\lambda+\frac{1}{2})\,\Gamma(\nu+2\lambda)\,(\sin\theta)^{1-2\lambda}}{\sqrt{\pi}\,\Gamma(\lambda)\,\Gamma(2\lambda)\,\Gamma(\nu+1)}$$
$$\times\int_0^\theta \cos(\nu+\lambda)t\ (\cos t-\cos\theta)^{\lambda-1}\,dt. \qquad \square$$

In the following lemma, if $\nu+\lambda=0$, the quotient $\frac{\sin(\nu+\lambda)t}{\nu+\lambda}$ should be replaced with t.

LEMMA 2.2. *If* $\lambda > 0$, $\nu+2\lambda \neq 0,-1,\ldots,$ *and* $0<\theta<\pi$, *then*

$$\left(C_\nu^\lambda\right)'(\cos\theta) = \frac{2^{\lambda+1}\,\Gamma(\lambda+\frac{3}{2})\,(\sin\theta)^{-1-2\lambda}}{\sqrt{\pi}\,\Gamma(\lambda)\,(\nu+\lambda)} \tag{10}$$

$$\times\;\left(C_\nu^\lambda\right)'(1)\int_0^\theta \sin t\,\sin(\nu+\lambda)t\ (\cos t-\cos\theta)^{\lambda-1}\,dt. \tag{11}$$

Proof. If $\nu = 0, -1, \ldots$, then by (5) and (7) both sides of (10)–(11) vanish. Thus we may assume that $\nu \neq 0, -1, \ldots$, which implies that $\left(C_\nu^\lambda\right)'(1) \neq 0$. The derivative $\left(C_\nu^\lambda\right)'(z)$ can be expressed through the Gegenbauer function of order $\lambda + 1$ as follows [1, 3.15.2 (30)]

$$\left(C_\nu^\lambda\right)'(z) = 2\lambda\, C_{\nu-1}^{\lambda+1}(z). \tag{12}$$

Consequently,

$$\frac{\left(C_\nu^\lambda\right)'(z)}{\left(C_\nu^\lambda\right)'(1)} = \frac{C_{\nu-1}^{\lambda+1}(z)}{C_{\nu-1}^{\lambda+1}(1)}. \tag{13}$$

An integral representation of $C_{\nu-1}^{\lambda+1}$ can be obtained by using (8)–(9) with parameters $\lambda + 1$ and $\nu - 1$. Substituting this representation into (13), we obtain

$$\left(C_\nu^\lambda\right)'(\cos\theta) = \frac{2^{\lambda+1}\Gamma(\lambda+\frac{3}{2})\,(\sin\theta)^{-1-2\lambda}}{\sqrt{\pi}\,\Gamma(\lambda+1)} \tag{14}$$

$$\times\ \left(C_\nu^\lambda\right)'(1) \int_0^\theta \cos(\nu+\lambda)t\ (\cos t - \cos\theta)^\lambda\, dt. \tag{15}$$

Integrating by parts gives

$$\int_0^\theta \cos(\nu+\lambda)t\ (\cos t - \cos\theta)^\lambda\, dt \tag{16}$$

$$= \frac{\lambda}{\nu+\lambda} \int_0^\theta \sin t\, \sin(\nu+\lambda)t\ (\cos t - \cos\theta)^{\lambda-1}\, dt. \tag{17}$$

Substituting (16)–(17) into (14)–(15) , we arrive at (10)–(11). □

LEMMA 2.3. *If* $\lambda > 0$, $\nu > 1$ *and* $0 < \theta \leqslant \frac{\pi}{\nu+\lambda}$, *then*

$$\left(C_\nu^\lambda\right)'(\cos\theta) < \left(C_\nu^\lambda\right)'(1).$$

Proof. We deduce from (5) that for every $z \in \mathbb{C}$

$$C_1^\lambda(z) = 2\lambda z,$$

which implies that

$$\left(C_1^\lambda\right)'(z) = 2\lambda. \tag{18}$$

It follows from (7) that $\left(C_\nu^\lambda\right)'(1) > 0$ and $\left(C_1^\lambda\right)'(1) > 0$. For $0 < t < \frac{\pi}{\nu+\lambda}$, we have

$$\sin(\nu+\lambda)t < \frac{\nu+\lambda}{1+\lambda}\,\sin(1+\lambda)t,$$

because the function $\frac{\sin z}{z}$ is decreasing on the interval $(0,\pi)$. Substituting this into (10)–(11) and using Lemma 2.2 with $\nu = 1$ gives

$$\begin{aligned}
\left(C_\nu^\lambda\right)'(\cos\theta) &< \frac{2^{\lambda+1}\Gamma(\lambda+\frac{3}{2})\,(\sin\theta)^{-1-2\lambda}}{\sqrt{\pi}\,\Gamma(\lambda)\,(1+\lambda)} \\
&\quad\times \left(C_\nu^\lambda\right)'(1)\int_0^\theta \sin t\,\sin(1+\lambda)t\,(\cos t-\cos\theta)^{\lambda-1}\,dt \\
&= \left(C_\nu^\lambda\right)'(1)\,\frac{\left(C_1^\lambda\right)'(\cos\theta)}{\left(C_1^\lambda\right)'(1)} \\
&= \left(C_\nu^\lambda\right)'(1).
\end{aligned}$$

In the last step, we have used (18). □

From (6) and (7), we infer that

$$\frac{\left(C_\nu^\lambda\right)'(1)}{C_\nu^\lambda(1)} = \frac{\nu(\nu+2\lambda)}{2\lambda+1}. \tag{19}$$

We define the function τ_λ as follows

$$\tau_\lambda(\nu) = \frac{\left(C_\nu^\lambda\right)'(1)}{C_\nu^\lambda(1)}\left(1-\cos\frac{\pi}{\nu+\lambda}\right) = \frac{\nu(\nu+2\lambda)}{2\lambda+1}\left(1-\cos\frac{\pi}{\nu+\lambda}\right). \tag{20}$$

LEMMA 2.4. *For a fixed* $\lambda > 0$, *the function*

$$\sigma_\lambda(\nu) = \frac{1}{C_\nu^\lambda(1)}\,C_\nu^\lambda\left(\cos\frac{\pi}{\nu+\lambda}\right) - 1 + \tau_\lambda(\nu) \tag{21}$$

increases on the interval $\nu > 1$.

Proof. Substituting (8)–(9) into (21), differentiating with respect to ν and setting $\theta = \frac{\pi}{\nu+\lambda}$, we obtain

$$\sigma_\lambda'(\nu) = \frac{2(\nu+\lambda)}{2\lambda+1}\,(1-\cos\theta) - \frac{\nu(\nu+2\lambda)}{(2\lambda+1)(\nu+\lambda)}\,\theta\sin\theta \tag{22}$$

$$+\frac{\partial}{\partial\nu}\left(\frac{1}{C_\nu^\lambda(1)}\,C_\nu^\lambda\right)(\cos\theta) + \frac{1}{C_\nu^\lambda(1)}\left(C_\nu^\lambda\right)'(\cos\theta)\,\frac{\partial\cos\theta}{\partial\nu}. \tag{23}$$

From (19), we deduce that

$$\frac{1}{C_\nu^\lambda(1)}\left(C_\nu^\lambda\right)'(\cos\theta)\,\frac{\partial\cos\theta}{\partial\nu} = \frac{\nu(\nu+2\lambda)}{2\lambda+1}\,\frac{1}{(C_\nu^\lambda)'(1)}\left(C_\nu^\lambda\right)'(\cos\theta)\,\frac{\theta\sin\theta}{\nu+\lambda}. \tag{24}$$

Since $\tan\frac{\theta}{2} > \frac{\theta}{2}$, we have $2(1-\cos\theta) > \theta\sin\theta$. Consequently,

$$\frac{2(\nu+\lambda)}{2\lambda+1}\,(1-\cos\theta) - \frac{\nu(\nu+2\lambda)}{(2\lambda+1)(\nu+\lambda)}\,\theta\sin\theta > \frac{\lambda^2}{(2\lambda+1)(\nu+\lambda)}\,\theta\sin\theta. \tag{25}$$

Differentiating (8)–(9) under the integral sign, we obtain

$$\frac{\partial}{\partial \nu}\left(\frac{1}{C_\nu^\lambda(1)}\,C_\nu^\lambda\right)(\cos\theta) = -\frac{2^\lambda\,\Gamma(\lambda+\frac{1}{2})\,(\sin\theta)^{1-2\lambda}}{\sqrt{\pi}\,\Gamma(\lambda)} \tag{26}$$

$$\times \int_0^\theta t\,\sin(\nu+\lambda)t\;(\cos t-\cos\theta)^{\lambda-1}\,dt. \tag{27}$$

If $0<t<\theta$, then $\frac{\sin\theta}{\theta} < \frac{\sin t}{t}$. Substituting this into (26)–(27) and using (10)–(11), we obtain

$$\frac{\partial}{\partial \nu}\left(\frac{1}{C_\nu^\lambda(1)}\,C_\nu^\lambda\right)(\cos\theta) > -\frac{2^\lambda\,\Gamma(\lambda+\frac{1}{2})\,(\sin\theta)^{-2\lambda}\,\theta}{\sqrt{\pi}\,\Gamma(\lambda)} \tag{28}$$

$$\times \int_0^\theta \sin t\,\sin(\nu+\lambda)t\;(\cos t-\cos\theta)^{\lambda-1}\,dt \tag{29}$$

$$= -\frac{\nu+\lambda}{2\lambda+1}\,\frac{\theta\sin\theta}{(C_\nu^\lambda)'(1)}\,\left(C_\nu^\lambda\right)'(\cos\theta). \tag{30}$$

Substituting (25), (28)–(30) and (24) into (22)–(23) gives

$$\begin{aligned}\sigma_\lambda'(\nu) &> \frac{\lambda^2\,\theta\sin\theta}{(2\lambda+1)(\nu+\lambda)} - \frac{\nu+\lambda}{2\lambda+1}\,\frac{\theta\sin\theta}{(C_\nu^\lambda)'(1)}\,\left(C_\nu^\lambda\right)'(\cos\theta)\\ &\quad+\frac{\nu(\nu+2\lambda)}{(2\lambda+1)(\nu+\lambda)}\,\frac{\theta\sin\theta}{(C_\nu^\lambda)'(1)}\,\left(C_\nu^\lambda\right)'(\cos\theta)\\ &= \frac{\lambda^2\,\theta\sin\theta}{(2\lambda+1)(\nu+\lambda)}\left(1-\frac{1}{(C_\nu^\lambda)'(1)}\,\left(C_\nu^\lambda\right)'(\cos\theta)\right).\end{aligned}$$

If follows from Lemma 2.3 that $\sigma_\lambda'(\nu)>0$. □

THEOREM 2.1. *If* $\lambda>0,\ \nu>1$ *and*

$$0 \leqslant w \leqslant \sigma_\lambda(\nu), \tag{31}$$

then there exists a unique μ *in the interval* $[0,\tau_\lambda(\nu)]$ *such that*

$$\frac{1}{C_\nu^\lambda(1)}\,C_\nu^\lambda\left(1-\frac{2\lambda+1}{\nu(\nu+2\lambda)}\,\mu\right)-1+\mu = w. \tag{32}$$

Moreover, if $\nu_0>1$ *and*

$$0 < w \leqslant \sigma_\lambda(\nu_0), \tag{33}$$

then $\mu=\mu(\nu)$ *exists for every* $\nu\geqslant\nu_0$, *and is a decreasing function of* ν.

Proof. For a fixed $\lambda>0$ and $\nu>1$, we define the function f_ν on the interval $[0,\tau_\lambda(\nu)]$ by the formula

$$f_\nu(\mu) = \frac{1}{C_\nu^\lambda(1)}\,C_\nu^\lambda\left(1-\frac{2\lambda+1}{\nu(\nu+2\lambda)}\,\mu\right)-1+\mu.$$

Differentiating with respect to μ and using (19) gives

$$f_\nu'(\mu) = 1 - \frac{2\lambda+1}{\nu(\nu+2\lambda)}\,\frac{1}{C_\nu^\lambda(1)}\,\left(C_\nu^\lambda\right)'\left(1 - \frac{2\lambda+1}{\nu(\nu+2\lambda)}\,\mu\right)$$
$$= 1 - \frac{1}{(C_\nu^\lambda)'(1)}\,\left(C_\nu^\lambda\right)'\left(1 - \frac{2\lambda+1}{\nu(\nu+2\lambda)}\,\mu\right).$$

We define a variable $\theta \in \left[0, \frac{\pi}{\nu+\lambda}\right]$, by the formula

$$\cos\theta = 1 - \frac{2\lambda+1}{\nu(\nu+2\lambda)}\,\mu. \tag{34}$$

In view of (20), the interval $0 \leqslant \theta \leqslant \frac{\pi}{\nu+\lambda}$ corresponds to the interval $0 \leqslant \mu \leqslant \tau_\lambda(\nu)$. From Lemma 2.3, we have $f_\nu'(\mu) > 0$ for $0 < \theta < \frac{\pi}{\nu+\lambda}$, so the function f_ν is increasing. Therefore, equation (32) has a unique solution whenever $f_\nu(0) \leqslant w \leqslant f_\nu(\tau_\lambda(\nu))$. We note that $f_\nu(0) = 0$ and $f_\nu(\tau_\lambda(\nu)) = \sigma_\lambda(\nu)$.

Let $\nu_0 > 1$, let w satify (33), and let $\nu \geqslant \nu_0$. From Lemma 2.4, we deduce that $f_\nu(\tau_\lambda(\nu)) \geqslant f_{\nu_0}(\tau_\lambda(\nu_0))$, so the solution $\mu = \mu(\nu)$ of (32) exists. Differentiating (32) with respect to ν, collecting the terms and using (19), we obtain

$$-\left[1 - \frac{1}{(C_\nu^\lambda)'(1)}\,\left(C_\nu^\lambda\right)'(\cos\theta)\right]\frac{d\mu}{d\nu} \tag{35}$$
$$= \frac{2(2\lambda+1)(\nu+\lambda)}{\nu^2(\nu+2\lambda)^2}\,\frac{\mu}{C_\nu^\lambda(1)}\,\left(C_\nu^\lambda\right)'(\cos\theta) + \frac{\partial}{\partial\nu}\left(\frac{1}{C_\nu^\lambda(1)}\,C_\nu^\lambda\right)(\cos\theta). \tag{36}$$

The assumption $w > 0$ implies that $\theta > 0$. In view of Lemma 2.3, the bracketed expression in (35) is positive. Substituting (19), (34), (10)–(11) and (26)–(27) into (36), we obtain

$$\frac{2(2\lambda+1)(\nu+\lambda)}{\nu^2(\nu+2\lambda)^2}\,\frac{\mu}{C_\nu^\lambda(1)}\,\left(C_\nu^\lambda\right)'(\cos\theta) + \frac{\partial}{\partial\nu}\left(\frac{1}{C_\nu^\lambda(1)}\,C_\nu^\lambda\right)(\cos\theta) \tag{37}$$
$$= \frac{2(\nu+\lambda)}{2\lambda+1}\,\frac{1-\cos\theta}{(C_\nu^\lambda)'(1)}\,\left(C_\nu^\lambda\right)'(\cos\theta) + \frac{\partial}{\partial\nu}\left(\frac{1}{C_\nu^\lambda(1)}\,C_\nu^\lambda\right)(\cos\theta) \tag{38}$$
$$= \frac{2^\lambda\,\Gamma(\lambda+\frac{1}{2})\,(\sin\theta)^{-1-2\lambda}}{\sqrt{\pi}\,\Gamma(\lambda)} \tag{39}$$
$$\times\left[2(1-\cos\theta)\int_0^\theta \sin t\,\sin(\nu+\lambda)t\,(\cos t - \cos\theta)^{\lambda-1}\,dt\right. \tag{40}$$
$$\left. - \sin^2\theta\int_0^\theta t\,\sin(\nu+\lambda)t\,(\cos t-\cos\theta)^{\lambda-1}\,dt\right]. \tag{41}$$

By definition, $0 \leqslant \theta \leqslant \frac{\pi}{\nu+\lambda} < \pi$. Therefore, if $0 < t < \theta$, then $\sin(\nu+\lambda)t > 0$ and $\frac{\sin t}{t} > \frac{\sin\theta}{\theta}$. Consequently,

$$2(1-\cos\theta)\sin t > 4\sin^2\frac{\theta}{2}\,\frac{\sin\theta}{\theta}\,t = \frac{\tan\frac{\theta}{2}}{\frac{\theta}{2}}\,t\sin^2\theta > t\sin^2\theta.$$

Thus the bracketed expression in lines (40)–(41) is positive. From (35)–(36), we infer that $\frac{d\mu}{d\nu} < 0$. □

3. Inequalities involving Gegenbauer polynomials

The following theorem is our main result.

THEOREM 3.1. *If $\lambda > 0$, $n \geqslant 2$ is an integer and $-1 \leqslant z \leqslant 1$, then* (1)–(2) *holds.*

The minimum function appearing in (2) is easy to evaluate. For $\lambda > 0$ and $n \geqslant 2$, we set

$$z_{n,\lambda} = 1 - \frac{4(\lambda+1)}{n(n+2\lambda)} \left(1 - \cos\frac{\pi}{\lambda+2}\right).$$

If $z_{n,\lambda} \leqslant z \leqslant 1$, then the minimum in (2) is attained at the first term. If $-1 \leqslant z \leqslant z_{n,\lambda}$, then the minimum in (2) is attained at the second term.

From (5), we deduce that

$$\frac{1}{C_2^\lambda(1)}\, C_2^\lambda(z) = \frac{2(\lambda+1)}{2\lambda+1} z^2 - \frac{1}{2\lambda+1}. \tag{42}$$

We note that $z_{2,\lambda} = \cos\frac{\pi}{\lambda+2}$. Comparing the coefficients at z^2, we see that if $n = 2$ and $\cos\frac{\pi}{\lambda+2} \leqslant z \leqslant 1$, then we have equality in (1)–(2).

From (20), (21) and (42), we derive an explicit form of $\sigma_\lambda(2)$

$$\sigma_\lambda(2) = \frac{1}{C_2^\lambda(1)}\, C_2^\lambda\left(\cos\frac{\pi}{\lambda+2}\right) - 1 + \frac{4(\lambda+1)}{2\lambda+1}\left(1 - \cos\frac{\pi}{\lambda+2}\right) \tag{43}$$

$$= \frac{2(\lambda+1)}{2\lambda+1}\left(1 - \cos\frac{\pi}{\lambda+2}\right)^2. \tag{44}$$

Proof of Theorem 3.1. For $n \geqslant 2$, we define the function g_n on the interval $[-1,1]$ by the formula

$$g_n(z) = \frac{1}{C_n^\lambda(1)}\, C_n^\lambda(z) - \left[1 + \frac{n(n+2\lambda)}{2\lambda+1}(z-1)\right]. \tag{45}$$

It is known [3, 18.14.1] that the maximum value of C_n^λ on the interval $[-1,1]$ is attained at $z = 1$. By (12), the same is true for the derivative $\left(C_\nu^\lambda\right)'$. In view of (19), we have

$$g_n'(z) = \frac{1}{C_n^\lambda(1)}\left(C_n^\lambda\right)'(z) - \frac{n(n+2\lambda)}{2\lambda+1}$$
$$= \frac{1}{C_n^\lambda(1)}\left(\left(C_n^\lambda\right)'(z) - \left(C_n^\lambda\right)'(1)\right) \leqslant 0.$$

This implies that g_n is strictly decreasing, since g_n is a non-constant polynomial. Consequently, $g_n(z) \geqslant g_n(1) = 0$. If $g_n(z) = 0$, then $z = 1$ and (1)–(2) holds. If

$g_n(z) \geqslant \sigma_\lambda(2)$, then (1)–(2) is trivially true in view of (43)–(44). It remains to consider the case when $0 < g_n(z) < \sigma_\lambda(2)$. We apply Theorem 2.1 with $w = g_n(z)$ and $v_0 = 2$ to the equation

$$\frac{1}{C_k^\lambda(1)} C_k^\lambda \left(1 - \frac{2\lambda+1}{k(k+2\lambda)} \mu\right) - 1 + \mu = g_n(z). \tag{46}$$

It follows that for $k = 2, 3, \ldots$ this equation has a unique solution μ_k in the interval $[0, \tau_\lambda(k)]$, and this solution is a decreasing function of k. It follows from (45) that $\mu_n = \frac{n(n+2\lambda)}{2\lambda+1}(1-z)$. From (42) and (46), we deduce that $\mu_2^2 = 8\frac{\lambda+1}{2\lambda+1} g_n(z)$. Since the sequence μ_k is non-negative and decreasing, we have

$$8\frac{\lambda+1}{2\lambda+1} g_n(z) = \mu_2^2 \geqslant \mu_n^2 = \frac{n^2(n+2\lambda)^2}{(2\lambda+1)^2}(1-z)^2.$$

This implies (1)–(2). □

Acknowledgements. We are grateful for the reviewers' comments and detailed suggestions.

The first author was supported by the Austrian Science Fund (FWF) grant I-1018-N25. The authors are supported by the Innovationsfonds "Forschung, Wissenschaft und Gesellschaft" of the Austrian Academy of Sciences on the project "Railway vibrations from tunnels".

REFERENCES

[1] HARRY BATEMAN, *Higher Transcendental Functions, Volume 1*, McGraw-Hill Book Company, 1953.

[2] T. HRYCAK, S. SCHMUTZHARD, *Evaluation of Chebyshev polynomials by a three-term recurrence in floating-point arithmetic*, BIT Numer. Math. **58**, (2018), 317–330.

[3] F. W. OLVER, D. W. LOZIER, R. F. BOISVERT, C. W. CLARK, *NIST Handbook of Mathematical Functions*, Cambridge University Press, 2010.

(Received March 27, 2018)

Tomasz Hrycak
Acoustics Research Institute
Austrian Academy of Sciences
Wohllebengasse 12-14, 1040 Vienna, Austria
e-mail: tomasz.hrycak@oeaw.ac.at

Sebastian Schmutzhard
Acoustics Research Institute
Austrian Academy of Sciences
Wohllebengasse 12-14, 1040 Vienna, Austria
e-mail: sschmutzhard@kfs.oeaw.ac.at